The Łojasiewicz exponent in non-degenerate deformations of surface singularities

Tadeusz Krasiński

(joint results with Szymon Brzostowski and Grzegorz Oleksik)

arXiv:2103.08078

University of Łódź

GKŁW Seminar „Singularity Theory"

March 26, 2021,

ZOOM Conference (via Internet)

Introduction

A formula for the Łojasiewicz exponent given in IMPANGA seminar in May 2020.

Introduction

A formula for the Łojasiewicz exponent given in IMPANGA seminar in May 2020.

$\mathcal{L}(f)$ = (in terms of the Newton polyhedron $N(f)$ of f)

Introduction

A formula for the Łojasiewicz exponent given in IMPANGA seminar in May 2020.

$\mathcal{L}(f)$ = (in terms of the Newton polyhedron $N(f)$ of f)

where $f = f(x, y, z)$ – a **non-degenerate** isolated **surface** singularity at $0\ in\ C^3$.

Introduction

The aim:

An application of the above formula to study behaviour of the Łojasiewicz exponent in families of surface singularities.

General introduction to the Łojasiewicz exponent

There are many versions (variants) of the Łojasiewicz exponent.

General introduction to the Łojasiewicz exponent

There are many versions (variants) of the Łojasiewicz exponent.

The main geometric (intuitive) idea is as follows:

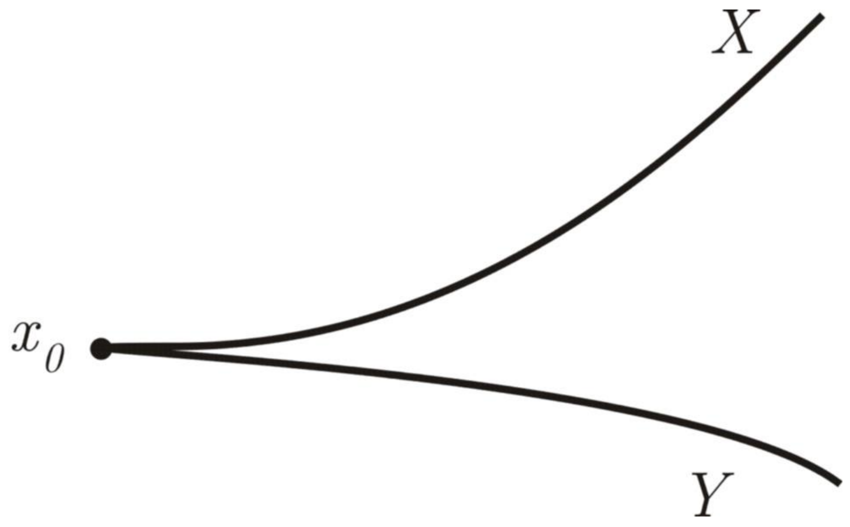
General introduction to the Łojasiewicz exponent

There are many versions (variants) of the Łojasiewicz exponent.

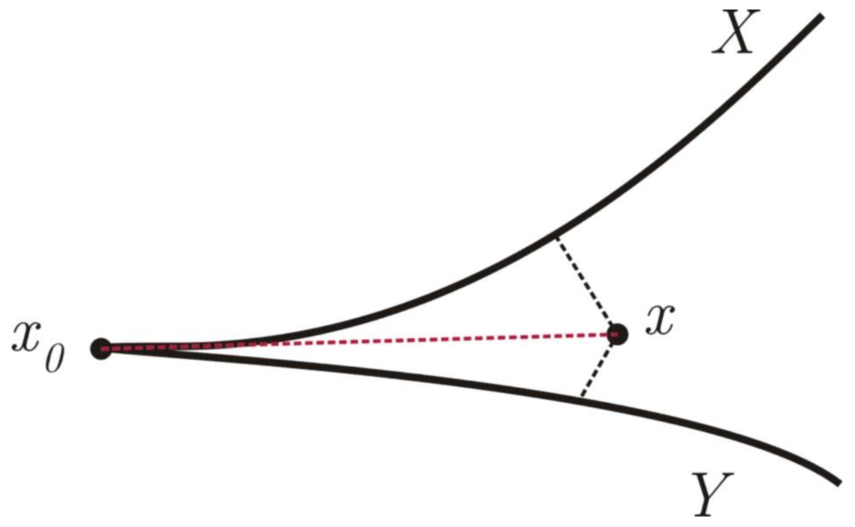
The main geometric (intuitive) idea is as follows:

We have two sets X and Y having a common point x_0 (for simplicity we assume x_0 is an isolated point of this intersection)

General introduction to the Łojasiewicz exponent

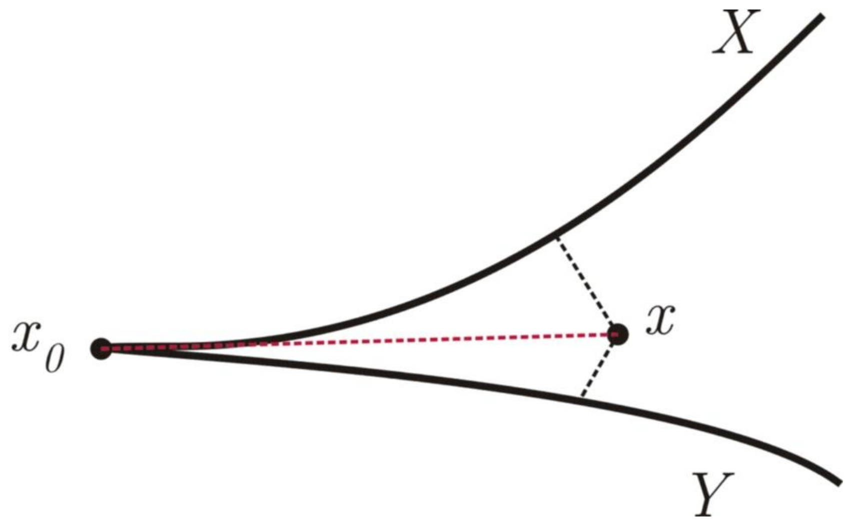


General introduction to the Łojasiewicz exponent



We compare $\rho(x, X) + \rho(x, Y)$ with $\rho(x, X \cap Y)$.

General introduction to the Łojasiewicz exponent

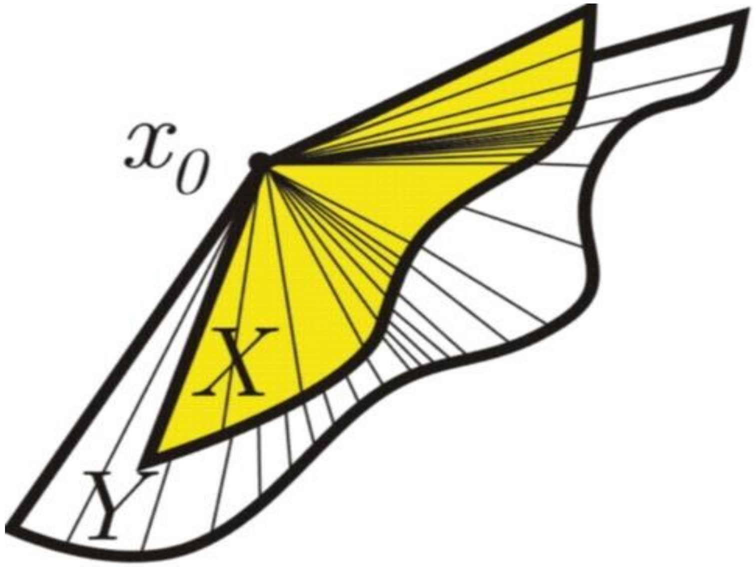


We compare $\rho(x, X) + \rho(x, Y)$ with $\rho(x, X \cap Y)$.

The Łojasiewicz exponent is the greatest $\lambda > 0$ such that

$$\rho(x, X) + \rho(x, Y) \sim \rho(x, X \cap Y)^{\lambda} \text{ when } x \to x_0$$

General introduction to the Łojasiewicz exponent



General introduction to the Łojasiewicz exponent

Two simple corollaries from this (intuitive) definition

General introduction to the Łojasiewicz exponent

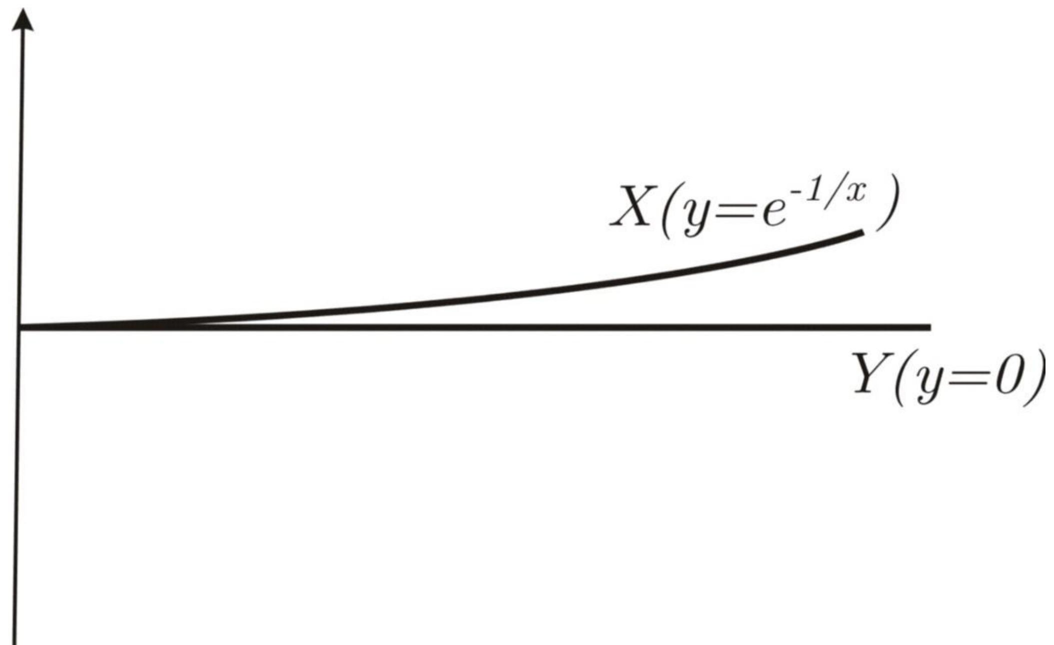
Two simple corollaries from this (intuitive) definitione:

Corollary 1. If sets X and Y are „infinitely” tangent (for instance in C^∞ category) then the Łojasiewicz exponent may not exist.

General introduction to the Łojasiewicz exponent

Two simple corollaries from this (intuitive) definition:

Corollary 1. If sets X and Y are „infinitely” tangent (for instance in C^{∞} category) then the Łojasiewicz exponent may not exist.



General introduction to the Łojasiewicz exponent

Corollary 2. If sets X and Y are submanifolds of R^n with the same tangent space at x_0 then the Łojasiewicz exponent at x_0 and the order of tangency of X and Y at x_0 satisfy the inequality

$$\nu(X,Y) \leq \mathcal{L}(X,Y)$$

General introduction to the Łojasiewicz exponent

Green curves realize the Łojasiewicz exponent (the greatest order of tangency).
Red curves realize the order of tangency (the least order of tangency).

Introduction

We are interested in the following, complex variant:

$$F = \nabla f = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right),$$

where $f: (C^n, 0) \to (C, 0)$ is an **isolated** complex singularity.

In this case we take $X := graph\, \left|\left|\boldsymbol{\nabla f}(\boldsymbol{z})\right|\right| \subset (C^n \times R)$ and $Y := (C^n \times \{0\})$. Of course we have $\{0\} = X \cap Y$ and $\rho(\boldsymbol{z}, X \cap Y) = ||\boldsymbol{z}||$

Introduction

Definition. The best exponent (the supremum) $\lambda \in \boldsymbol{R}$ for which there exists there exists a holomorphic curve $\Phi(t), \Phi(\mathbf{0}) = \mathbf{0}$ such that

$$||\nabla f\big(\Phi(t)\big)|| \sim ||\Phi(t)||^{\lambda}.$$

is the **Łojasiewicz exponent of** f and is denoted by $\mathcal{L}(f)$.

Introduction

Definition. The best exponent (the supremum) $\lambda \in \boldsymbol{R}$ for which there exists there exists a holomorphic curve $\Phi(t), \Phi(\mathbf{0}) = \mathbf{0}$ such that

$$||\boldsymbol{\nabla} f\big(\Phi(t)\big)|| \sim ||\Phi(t)||^{\lambda}.$$

is the **Łojasiewicz exponent of** f and is denoted by $\mathcal{L}(f)$.

Equivalently

Introduction

Definition. The best exponent (the infimum) $\lambda \in \boldsymbol{R}$ such that the following inequality holds

$$\left|\left|\boldsymbol{\nabla f}(\boldsymbol{z})\right|\right| \geq \boldsymbol{C}\ ||\boldsymbol{z}||^{\lambda}$$

in a neighbourhood of the origin in $\boldsymbol{C}^n$ is the **Łojasiewicz exponent of** f and is denoted by $\mathcal{L}(f)$.

Introduction

There are three important topics:

1. In which category is the Łojasiewicz exponent invariant?

Introduction

There are three important topics:

1. In which category is the Łojasiewicz exponent invariant?

2. Find effective formulas for the Łojasiewicz exponent.

Introduction

There are three important topics:

1. In which category is the Łojasiewicz exponent invariant?
2. Find effective formulas for the Łojasiewicz exponent.
3. Explain behaviour of the Łojasiewicz exponent in families of singularities.

Introduction

1. In which category is the Łojasiewicz exponent invariant?

Fact 1. $\mathcal{L}(f)$ is an invariant in C^1 category

$$\mathcal{L}(f \circ F) = \mathcal{L}(f),$$

where F is a local diffeomorphism. Obvious

Introduction

1. In which category is the Łojasiewicz exponent invariant?

Fact 1. $\mathcal{L}(f)$ is an invariant in C^1 category

$$\mathcal{L}(f \circ F) = \mathcal{L}(f),$$

where F is a local diffeomorphism. Obvious.

Fact 2. (Bivia-Ausina, Fukui) $\mathcal{L}(f)$ is an invariant in $\mathrm{bi}-\mathrm{lipschitz}$ category

$$\mathcal{L}(f \circ F) = \mathcal{L}(f),$$

where F is a local bi-lipschitz homeomorphism.

Introduction

1. In which category is the Łojasiewicz exponent invariant?

Open question. Is $\mathcal{L}(f)$ an invariant of homeomorphisms i.e. in C^0 category?

Introduction

1. In which category is the Łojasiewicz exponent invariant?

Open question. Is $\mathcal{L}(f)$ an invariant of homeomorphisms i.e. in C^0 category?

(True for plane curve singularities i.e. $f\colon (C^2,0) \to (C,0)$)

Introduction

2. Find effective formulas for the Łojasiewicz exponent.

Many formulas in various terms, dimensions for various classes of singularities.

Introduction

3. Explain behaviour of the Łojasiewicz exponent in families of singularities.

Introduction

3. Explain behaviour of the Łojasiewicz exponent in families of singularities.

This is the topic of this lecture.

The main result

In general, the Łojasiewicz exponent has no property of semi-continuity in families of isolated singularities.

The main result

In general, the Łojasiewicz exponent has no property of semi-continuity in families of isolated singularities.

Example 1.

$$f_s := x^2 + sy^2 + y^3, \qquad s \in C,$$

The main result

In general, the Łojasiewicz exponent has no property of semi-continuity in families of isolated singularities.

Example 1.

$$f_s := x^2 + sy^2 + y^3, \qquad s \in C,$$

$$\nabla f_0 = (2x, 3y^2) \qquad \text{for s} = 0$$
$$\nabla f_s = (2x, 2sy + 3y^2) \quad \text{for s} \neq 0$$

The main result

In general, the Łojasiewicz exponent has no property of semi-continuity in families of isolated singularities.

Example 1.

$$f_s := x^2 + sy^2 + y^3, \qquad s \in C,$$

$$\nabla f_0 = (2x, 3y^2) \qquad \text{for s} = 0$$
$$\nabla f_s = (2x, 2sy + 3y^2) \quad \text{for s} \neq 0$$

Hence

$$\mathcal{L}(f_0) = 2 \quad (\text{take } \Phi(t) = (0, t))$$
$$\mathcal{L}(f_s) = 1 \quad (\text{take } \Phi(t) = (0, t))$$

The main result

In general, the Łojasiewicz exponent has no property of semi-continuity in families of isolated singularities.

Example 2.

$$f_s := xy^5 + sx^2 + x^8, \quad s \in C,$$

The main result

In general, the Łojasiewicz exponent has no property of semi-continuity in families of isolated singularities.

Example 2.

$$f_s := xy^5 + sx^2 + x^8, \quad s \in C,$$

$$\nabla f_0 = (y^5 + 8x^7, 5xy^4) \qquad \text{for } s = 0$$

$$\nabla f_s = (y^5 + 2sx + 8x^7, 5xy^4) \quad \text{for } s \neq 0.$$

The main result

In general, the Łojasiewicz exponent has no property of semi-continuity in families of isolated singularities.

Example 2.

$$f_s := xy^5 + sx^2 + x^8, \quad s \in C,$$

$$\nabla f_0 = (y^5 + 8x^7, 5xy^4) \qquad \text{for } s = 0$$
$$\nabla f_s = (y^5 + 2sx + 8x^7, 5xy^4) \quad \text{for } s \neq 0.$$

Hence

$$\mathcal{L}(f_0) = 7 \qquad (\text{take } \Phi(t) = (t, 0)),$$
$$\mathcal{L}(f_s) = 9 \; (\text{take } \Phi(t) = (-\frac{t^5}{2s}, t)).$$

The main result

B. Teissier (1977) proved:

Theorem. If (f_s) is μ-constant family of isolated singularities (i.e. the Milnor number is constant in this family) then $\mathcal{L}(f_s)$ is semi-continuous from below.

The main result

B. Teissier (1977) proved:

Theorem. If (f_s) is μ-constant family of isolated singularities (i.e. the Milnor number is constant in this family) then $\mathcal{L}(f_s)$ is semi-continuous from below.

Remark. The Teissier's result was generalized by Płoski (2010) to mappings (instead of a family of gradient mappings we have a family of mappings with constant multiplicity).

The main result

B. Teissier (1977) posed also a hypothesis:

Conjecture. $\mathcal{L}(f_s)$ is constant in μ-constant familes of isolated singularities.

The main result

B. Teissier (1977) posed also a hypothesis:

Conjecture. $\mathcal{L}(f_s)$ is constant in μ-constant familes of isolated singularities.

For plane curve singularities $(n = 2)$ the conjecture is true.

The main result

B. Teissier (1977) posed also a hypothesis:

Conjecture. $\mathcal{L}(f_s)$ is constant in μ-constant familes of isolated singularities.

For plane curve singularities $(n = 2)$ the conjecture is true.

Obvious, because μ-constant family of plane curve singularities is topologically trivial and $\mathcal{L}(f_s)$ is a topological invariant for such singularities.

The main result

Conjecture. In μ-constant family of isolated singularities $\mathcal{L}(f_s)$ is constant.

Theorem. In μ-constant family of **non-degenerate** isolated **surface** singularities $\mathcal{L}(f_s)$ is constant.

The main result

Conjecture. In μ-constant family of isolated singularities $\mathcal{L}(f_s)$ is constant.

Theorem. In μ-constant family of **non-degenerate** isolated **surface** singularities $\mathcal{L}(f_s)$ is constant.

Surface singularity:

$$f_s(x, y, z): (C^3, 0) \to (C, 0), \quad n = 3.$$

The main result

Conjecture. In μ-constant family of isolated singularities $\mathcal{L}(f_s)$ is constant.

Theorem. In μ-constant family of **non-degenerate** isolated **surface** singularities $\mathcal{L}(f_s)$ is constant.

Surface singularity:

$$f_s(x,y,z)\colon (C^3,0) \to (C,0), \quad n = 3.$$

Family of non-degenerate isolated singularities

Each f_s is non-degenerate (in the Kushnirenko sense).

The idea of proof

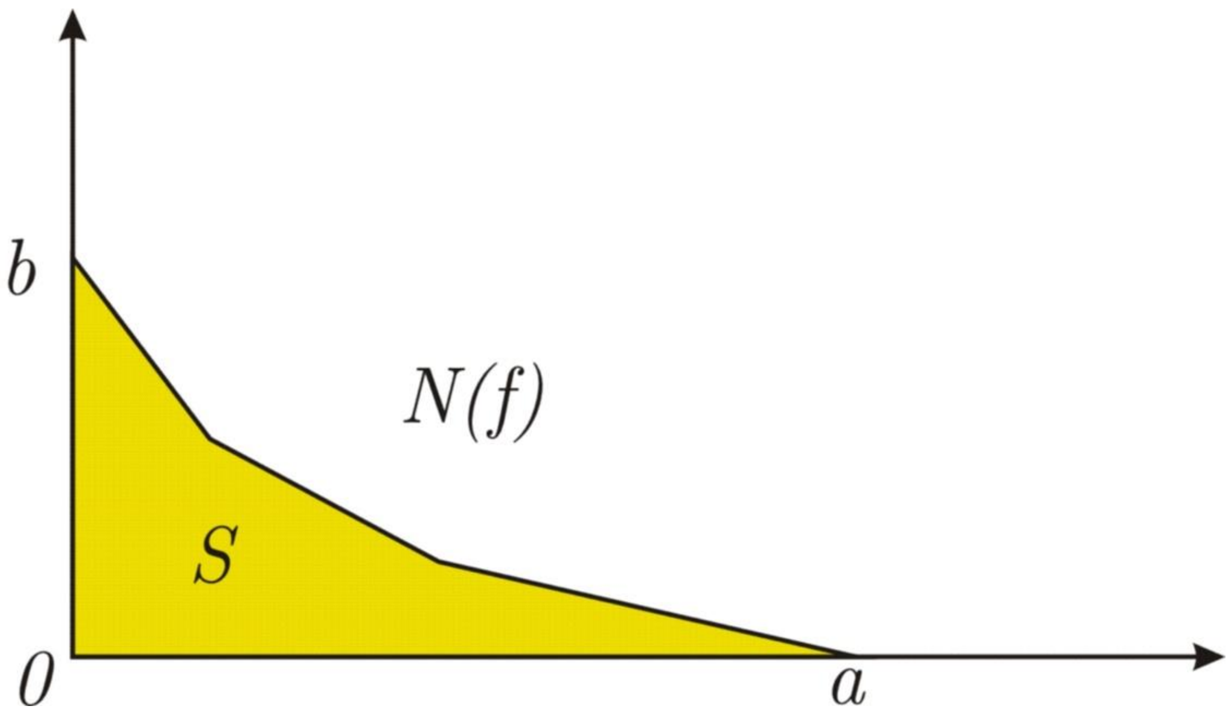
The main result follows from 3 other results:

The idea of proof

The main result follows from 3 other results:

1. **The Kushnirenko result (1976)** (n-dimensional). If f is a non-degenerate isolated singularity then $\mu(f) = \nu(f)$, where $\nu(f)$ is the Newton number of f (= effective, discrete invariant which we read off from the Newton polyhedron $N(f)$ of f).

The idea of proof



$$\nu(f) = 2S - a - b + 1$$

The idea of proof

The main result follows from 3 other results:

1. **The Kushnirenko result (1976)** (n-dimensional). If f is a non-degenerate isolated singularity then $\mu(f) = \nu(f)$, where $\nu(f)$ is the Newton number of f (= effective, discrete invariant which we read off from the Newton polyhedron $N(f)$ of f).

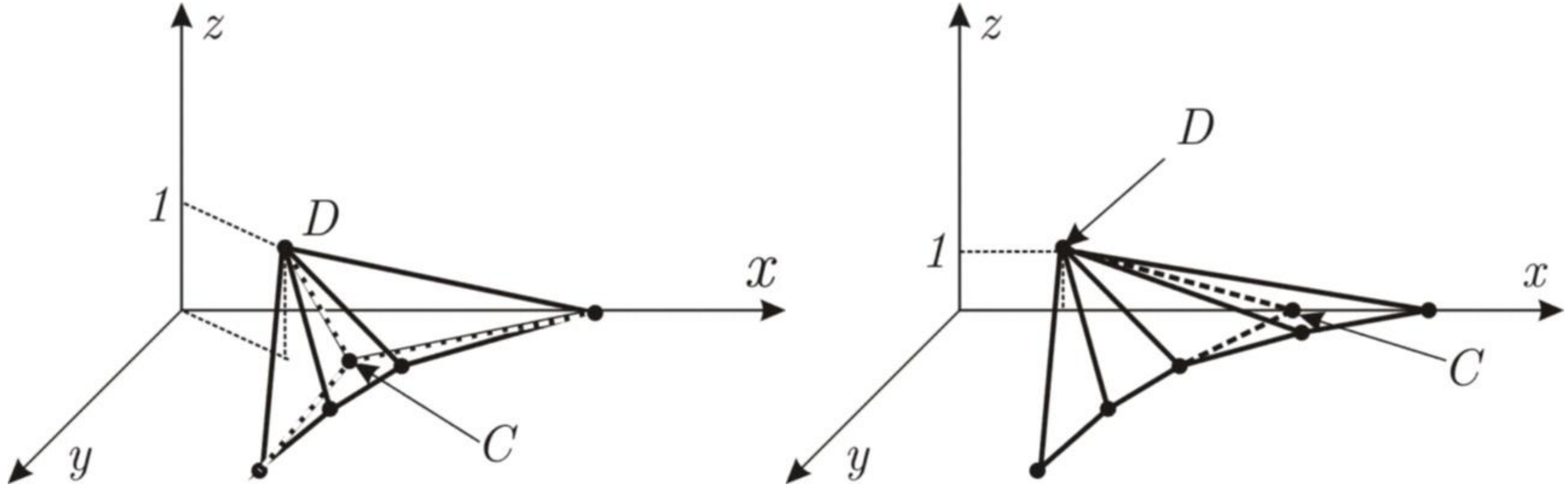
From this we get

$$\nu(f_s) = const.$$

The idea of proof

2. **Brzostowski, Krasiński, Walewska (2019)** (3-dimensional). For two surface singularities f and g if the Newton polyhedrons $N(f)$ and $N(g)$ satisfy $N(f) \subset N(g)$ and $\nu(f) = \nu(g)$ then $N(f)$ and $N(g)$ differ in a very explicit way (they differ on some pyramids with basis in coordinate planes and height one).

The idea of proof



The idea of proof

2. **Brzostowski, Krasiński, Walewska (2019)** (3-dimensional). For two surface singularities f and g if the Newton polyhedrons $N(f)$ and $N(g)$ satisfy $N(f) \subset N(g)$ and $\nu(f) = \nu(g)$ then $N(f)$ and $N(g)$ differ in a very explicit way.

From this we get

$N(f_s)$ and $N(f_0)$ differ in a very explicit way

(because always $N(f_0) \subset N(f_s)$).

The idea of proof

3. **Brzostowski, Krasiński, Oleksik (2020 arXiv)** (3-dimensional). An effective formula for the Łojasiewicz exponent of a non-degenerate surface singularity f in terms of the Newton polyhedron $N(f)$.

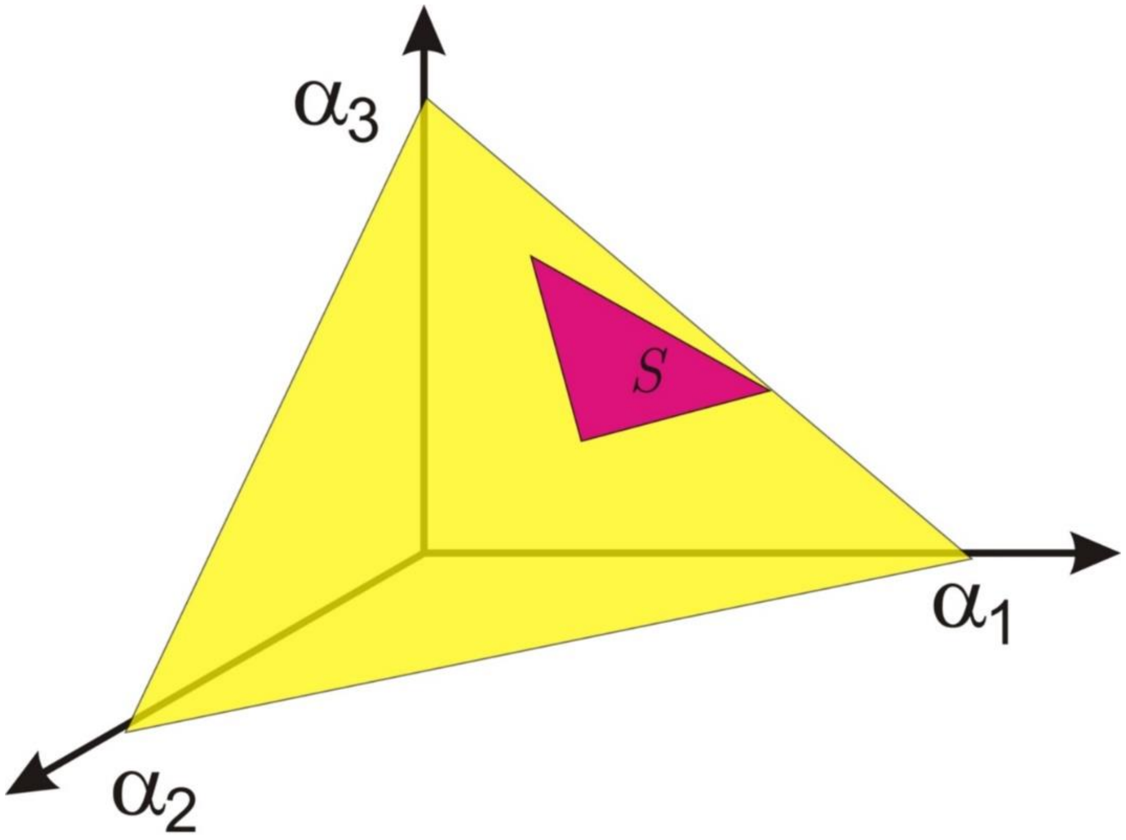
The idea of proof

3. **Brzostowski, Krasiński, Oleksik (2020 arXiv)** (3-dimensional). An effective formula for the Łojasiewicz exponent of a non-degenerate surface singularity f in terms of the Newton polyhedron $N(f)$.

$$\mathcal{L}(f) = \max\left(\alpha(S): S \epsilon \partial N(f) - E_f\right) - 1.$$

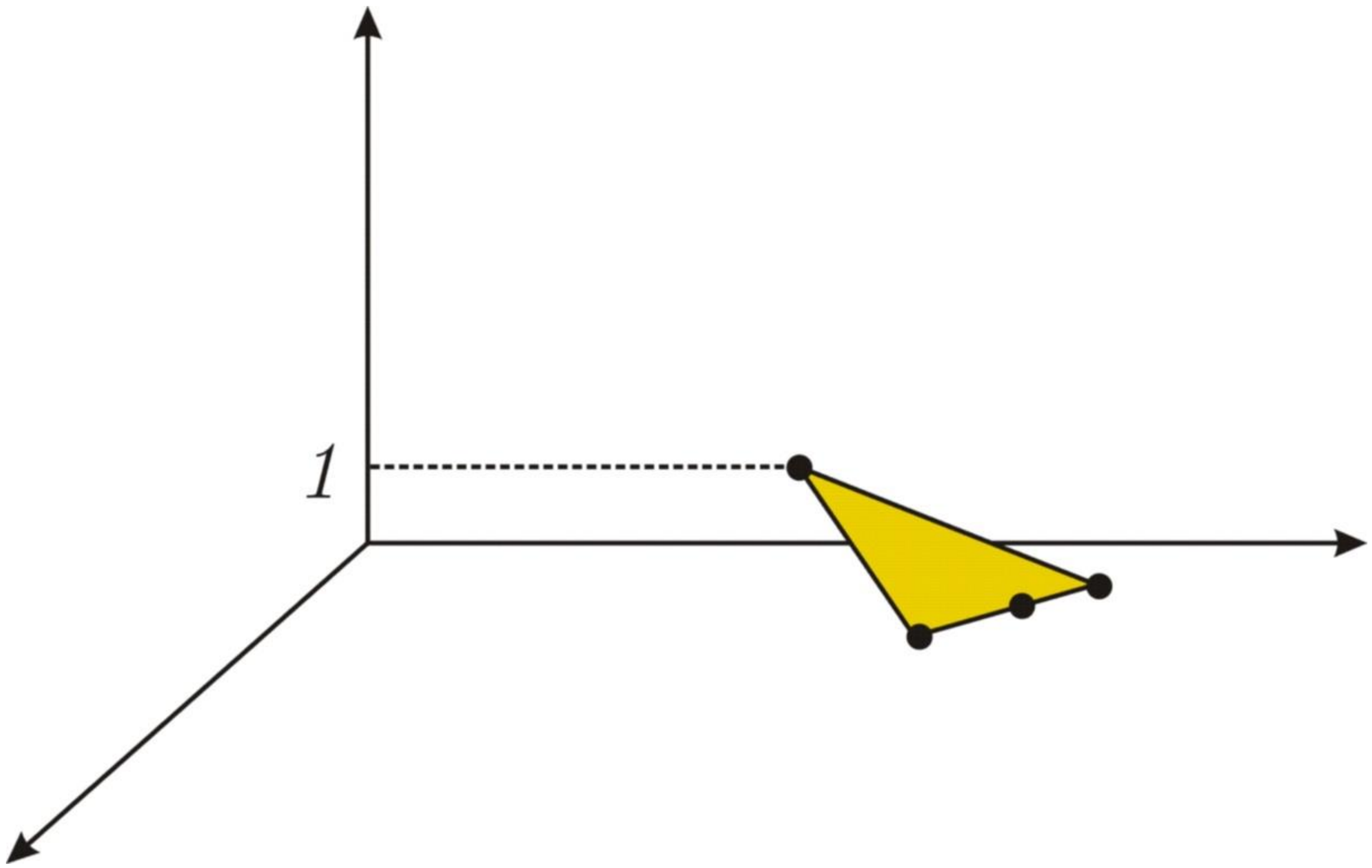
where E_f - exceptional faces.

The idea of proof



$$\alpha(S) = \max(\alpha_1, \alpha_2, \alpha_3)$$

The idea of proof



An exceptional face

The idea of proof

3. **Brzostowski, Krasiński, Oleksik (2020 arXiv)** (3-dimensional). An effective formula for the Łojasiewicz exponent of a surface singularity f in terms of the Newton polyhedrons $N(f)$.

$$\mathcal{L}(f) = \max\left(\alpha(S) : S \,\epsilon \partial N(f) - E_f\right) - 1.$$

E_f - exceptional faces.

From this formula follows $\mathcal{L}(f_s) = \mathcal{L}(f_0)$ (because the difference $N(f_s)$ and $N(f_0)$ does not influence on this formula).

Generalization

A generalization of the main theorem to n-dimensional case is easy provided we will get in n-dimensional case three results we used in the proof:

Generalization

A generalization of the main theorem to n-dimensional case is easy provided we will get in n-dimensional case three results we used in the proof:

1. The Kushnirenko result (1976) (n-dimensional). If f is a non-degenerate isolated singularity then $\mu(f) = \nu(f)$, where $\nu(f)$ is the Newton number of f (= effective, discrete invariant which we read off from the Newton polyhedron $N(f)$ of f).

Generalization

A generalization of the main theorem to n-dimensional case is easy provided we will get in n-dimensional case three results we used in the proof:

2. Brzostowski, Krasiński, Walewska (2019) (3-dimensional). For two surface singularities f and g if the Newton polyhedrons $N(f)$ and $N(g)$ satisfy $N(f) \subset N(g)$ and $\nu(f) = \nu(g)$ then $N(f)$ and $N(g)$ differ in a very explicit way.

This result has been recently generalized to n-dimensional case by Leyton-Alvarez, Mourtada and Spivakovsky (2020, arXiv).

Generalization

A generalization of the main theorem to n-dimensional case is easy provided we will get in n-dimensional case three results we used in the proof:

3. **Brzostowski, Krasiński, Oleksik (2020 arXiv)** (3-dimensional). An effective formula for the Łojasiewicz exponent of a surface singularity f in terms of the Newton polyhedrons $N(f)$.

We are working over such formula in n-dimensional case.

The end

Thank you for your attention.

Out[26]=