Invariants for bi-Lipschitz equivalence of ideals and related topics

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1. Bi-Lipschitz equivalence

Let $a(x)$ and $b(x)$ be two function germs $(\mathbb{C}^n, x_0) \to \mathbb{R}$, where $x_0 \in \mathbb{C}^n$. Then

- $a(x) \lesssim b(x)$ *near* x_0 means that there exists a positive constant $C > 0$ and an open neighbourhood U of x_0 in $\mathbb{C}^n$ such that $a(x) \leqslant C\, b(x)$, for all $x \in U$.
- $a(x) \sim b(x)$ *near* x_0 means that $a(x) \lesssim b(x)$ near x_0 and $b(x) \lesssim a(x)$ near x_0.

For an n-tuple $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, we write $\|x\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$.

A map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is said to be *Lipschitz* if

$$\|f(x) - f(x')\| \lesssim \|x - x'\| \text{ near } 0.$$

We say that a homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is *bi-Lipschitz* if φ and φ^{-1} are Lipschitz.

This notion leads to the known equivalence relations between maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$: bi-Lipschitz $\mathcal{R}$-equivalence, bi-Lipschitz $\mathcal{A}$-equivalence and bi-Lipschitz $\mathcal{K}$-equivalence.

Two given subsets X_1 and X_2 of $(\mathbb{C}^n, 0)$ are called *bi-Lipschitz equivalent* if there is a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $\varphi(X_1) = X_2$.

If I is an ideal of $\mathcal{O}_n$, then we denote by $\overline{I}$ the integral closure of I.

Definition

Let I and J be ideals of $\mathcal{O}_n$. We say that I and J are *bi-Lipschitz equivalent* if there exist two families $f_1, \ldots, f_p$ and $g_1, \ldots, g_q$ of functions of $\mathcal{O}_n$ such that

(a) $\langle f_1, \ldots, f_p \rangle \subseteq I$ and $\overline{\langle f_1, \ldots, f_p \rangle} = \overline{I}$,

(b) $\langle g_1, \ldots, g_q \rangle \subseteq J$ and $\overline{\langle g_1, \ldots, g_q \rangle} = \overline{J}$,

(c) there is a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\|(f_1(x), \ldots, f_p(x))\| \sim \|(g_1(\varphi(x)), \ldots, g_q(\varphi(x)))\| \qquad \text{near } 0.$$

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be analytic map germs. Here we remark some obvious consequences:

- If f and g are bi-Lipschitz $\mathcal{K}$-equivalent, then the ideals generated by their components are bi-Lipschitz equivalent.
- If two ideals are bi-Lipschitz equivalent, then their zero sets are bi-Lipschitz equivalent.

Let I and J be ideals of $\mathcal{O}_n$. Let $\{f_1, \ldots, f_p\}$ be a generating system of I and let $\{g_1, \ldots, g_q\}$ be a generating system of J.

Let us consider the maps $f = (f_1, \ldots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $g = (g_1, \ldots, g_q) : (\mathbb{C}^n, 0) \to (\mathbb{C}^q, 0)$. The *Łojasiewicz exponent of I with respect to J*, denoted by $\mathcal{L}_J(I)$, is defined as the infimum of the set

$$\left\{\alpha \in \mathbb{R}_{\geqslant 0} : \|g(x)\|^{\alpha} \lesssim \|f(x)\| \text{ near } 0\right\}. \tag{1}$$

By convention, we set $\inf \emptyset = \infty$. So if the above set is empty, then $\mathcal{L}_J(I) = \infty$.

It is well known that $\mathcal{L}_J(I)$ is finite if and only if $V(I) \subseteq V(J)$. When $\mathcal{L}_J(I)$ is finite, then this is a rational number.

Let us suppose that the ideal I has finite colength. When $J = \mathbf{m}_n$, then we denote the number $\mathcal{L}_J(I)$ by $\mathcal{L}_0(I)$. That is

$$\mathcal{L}_0(I) = \inf\left\{\alpha \in \mathbb{R}_{\geqslant 0} : \|x\|^{\alpha} \lesssim \|f(x)\| \text{ near } 0\right\}.$$

We refer to $\mathcal{L}_0(I)$ as the *Łojasiewicz exponent of I*.

Theorem

Let I and J be ideals of $\mathcal{O}_n$. Suppose that I and J are bi-Lipschitz equivalent. Then

- $\mathbf{m}^r I^s$ and $\mathbf{m}^r J^s$ are bi-Lipschitz equivalent, for all $r, s \in \mathbb{Z}_{\geqslant 1}$.
- $\mathrm{ord}(I) = \mathrm{ord}(J)$. If I and J have finite colength, then $\mathcal{L}_0(I) = \mathcal{L}_0(J)$.

Theorem

Let $f, g \in \mathcal{O}_n$. Let us suppose that f and g are bi-Lipschitz $\mathcal{A}$-equivalent. Then $J(f)$ and $J(g)$ are bi-Lipschitz equivalent. In particular, $\mathrm{ord}(f) = \mathrm{ord}(g)$ and $\mathcal{L}_0(J(f)) = \mathcal{L}_0(J(g))$.

2. Log canonical threshold of ideals

Definition

Let I be an ideal of $\mathcal{O}_n$ and let $g_1, \ldots, g_r$ be a generating system of I. Then the *log canonical threshold of* I, denoted by $\operatorname{lct}(I)$, is defined as

$$\operatorname{lct}(I) = \sup\left\{ s \in \mathbb{R}_{\geqslant 0} : \frac{1}{\left(|g_1(z)|^2 + \cdots + |g_r(z)|^2\right)^s} \text{ is locally integrable at } 0 \right\}.$$

The computation of $\operatorname{lct}(I)$ is a non-trivial problem. It can also obtained by means of the information supplied by a log resolution of I.

The *Arnold index* of I, denoted by $\mu(I)$, is defined as

$$\mu(I) = \frac{1}{\operatorname{lct}(I)}.$$

In particular, if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a non-identically zero holomorphic function germ, the *log canonical threshold of* f or *complex singularity exponent* of f, denoted by $\operatorname{lct}(f)$, is defined as $\operatorname{lct}(\langle f \rangle)$. That is:

$$\operatorname{lct}(f) = \sup\left\{ s \in \mathbb{R}_{\geqslant 0} : \frac{1}{|f(z)|^{2s}} \text{ is locally integrable at } 0 \right\}.$$

Here we recall some known facts.

- If $f \in \mathcal{O}_n$, $f(0) = 0$, then $\operatorname{lct}(f) \in \mathbb{Q} \cap [0, 1]$.
- For any analytic function $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ we have $\operatorname{lct}(f) = \frac{1}{\operatorname{ord}(f)}$.
- Skoda inequality: given an ideal $I \subseteq \mathbf{m}_n$, then $\frac{1}{\operatorname{ord}(I)} \leqslant \operatorname{lct}(I) \leqslant \frac{n}{\operatorname{ord}(I)}$.
- For any $f, g \in \mathcal{O}_n$:
$$\operatorname{lct}(f + g) \leqslant \operatorname{lct}(f) + \operatorname{lct}(g)$$
$$\operatorname{lct}(fg) \leqslant \min\{\operatorname{lct}(f), \operatorname{lct}(g)\}.$$
- Theorem (Varchenko, 1982): Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic deformation such that f_t has an isolated singularity, for all t. If this deformation is μ-constant, then $\operatorname{lct}(f_t)$ is also constant.
- Theorem (de Fernex-Ein-Mustață, 2004): Let I be an ideal of finite colength of $\mathcal{O}_n$. Then
$$e(I) \geqslant \left(\frac{n}{\operatorname{lct}(I)}\right)^n$$
and equality holds if and only if there exists some integer $q \geqslant 1$ such that $\overline{I} = m^q$. In this case $q = \frac{n}{\operatorname{lct}(I)} = \operatorname{ord}(I)$.

Theorem

Let $f, g \in \mathbf{m}_n$ and let I and J be proper ideals of $\mathcal{O}_n$.

(a) If f and g are bi-Lipschitz $\mathcal{K}$-equivalent, then $\operatorname{lct}(f) = \operatorname{lct}(g)$.
(b) If I and J are bi-Lipschitz equivalent, then $\operatorname{lct}(I) = \operatorname{lct}(J)$.

Theorem

Let I, J be ideals of $\mathcal{O}_n$ such that $V(I) \subseteq V(J)$. Then

$$\frac{\operatorname{lct}(J)}{\operatorname{lct}(I)} \leqslant \mathcal{L}_J(I).$$

In particular (taking $J = \mathbf{m}_n$) we have

$$\frac{n}{\mathcal{L}_0(I)} \leqslant \operatorname{lct}(I).$$

Corollary

Let I be an ideal of $\mathcal{O}_n$ such that $V(I) \subseteq V(x_1 \cdots x_n)$. Then

$$\frac{1}{\mathcal{L}_{x_1 \cdots x_n}(I)} \leqslant \operatorname{lct}(I)$$

and equality holds when $\overline{I}$ is monomial.

Let $\Gamma_+ \subseteq \mathbb{R}^n_+$ be a Newton polyhedron. We define

$$\mu(\Gamma_+) = \min\{\mu \in \mathbb{R}_{\geqslant 0} : \mu(1, \dots, 1) \in \Gamma_+\}$$

and $P_{\Gamma_+} = \mu(\Gamma_+)(1, \dots, 1)$.

Let J be a proper monomial ideal of $\mathcal{O}_n$. By a result of Howald we know that

$$\mathrm{lct}(J) = \frac{1}{\mu(\Gamma_+(J))}. \tag{2}$$

That is, $\mu(J) = \mu(\Gamma_+(J))$. Let us define $P_J = P_{\Gamma_+(J)}$.

It is also known that if $f \in \mathcal{O}_n$ verifies that $\Gamma_+(f) = \Gamma_+(J)$ and f is Newton non-degenerate (is the sense of Kouchnirenko), then $\mathrm{lct}(f) = \min\{1, \mathrm{lct}(J)\}$.

Example:

For any $a \in \mathbb{Z}_{\geqslant 3}$, let us consider the function of $\mathcal{O}_2$ given by $f_a = x^3y^2 + y^a$ and the ideal $I_a = \langle x^3y^2, y^a \rangle \subseteq \mathcal{O}_2$. We observe that f_a is a Newton non-degenerate function, for all $a \in \mathbb{Z}_{\geqslant 3}$. Thus, we have that

$$\mathrm{lct}(f_a) = \min\{1, \mathrm{lct}(I_a)\} = \frac{a+1}{3a},$$

for all $a \in \mathbb{Z}_{\geqslant 3}$. If $a, b \in \mathbb{Z}_{\geqslant 3}$, then we conclude that f_a is bi-Lipschitz $\mathcal{K}$-equivalent to f_b if and only if $a = b$

Example:

Let us consider the monomial ideals of $\mathcal{O}_2$ given by

$$I = \langle x^{11}, x^8y^5, x^6y^9, y^{30} \rangle$$
$$J = \langle x^{11}, x^8y^4, x^6y^{10}, y^{30} \rangle.$$

Then we observe that $\mathrm{ord}(I) = \mathrm{ord}(J) = 11$, $\mathcal{L}_0(I) = \mathcal{L}_0(J) = 30$ and $\mathrm{lct}(I) = \mathrm{lct}(J) = \frac{1}{7}$. However, we find that $\mathrm{lct}(\mathbf{m}_2 I) = \frac{3}{22}$ and $\mathrm{lct}(\mathbf{m}_2 J) = \frac{4}{29}$. Therefore I and J are not bi-Lipschitz equivalent.

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ, then we denote by $D(f)$ the Jacobian matrix of f. Let $N(f)$ be the matrix given by

$$N(f) = \begin{bmatrix} x_1 \frac{\partial f_1}{\partial x_1} & \cdots & x_n \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ x_1 \frac{\partial f_p}{\partial x_1} & \cdots & x_n \frac{\partial f_p}{\partial x_n} \end{bmatrix}. \tag{3}$$

Definition (Veys-Zúñiga)

Let $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ. Then f is *strongly non-degenerate* if and only if, for any compact face Δ of $\Gamma_+(f)$, we have

$$f_\Delta^{-1}(0) \cap \left\{x \in \mathbb{C}^n : \operatorname{rank}(D(f_\Delta)(x)) < \min\{n, p\}\right\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\},$$

where $f_\Delta = (f_{1,\Delta}, \dots, f_{p,\Delta})$, $f_{i,\Delta} = (f_i)_\Delta$, for all $i = 1, \dots, p$, and $D(f_\Delta)$ denotes the Jacobian matrix of f_Δ.

Proposition

Let $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a complex analytic map. Let $J = \langle x^k : k \in \Gamma_+(f) \rangle$. Then the following conditions are equivalent:

(a) f is strongly non-degenerate

(b) the ideal $\langle f_1, \dots, f_p \rangle \overline{J^{p-1}} + \mathbf{I}_p(N(f))$ is Newton non-degenerate.

Theorem (Veys-Zúñiga)

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ and let us consider the ideal of $\mathcal{O}_n$ given by $J = \langle x^k : k \in \Gamma_+(f) \rangle$. If f is strongly non-degenerate and $\mathrm{lct}(J) \leqslant p$, then $\mathrm{lct}(\langle f_1, \dots, f_p \rangle) = \mathrm{lct}(J)$.

Corollary

Let I be a proper ideal of $\mathcal{O}_n$ and let $p \in \mathbb{Z}_{\geqslant 1}$. Suppose that

- $\mathrm{lct}(I^0) \leqslant p$
- there exists a map $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ such that $f_1 \dots, f_p \in I$ and f is strongly non-degenerate.
- $P_{I^0} \in \Gamma_+(f)$.

Then $\mathrm{lct}(I) = \mathrm{lct}(I^0)$.
In particular, if $\mathrm{lct}(I^0) \leqslant 1$ and there exists some $g \in I$ such that g is Newton non-degenerate and $\Gamma_+(g) = \Gamma_+(I)$, then $\mathrm{lct}(I) = \mathrm{lct}(I^0)$.

Example:

Let $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the analytic deformation given by the Briançon-Speder example. That is

$$f_t(x, y, z) = x^5 + z^{15} + y^7 z + txy^6$$

for all $(x, y, z) \in \mathbb{C}^3$, $t \in \mathbb{C}$. We recall that, if $w = (3, 2, 1)$, then f_t is weighted homogeneous with respect to w, $d_w(f_t) = 15$ and f_t has an isolated singularity at the origin, for all $|t| \ll 1$.

Let $J_1 = \langle x^k : k \in \Gamma_+(J(f_0)) \rangle$ and let $J_2 = \langle x^k : k \in \Gamma_+(J(f_t)) \rangle$, for $t \neq 0$. Then $J_1 \subseteq J_2$ and it is easy to check that

$$J_1 = \overline{\langle x^4, y^7, z^{14}, y^6 z \rangle} \qquad \text{and} \qquad J_2 = \overline{\langle x^4, y^6, z^{14} \rangle}.$$

The family f_t is not μ^*-constant and $\mathcal{L}_0^*(\nabla f_t)$ is not constant, since

$$\mu^*(f_t) = \begin{cases} (364, 28, 4) & \text{if } t = 0 \\ (364, 26, 4) & \text{if } t \neq 0. \end{cases} \qquad \mathcal{L}_0^*(\nabla f_t) = \begin{cases} (14, 7, 4) & \text{if } t = 0 \\ (14, 6.5, 4) & \text{if } t \neq 0. \end{cases}$$

Hence we observe that f_t is a Hickel singularity if and only if $t \neq 0$.

The ideal $J(f_0)$ is Newton non-degenerate. Therefore $\mathrm{lct}(J(f_0)) = \frac{10}{21}$, by Howald's result.

By the lower semi-continuity of the log canonical threshold we have that $\mathrm{lct}(J(f_0)) \leqslant \mathrm{lct}(J(f_t))$, for all $|t| \ll 1$. The inclusion $J(f_t) \subseteq J_2$ implies that $\mathrm{lct}(J(f_t)) \leqslant \mathrm{lct}(J_2) = \frac{41}{84}$.

Let $t \in \mathbb{C} \smallsetminus \{0\}$ such that $|t| < 1$. Let us define the function

$$g = \frac{\partial f_t}{\partial x} + \frac{\partial f_t}{\partial y} + \frac{\partial f_t}{\partial z}.$$

It is straightforward to see that $g \in J(f_t)$ and $\Gamma_+(g) = \Gamma_+(J_2)$. Moreover, g is Newton non-degenerate. Therefore, we obtain that

$$\mathrm{lct}(J(f_t)) = \mathrm{lct}(J_2) = \frac{41}{84}.$$

Then f_0 is not bi-Lipschitz $\mathcal{A}$-equivalent to f_t, if $t \neq 0$, $|t| \ll 1$.

Example:

Let $\alpha \in \mathbb{Z}_{\geqslant 3}$ such that α is odd and let $\beta \in \mathbb{Z}_{\geqslant 1}$ such that $3\alpha = 2\beta + 1$. Let us consider the deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$f_t(x, y, z) = x^3 + y^\beta z + z^{3\alpha} + y^{\beta+1} + txy^\alpha$$

for all $(x, y, z) \in \mathbb{C}^3$, $t \in \mathbb{C}$. This deformation f_t is a slight modification of an example given in Briançon-Speder's paper.

We observe that:

- f_t is semi-weighted homogeneous with respect to $(\alpha, 2, 1)$, $d_w(f) = 3\alpha$
- $p_w(f_t) = x^3 + y^\beta z + z^{3\alpha} + txy^\alpha$, $d_w(y^{\beta+1}) = d_w(f) + 1$
- $p_w(f_t) = x^3 + y^\beta z + z^{3\alpha} + txy^\alpha$
- $J(f_0)$ is Newton non-degenerate and $J(f_t)$ is not, if $t \neq 0$.

Let us write $\alpha = 2k + 1$, where $k \in \mathbb{Z}_{\geqslant 1}$. Using the fact that f_t is Newton non-degenerate, for all t, we deduce that

$$\mu^*(f_t) = \begin{cases} \left(36k^2 + 18k + 2,\ 6k + 2,\ 2\right) & \text{if } t = 0 \\ \left(36k^2 + 18k + 2,\ 6k + 1,\ 2\right) & \text{if } t \neq 0. \end{cases}$$

Thus $\mu^{(2)}(f_t)$ is not constant.

Moreover

$$\mathcal{L}_0^*(\nabla f_t) = \begin{cases} \left(6k+2,\, 3k+1,\, 2\right) & \text{if } t = 0 \\ \left(6k+2,\, 3k+\frac{1}{2},\, 2\right) & \text{if } t \neq 0. \end{cases}$$

Let us define the ideals $K_1 = \langle x^k : k \in \Gamma_+(f_0)\rangle$ and $K_2 = \langle x^k : k \in \Gamma_+(f_t)\rangle$, for $t \neq 0$. An elementary combinatorial analysis shows that

$$K_1 = \overline{\left\langle x^3, y^{3k+1}z, z^{6k+3}, y^{3k+2}\right\rangle} \qquad K_2 = \overline{K_1 + \left\langle xy^{2k+1}\right\rangle}.$$

If $t \neq 0$, we observe that

$$e(J(f_t)) = 36k^2 + 18k + 2 = (6k+2)\left(3k + \frac{1}{2}\right)2$$

then f_t is a Hickel singularity, if $t \neq 0$, whereas f_0 is not Hickel. We also observe that

$$\operatorname{lct}(f_0) = \operatorname{lct}(K_1) = \frac{2k+4}{6k+3} = \operatorname{lct}(K_2) = \operatorname{lct}(f_t)$$

if $t \neq 0$.

Moreover, since $J(f_0)$ is Newton non-degenerate, we deduce that

$$\mathrm{lct}(J(f_0)) = \mathrm{lct}\left(\langle x^2, y^{3k+1}, z^{6k+2}, y^{3k}z\rangle\right) = \frac{9k^2+12k+1}{18k^2+6k}.$$

If we fix $t \neq 0$, then the function

$$g = \frac{\partial f_t}{\partial x} + \frac{\partial f_t}{\partial y} + \frac{\partial f_t}{\partial z}$$

is Newton non-degenerate and $\Gamma_+(g) = \Gamma_+(J(f_t)) = \Gamma_+(x^2, y^{2k+1}, z^{6k+2})$. Thus

$$\mathrm{lct}(J(f_t)) = \mathrm{lct}\left(\langle x^2, y^{2k+1}, z^{6k+2}\rangle\right) = \frac{1}{2} + \frac{1}{2k+1} + \frac{1}{6k+2} = \frac{6k^2+13k+4}{12k^2+10k+2}.$$

Then $\mathrm{lct}(J(f_0)) = \mathrm{lct}(J(f_t)$ if and only if $k = \frac{1\pm\sqrt{7}}{6}$. That is $\mathrm{lct}(J(f_0)) \neq \mathrm{lct}(J(f_t))$, if $|t| \ll 1$, $t \neq 0$. This shows that the deformation f_t is not bi-Lipschitz $\mathcal{A}$-trivial.

- We say that an ideal I of $\mathcal{O}_n$ is *diagonal*, when there exist positive integers $a_1, \dots, a_n$ such that $\overline{I} = \overline{\langle x_1^{a_1}, \dots, x_n^{a_n} \rangle}$.
- If $a_n \geqslant \cdots \geqslant a_1$, then we recall that $\mathcal{L}_0^*(I) = (a_n, \dots, a_1)$.
- It is clear that any diagonal ideal is Hickel. The converse does not hold, as can be easily checked for the ideal $I = \langle x^3, xy, y^3 \rangle \subseteq \mathcal{O}_2$.
- If I is an ideal of $\mathcal{O}_n$ of finite colength, then we define the *Demailly-Pham number of* I, which we denote by $\mathrm{DP}(I)$, as
$$\mathrm{DP}(I) = \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} \cdots + \frac{e_{n-1}(I)}{e_n(I)}.$$
- Theorem (Demailly-Pham): Let I be an ideal of finite colength. Then $\mathrm{DP}(I) \leqslant \mathrm{lct}(I)$.
- As a consequence we have:
$$\frac{1}{\mathcal{L}_0^{(1)}(I)} + \frac{1}{\mathcal{L}_0^{(2)}(I)} + \cdots + \frac{1}{\mathcal{L}_0^{(n)}(I)} \leqslant \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} = \mathrm{DP}(I) \leqslant \mathrm{lct}(I).$$
- Theorem (B.-A.): If I is an ideal of finite colength of $\mathcal{O}_n$ such that $\mathrm{lct}(I) = \mathrm{lct}(I^0)$, then $\mathrm{DP}(I) = \mathrm{lct}(I)$ if and only if I is a diagonal ideal.

Theorem

Let I and J be ideals of $\mathcal{O}_3$ of finite colength. If I and J are bi-Lipschitz equivalent and I is diagonal, then $\mathcal{L}_0^{(2)}(J) \geqslant \mathcal{L}_0^{(2)}(I)$.

Theorem

Let us consider an analytic map $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be the map given by $f_t(x) = F(t, x)$ and let I_t denote the ideal of $\mathcal{O}_n$ generated by the component functions of f_t, for all $|t| \ll 1$. Let us assume that I_t is an ideal of finite colength, for all $|t| \ll 1$, and I_0 is diagonal. If I_t is bi-Lipschitz trivial, then $e_i(I_t)$ is constant, for all $i = 1, \dots, n$ and all $|t| \ll 1$.

Corollary

Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic deformation such that f_t has an isolated singularity for all $|t| \ll 1$. Let us suppose that this deformation is bi-Lipschitz $\mathcal{A}$-trivial. If $J(f_0)$ is diagonal, then $\mu^*(f_t)$ is constant, for $|t| \ll 1$.

Theorem

Let us fix an analytic family of functions $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that f_t is Hickel, for all $|t| \ll 1$. If, in addition, $J(f_0)$ is diagonal and the family f_t is bi-Lipschitz trivial, then $\mathcal{L}_0^*(J(f_0)) = \mathcal{L}_0^*(J(f_t))$, for all $|t| \ll 1$.

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