Almost non-degenerate functions and a Zariski pair of links

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Introduction

Consider an analytic function $f(\mathbf{z}) = \sum_\nu a_\nu \mathbf{z}^\nu$ of n variables defined in a neighborhood of the origin of $\mathbb{C}^n$. Assume that we are given a good resolution

$$\hat{\pi} : X \to \mathbb{C}^n$$

of the function f and let $E_1, \dots, E_s$ be the exceptional divisors of $\hat{\pi}$, that is $\hat{\pi}^* f^{-1}(V) = \widetilde{V} \cup_{j=1}^s E_j$ where $\widetilde{V}$ is the strict transform of the hypersurface $V = f^{-1}(0)$. Consider the open dense subset

$$E_j'' = E_j \cap \hat{\pi}^{-1}(0) \setminus \widetilde{V} \cup_{i \neq j} E_i.$$

Let m_j be the multiplicity of $\hat{\pi}^* f$ along E_j. By A'Campo [1], the zeta function of the Milnor monodromy at the origin is given as

$$(AC) \qquad \zeta(t) = \prod_{j=1}^{s} (1 - t^{m_j})^{-\chi(E_j'')}.$$

Suppose that $f(\mathbf{z})$ is Newton non-degenerate. Then using a toric modification $\hat{\pi} : X \to \mathbb{C}^n$ which is admissible with the dual Newton diagram $\Gamma^*(f)$, the zeta function can be computed combinatorially (Varchenko, [16]). More precisely the zeta function is given as

$$(V) \quad \zeta(t) \ = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{Q \in \mathcal{P}_I} (1 - t^{d(Q;f^I)})^{-\chi(Q)}$$

$\mathcal{P}_I$ is the set of primitive integer weight vectors of the coordinate subspace $\mathbb{C}^I$ which correspond to the maximal faces of $\Gamma(f^I)$. For $I = \{1, \dots, n\}$, we simply denote $\mathcal{P}$.

The number $\chi(Q)$ in (V) is defined as follows.

$$\chi(Q) = (-1)^{|I|-1}|I|!\mathrm{Vol}_{|I|}C(\Delta(Q; f^I), O_I)/d(Q; f^I).$$

Here $C(\Delta(Q; f^I), O_I)$ is the cone over $\Delta(Q; f^I)$ with a vertex at the origin O_I of $\mathbb{C}^I$. See [16, 13] or Chapter 3 of [11] for details. Ia using a toric modification is convenient in such a situation, as it express the geometry more directly.

Fundamental Lemma

Lemma 1 (Kouchnirenko [6], Oka [14])

Assume that $h(\mathbf{y})$ *is Newton non-degenerate as a polynomial and let* $V(h)^* = \{\mathbf{y} \in \mathbb{C}^{*m} \mid h(\mathbf{y}) = 0\}$. *Then the Euler characteristic is given as*

$$\chi(V(h)^*) = (-1)^{m-1} m! \mathrm{Vol}_m \Delta(h).$$

Purpose of this paper

The purpose of this talk is to generalize Varchenko's formula for certain functions which have some Newton degenerate faces.
As an application of Main Theorem, we give a Zariski pair of links. Namely we give two hypersurfaces of in $\mathbb{C}^3$ with the same zeta function with different tangent cones in §4. In §5, we give a formula of the Sift Formula of Milnor number.

A good resolution of a function

Let f be an analytic function defined in a neighborhood U of the origin of $\mathbb{C}^n$. Let X be a complex manifold of dimension n and $\hat{\pi} : X \to U$ is a proper holomorphic function. $\hat{\pi} : X \to U$ is called a good resolution of f if it satisfies the following:
(1) $\hat{\pi}$ is biholomorphic on the restriction to $X \setminus \hat{\pi}^{-1}(V) \to U \setminus V$, $V = f^{-1}(0)$.
Assume that the divisor $(\hat{\pi}^* f)$ is given as $\widetilde{V} + \sum_{i=1}^k m_i E_i$ where $\widetilde{V}$ is the strict transform of $V = f^{-1}(0)$ and m_i is the multiplicity of $\hat{\pi}^* f$ along E_i. Let $\widetilde{V}_i$, $i = 1, \ldots, m$ be the irreducible components of $\widetilde{V}$.
(2) Each irreducible component $\widetilde{V}_i$ and the divisors $E_1, \ldots, E_k$ are non-singular and $\widetilde{V} \cup_{i=1}^k E_i$ has, at most, ordinary normal crossing singularities.

The Newton boundary and the dual Newton diagram

Let $f(\mathbf{z}) = \sum_\nu a_\nu \mathbf{z}^\nu$ be a given holomorphic function defined by a convergent series. The Newton polygon $\Gamma^+(f)$ with respect to the given coordinates $\mathbf{z} = (z_1, \dots, z_n)$ is the convex hull of the union $\cup_{\nu, a_\nu \neq 0}\{\nu + \mathbb{R}^n_{\geq 0}\}$ and the Newton boundary $\Gamma(f)$ is defined by the union of compact faces of $\Gamma^+(f)$. An integral point $\nu = (\nu_1, \dots, \nu_n) \in \Gamma^+(f)$ corresponds to the monomial $\mathbf{z}^\nu = z_1^{\nu_1} \dots z_n^{\nu_n}$ For a positive weight vector $P = {}^t(p_1, \dots, p_n)$, we consider the canonical linear function ℓ_P on $\Gamma^+(f)$ which is defined by $\ell_P(\nu) = \sum_{i=1}^n \nu_i p_i$. This is nothing but the degree mapping $\deg_P \mathbf{z}^\nu = \sum_{i=1}^n p_i \nu_i$. The minimal value of ℓ_P is denoted by $d(P; f)$. Put $\Delta(P; f) := \{\nu \in \Gamma^+(f) \mid \ell_P(\nu) = d(P)\}$. We will use the simplified notations $d(P)$ and $\Delta(P)$ if any ambiguity seems unlikely.

Non-degeneracy and Dual Newton diagram

Two weight vectors are $P \sim Q$ if and only if $\Delta(P) = \Delta(Q)$ and this equivalent relation gives a conical decomposition of the positive weight vectors and we denote it as $\Gamma^*(f)$ and call it the dual Newton diagram of f.

f is Newton non-degenerate on a face Δ of $\Gamma(f)$ if $f_\Delta : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical points. f is Newton non-degenerate if it is non-degenerate on every face $\Delta \subset \Gamma(f)$ of any dimension. The closure of an equivalent class can be irreduntantly expressed as

$$\mathrm{Cone}(P_1, \dots, P_k) := \left\{ \sum \lambda_i P_i \mid \lambda_i \geq 0 \right\}$$

where $P_1, \dots, P_k$ are chosen to be primitive integer vectors. That is, k is minimal among any possible such expressions.

Simplicial, regular fan

A cone $\sigma = \mathrm{Cone}\,(P_1, \ldots, P_k)$ is simplicial if dim $\sigma = k$ and σ is regular if $P_1, \ldots, P_k$ are primitive integer vectors which can be extended to a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Recall that f is convenient if $\Gamma(f)$ touches with every coordinate axis. We say f is pseudo-convenient if f is written as $f(\mathbf{z}) = \mathbf{z}^{\nu_0} f'(\mathbf{z})$ where $f'(\mathbf{z})$ convenient and ν_0 is a positive integer vector.

Toric modification

A regular simplicial cone subdivision Σ^* of the space of positive weight vectors $N_{\mathbb{R}}^+ = \mathbb{R}_+^n$ is admissible with the dual Newton diagram $\Gamma^*(f)$ if Σ^* is a subdivision of $\Gamma^*(f)$. For such a regular simplicial cone subdivision, we associate a modification $\hat{\pi} : X \to \mathbb{C}^n$ as follows: let $\mathcal{S}$ be the set of n-dimensional cones in Σ^*. For each $\sigma = \mathrm{Cone}(P_1, \dots, P_n) \in \mathcal{S}$, we identify σ with the unimodular matrix:

$$\sigma = \begin{pmatrix} p_{11}, & \dots & p_{1n} \\ \vdots & \vdots & \vdots \\ p_{n1} & \dots & p_{nn} \end{pmatrix}$$

where $P_j = {}^t(p_{1j}, \dots, p_{nj})$. To each $\sigma \in \mathcal{S}$, we associate an affine coordinate chart $(\mathbb{C}_\sigma^n, \mathbf{u}_\sigma)$ where $\mathbf{u}_\sigma = (u_{\sigma 1}, \dots, u_{\sigma n})$. The modification $\hat{\pi}$ is defined as follows.

Exceptional divisors

For each $\sigma \in \mathcal{S}$, we associate a birational mapping

$$\hat{\pi}_\sigma : \mathbb{C}^n_\sigma \to \mathbb{C}^n, \quad z_i = u_{\sigma 1}^{p_{i1}} \dots u_{\sigma n}^{p_{in}},\ i = 1, \dots, n$$

and X is the complex manifold obtained by gluing $\mathbb{C}^n_\sigma$ and $\mathbb{C}^n_\tau$ by $\hat{\pi}_\tau^{-1} \circ \hat{\pi}_\sigma : \mathbb{C}^n_\sigma \to \mathbb{C}^n_\tau$ where it is well-defined. This defines the modification $\hat{\pi} : X \to \mathbb{C}^n$ which is proper and the restriction $\hat{\pi}$ to the torus $\mathbb{C}^{*n}_\sigma$ is an isomorphism onto the torus $\mathbb{C}^{*n} = (\mathbb{C}^*)^n$ in the base space. If $\sigma = \mathrm{Cone}(P_1, \dots, P_n)$ and $\tau = \mathrm{Cone}(Q_1, \dots, Q_n)$ have a same vertex $Q_1 = P_1$, the hyperplane $u_{\sigma 1} = 0$ glues canonically with the hyperplane $\{u_{\tau 1} = 0\}$. Thus any vertex P of Σ^*, gluing the hyperplanes on every such toric coordinates with $P_1 = P$, defines a divisor in X, and we denote this divisor by $\hat{E}(P)$.

Note that if f is pseudo-convenient, there exists a regular simplicial decomposition Σ^* whose vertices are strictly positive except for the canonical weight vectors $e_i = {}^t(0, \dots, \overset{i}{\overbrace{1}}, \dots, 0), i = 1, \dots, n$. For simplicity we assume hereafter that $f(\mathbf{z})$ has a convenient or pseudo-convenient Newton boundary and the vertices of Σ^* are strictly positive except the canonical weight vectors $e_i, i = 1, \dots, n$. Recall $\hat{E}(e_i)$ is bijectively mapped onto $\{z_i = 0\}$. Then $\hat{\pi} : X \setminus \hat{\pi}^{-1}(0) \to \mathbb{C}^n \setminus \{0\}$ is biholomorphic. Let $\mathcal{V}^+$ be the set of strictly positive vertices of Σ^*. Then the exceptional divisors of $\hat{\pi} : X \to \mathbb{C}^n$ corresponds bijectively to the vertices of $\mathcal{V}^+$.

Intersection criterion

The pull-back $\hat{\pi}^* f$ of f is expressed in the toric chart $\mathbb{C}^n_\sigma$ with $\sigma = \mathrm{Cone}(P_1, \dots, P_n)$ as follows:

$$\hat{\pi}^* f(\mathbf{u}_\sigma) = (\prod_{i=1}^{n} u_{\sigma,i}^{d(P_i)}) \widetilde{f}(\mathbf{u}_\sigma)$$

and $\widetilde{f}(\mathbf{u}_\sigma)$ is the defining function of the strict transform $\widetilde{V}$ of V. The intersection $E(P) := \widetilde{V} \cap \hat{E}(P)$ is defined in $\hat{E}(P)$ by $\widetilde{f}(0, u_{\sigma 2}, \dots, u_{\sigma n}) = 0$. $E(P)$ is an exceptional divisor of the restriction $\pi := \hat{\pi}|_{\widetilde{V}} : \widetilde{V} \to V$. We recall that $\pi^{-1}(O)$ is the union of $E(P)$ such that $P \in \mathcal{V}^+$ and $\Delta(P) \geq 1$. Two exceptional divisors $\hat{E}(P)$ and $\hat{E}(Q)$ intersect if and only if there is a $\sigma \in \mathcal{S}$ such that $\sigma = \mathrm{Cone}(P, Q, P_3, \dots, P_n)$. However for $E(P) \cap E(Q) \neq \emptyset$ it is also necessary that $\dim \Delta(P) \cap \Delta(Q) \geq 1$. See Proposition (1.3.2), in Chapter II ([11]).

A'Campo-Varchenko formula

If f is Newton non-degenerate, any admissible toric modification $\hat{\pi}: X \to \mathbb{C}^n$ gives a good resolution of f and by (AC) the zeta function is written as

$$\zeta(t) = \prod_{P\in\mathcal{V}^+} (1-t^{d(P)})^{-\chi(\hat{E}(P)'')} \qquad (1)$$

$$\hat{E}(P)'' = \hat{E}(P) \setminus \left(\widetilde{V} \cup_{Q\in\mathcal{V}^+, Q\neq P} \hat{E}(Q)\right) \qquad (2)$$

where $\widetilde{V}$ is the strict transform of $V = f^{-1}(0)$ on X.
Let $\hat{E}(P)^{*I} := \hat{E}(P) \cap \mathbb{C}^{*I}$ and $E(P)^I = E(P) \cap \mathbb{C}^{*I}$. Then we use the toric decomposition $\hat{E}(P) = \cup_I \hat{E}(P)^{*I}$ and the equality

$$\chi(E(P)^{*I}) = \chi(Q) = (-1)^{|I|-1}|I|!\mathrm{Vol}_{|I|}\mathrm{C}(\Delta(\mathrm{Q})\cap\mathbb{R}^{\mathrm{I}}, \mathrm{O})/\mathrm{d}(\mathrm{Q})$$

for the computation of the Euler characteristic $\chi(\hat{E}(P)'')$ where $Q \in N^I$ is chosen to be a primitive integer vector such that $\Delta(Q) = \Delta(P) \cap \mathbb{R}^I$ (See [6, 14]).

Almost non-degenerate functions

Consider a function $f(\mathbf{z}) = \sum_\nu a_\nu \mathbf{z}^\nu$ which is expanded in the Taylor series and let $\Gamma(f)$ be the Newton boundary. Let $\hat{\pi} : X \to \mathbb{C}^n$ be a toric modification with respect to Σ^* which is a simplicial regular subdivision of the dual Newton diagram $\Gamma^*(f)$. Let $\mathcal{M}$ be the set of maximal dimensional faces of $\Gamma(f)$ and let $\mathcal{M}_0$ be the subset of $\mathcal{M}$ so that for $\Delta \in \mathcal{M}$, $f_\Delta : \mathbb{C}^{*n} \to \mathbb{C}$ is degenerate if and only if $\Delta \in \mathcal{M}_0$. For P which corresponds to a maximal face $\Delta \in \mathcal{M}$, we denote the exceptional divisor by $\hat{E}(P)$ which corresponds to P.

Conditions(A1) and (A2)

We say that f is an almost non-degenerate function if it satisfies the following conditions.
(A1) For $\Delta \in \mathcal{M}_0$, $f_\Delta : \mathbb{C}^{*n} \to \mathbb{C}$ has a finite number of 1-dimensional critical loci which are $\mathbb{C}^*$-orbits through the origin. (For any face Δ of $\Gamma(f)$ with either $\Delta \in \mathcal{M} \setminus \mathcal{M}_0$ or $\dim \Delta \leq n-2$, f is Newton non-degenerate on Δ.)
Here we recall that $f_\Delta(\mathbf{z})$ is a weighted homogeneous polynomial with respect to the weight vector P and there is an associated $\mathbf{C}^*$-action defined by $t \circ (z_1, \dots, z_n) = (t^{p_1} z_1, \dots, t^{p_n} z_n)$, $t \in \mathbf{C}^*$ where $\Delta(P) = \Delta$ and $P = {}^t(p_1, \dots, p_n)$. Critical points loci of f_Δ are stable under this action.

Local description of $E(P)$

Let $\sigma = \mathrm{Cone}(P_1, \dots, P_n)$ be a simplicial cone in Σ^* such that $\Delta(P_1) = \Delta \in \mathcal{M}_0$. Let $\mathbf{u}_\sigma = (u_{\sigma 1}, \dots, u_{\sigma n})$ be the corresponding toric coordinate chart. The strict transform $\widetilde{V}$ of $V(f)$ is defined by $\widetilde{f}(\mathbf{u}_\sigma) = 0$ where $\widetilde{f}$ is defined by the equality:

$$\hat{\pi}^* f = \left(\prod_{i=1}^{n} u_{\sigma,i}^{d(P_i)}\right) \widetilde{f}(\mathbf{u}_\sigma) = 0$$

and $E(P_1) \subset \hat{E}_0$ is defined by $\{\mathbf{u}_\sigma \mid u_{\sigma 1} = 0, g_\Delta(u_{\sigma 2}, \dots, u_{\sigma n}) = 0\}$ where $g_\Delta(u_{\sigma,2}, \dots, u_{\sigma n}) := \widetilde{f}(0, u_{\sigma 2}, \dots, u_{\sigma n})$.

Admissible coordinates

The assumption (A1) implies $E(P_1)$ has a finite number of isolated singular points. Let $S(\Delta)$ be the set of the singular points of $E(P_1)$. Take any $q \in S(\Delta)$ and assume $q = (0, \beta_2, \dots, \beta_n)$ in $\mathbb{C}_\sigma^n$. An admissible coordinate chart at q is an analytic coordinate chart $(U_q, \mathbf{w})$, $\mathbf{w} = (w_1, \dots, w_n)$ centered at q where U_q is an open neighborhood of q and $(w_2, \dots, w_n)$ is an analytic coordinate change of $(u_{\sigma 2}, \dots, u_{\sigma n})$, but $w_1 = u_{\sigma 1}$ and we do not change $u_{\sigma 1}$. (In many cases, we can take $w_i = u_{\sigma,i} - \beta_i$, $i = 2, \dots, n$.) We fix $w_1 = u_{\sigma 1}$ as $u_{\sigma 1} = 0$ is the defining function of $\hat{E}(P_1)$.

A2 condition

As a second condition, we assume
(A2) *For any* $\Delta \in \mathcal{M}_0$ *and* $q \in S(\Delta)$, *there exists an admissible coordinate* $(U_q, \mathbf{w})$ *centered at* q *such that* $\hat{\pi}^* f(\mathbf{w})$ *is Newton non-degenerate and pseudo-convenient with respect to this coordinates* $(U_q, \mathbf{w})$.

Milnor fibration:First modification of f

We assume that $f(z)$ is an almost non-degenerate function and we take an admissible toric modification as in the previous section. We consider the tubular Milnor fibration

$$(\star) \quad f : U(\varepsilon,\delta)^* \to D_\delta^*,\; U(\varepsilon,\delta) = \{\mathbf{z} \,|\, 0 < |f(z)| \le \delta,\; \|\mathbf{z}\| \le \varepsilon\},\; \delta \ll \varepsilon.$$

This fibration is isomorphically lifted on X so that

$$(\star 2) \quad \hat{f} : \hat{U}(\varepsilon,\delta)^* \to D_\delta^*$$

is equivalent to the Milnor fibration $(\star)$. Here $\hat{U}(\varepsilon,\delta)^* := \hat{\pi}^{-1}(U(\varepsilon,\delta))$ and $\hat{f} := f \circ \hat{\pi}$. This fibration $(\star 2)$ can be decomposed as the union of the fibrations along $\hat{E}(P)$ for $P \in \mathcal{V}^+$ and local Milnor fibrations of $\hat{f}$ at $q \in S(\Delta)$, $\Delta \in \mathcal{M}_0$.

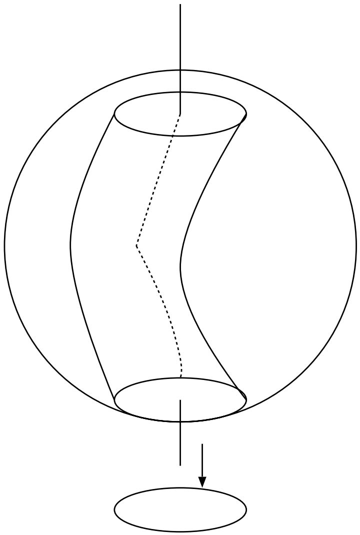


Figure: Milnor Fibation

Local fibration at q

At each $q \in S(\Delta)$, we take a small ball $B_{\varepsilon'}(q)$ and we consider the local Milnor fibration of the function $\hat{f}(\mathbf{w}) := \hat{\pi}^* f(\mathbf{w})$ at q:

$$\hat{f} : U_q(\varepsilon', \delta)^* \to D_\delta^*,$$

where $q \in S(\Delta)$ and

$$U_q(\varepsilon', \delta)^* = \{\mathbf{w} \in U_q \mid 0 < |\hat{f}(\mathbf{w})| \le \delta, \ \|\mathbf{w}\| \le \varepsilon'\}, \ \delta \ll \min\{\varepsilon', \varepsilon\}.$$

We assume δ is small enough so that we can use the same δ in $(\star)$ for the local Milnor fibrations at q. This means that for any $\eta \neq 0$, $|\eta| \le \delta$, the level hypersurface $\hat{f}^{-1}(\eta)$ intersects transversely with the sphere $S_{\varepsilon'}^{2n-1}(q)$. Note that $\hat{f}$ does not have an isolated singularity at q but Newton non-degenerate.

$U_q(\varepsilon', \delta)$

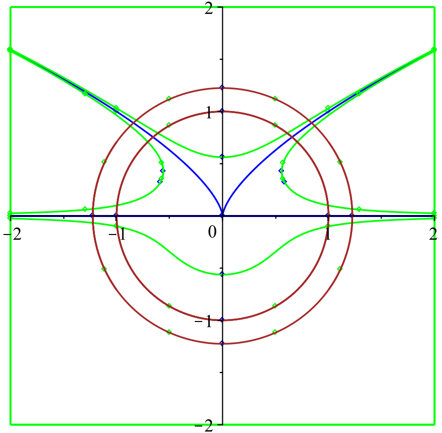


Figure: $U_q(\varepsilon', \delta)$

Here we assume that **w** is an admissible coordinate at q. Let us consider the decomposition of of the total space of the lifted Milnor fibration $\hat{U}(\varepsilon,\delta)^*$,

$$\hat{U}(\varepsilon,\delta)^* = \hat{U}(\varepsilon,\delta)' \cup \left(\cup_{q\in\Delta,\Delta\in\mathcal{M}_0} U_q(\varepsilon',\delta)\right)$$
$$\text{where } \hat{U}(\varepsilon,\delta)' := U(\varepsilon,\delta)^* \setminus \cup \left(\cup_{q\in\Delta,\Delta\in\mathcal{M}_0} U_q(\varepsilon'',\delta)\right)$$

and we consider the corresponding decomposition of the Milnor fibration. We take ε'' a bit smaller than ε'. Assume $\{q_1,\dots,q_m\} = \{q \in S(\Delta) \,|\, \Delta \in \mathcal{M}_0\}$ and consider the sequence of subspaces

$$\hat{U}_0(\varepsilon,\delta)' \subset \hat{U}_1(\varepsilon,\delta)' \subset \cdots \subset \hat{U}_m(\varepsilon,\delta) = \hat{U}(\varepsilon,\delta)$$
$$\text{where} \quad \hat{U}_i(\varepsilon,\delta) := \hat{U}(\varepsilon,\delta)' \bigcup_{j\le i} U_{q_j}(\varepsilon'\delta)).$$

Computation of $\zeta'(t)$

Let $\zeta'(t)$ be the zeta function of the restriction $\hat{f} : \hat{U}(\varepsilon, \delta)' \to D_\delta^*$ and let $\zeta_q(t)$ be the zeta function of the local Milnor fibration of $\hat{f}$ at q. ($\hat{f}$ is the total pull-back of f). We denote the set of primitive weight vectors by $\mathcal{P}$ and $\mathcal{P}_0$ which correspond to $\mathcal{M}$ and $\mathcal{M}_0$ respectively. Then we have

Lemma 2

Assume that $f(\mathbf{z})$ is an almost non-degenerate function as above. Then the zeta function of f is given as the product

$$\zeta(t) = \zeta'(t) \prod_{q \in S(\Delta)} \zeta_q(t) \tag{3}$$

where each product factor $\zeta_q(t)$ can be computed by Varchenko's formula (V) and $\zeta'(t)$ is given as

$$\begin{aligned} \zeta'(t) &= \prod_{I \subsetneq \{1,\dots,n\}} \zeta_I(t) \prod_{P \in \mathcal{P} \setminus \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(P)} \\ &\times \prod_{P \in \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(P) + (-1)^{n-1} \sum_{q \in S(\Delta(P))} \mu_q}. \end{aligned} \tag{4}$$

Outline of the proof

Proof: We give an outline of a proof of the assertion by an inductive argument on i, showing the zeta function $\zeta^{(i+1)}(t)$ of $\hat{f} : \hat{U}_{i+1}(\varepsilon, \delta) \to D_\delta^*$ is given as

$$\zeta^{(i+1)}(t) = \zeta^{(i)}(t)\zeta_{q_{i+1}}(t), \ 0 \le i \le m. \tag{5}$$

For the proof, we use the following Sublemma 26 and Proposition 27.

Two basic tools

Sublemma*[Lemma (5.3), [11]] Let $U \subset \hat{\pi}^{-1}(U(\varepsilon, \delta))$ and suppose that there is a manifold M and a submersion $p : U \to M$ so that $p \times \hat{f} : U \to M \times D_\delta^*$ is a locally trivial fibration. Its restriction to $p^{-1}(m)$, $\hat{f} : p^{-1}(m) \to D_\delta^*$, with $m \in M$ is also a fibration. Let $\zeta(t)$ and $\zeta_M^\perp(t)$ be the respective zeta functions of the fibrations $\hat{f} : U \to D_\delta^*$ and $\hat{f} : p^{-1}(m) \to D_\delta^*$. Then we have the equality: $\zeta(t) = (\zeta_M^\perp(t))^{\chi(M)}$.*

The assertion is trivial when $p : U \to M$ is a trivial fibration. Then we apply Mayer-Vietoris argument. $\zeta_M^\perp(t)$ is called the normal zeta function along M.

Second tool

[Proposition (2.8), [11]] Let $U = U_1 \cup U_2$ be an open covering of the fibration $p : U \to D^*_\delta$ where the restriction of p to U_1, U_2 and $U_{12} := U_1 \cap U_2$ is also fibration. Consider four fibrations. Let F, F_1, F_2, F_{12} be the respective fibers and let $\zeta, \zeta_1(t), \zeta_2(t), \zeta_{12}$ be their zeta functions. Then

$$\chi(F) = \chi(F_1) + \chi(F_2) - \chi(F_{12}), \quad \zeta(t) = \zeta_1(t)\zeta_2(t)\zeta_{12}(t)^{-1}.$$

The assertion follows easily from the Mayer-Vietoris argument.

We apply Proposition 27 to the union $\hat{U}_{i+1}(\varepsilon,\delta) = \hat{U}_i(\varepsilon,\delta) \cup U_{q_{i+1}}(\varepsilon',\delta)$. Let $W_{i+1} = U_{q_{j+1}}(\varepsilon',\delta) \setminus U_{q_{i+1}}(\varepsilon'',\delta) = \hat{U}_i(\varepsilon,\delta) \cap U_{q_{i+1}}(\varepsilon',\delta)$. The proof of the inductive assertion (5) follows from the next assertion.

Assertion 0.1

The contribution to the zeta function from $W_{i+1} = \hat{U}_i(\varepsilon,\delta) \cap U_{q_{i+1}}(\varepsilon',\delta)$ is trivial.

Assuming Assertion 0.1, Lemma 2 follows by the inductive argument.

Zeta function $\zeta'(t)$ and the proof of (4)

Recall that $\zeta'(t)$ is the zeta function for the first toric modification, outside of the singular points $\{q \mid q \in S(\Delta),\ \Delta \in \mathcal{M}_0\}$. Applying A'Campo's formula and Varchenko's description, we get a formula for the zeta function $\zeta'(t)$ which is given as follows.
For $\Delta \in \mathcal{M} \setminus \mathcal{M}_0$, the calculation is the same as in the proof of Theorem (5.3), [11]. Assume that $\Delta \in \mathcal{M}_0$. Let P be the primitive weight vector which corresponds to Δ and take a toric chart $\mathbb{C}^n_\sigma$, $\sigma = \mathrm{Cone}(P_1, \dots, P_n)$ with $P = P_1$ and let $\mathbf{u}_\sigma = (u_{\sigma 1}, \dots, u_{\sigma n})$ the toric coordinate of this chart. Then as in the previous section, the exceptional divisor $\hat{E}(P)$ is defined by $u_{\sigma 1} = 0$ and $E(P) := \hat{E}(P) \cap \widetilde{V}$ is defined in $\hat{E}(P)$ by $g_P(u_{\sigma 2}, \dots, u_{\sigma n}) = 0$ where

$$\hat{\pi}^* f(\mathbf{u}_\sigma) = \left(\prod_{i=1}^{n} u_{\sigma,i}^{d(P_i;f)}\right) \widetilde{f}(\mathbf{u}_\sigma)$$

$$g_P(u_{\sigma 2}, \dots, u_{\sigma n}) := \widetilde{f}(0, u_{\sigma 2}, \dots, u_{\sigma n}).$$

Let μ_q be the Milnor number of (g_P, q) as a germ of a hypersurface at $q \in \hat{E}(P)$. We take a small ball $B_q(\varepsilon)$ centered at q for $q \in S(\Delta)$. Let us consider a small perturbation family $f_s(\mathbf{z})$, $|s| \le 1$ of the coefficients of f so that $f = f_0$, $\Gamma(f_s) = \Gamma(f)$ for any s and f_s is non-degenerate for $s \neq 0$. More precisely, we need only move a bit the coefficients of f_Δ, $\Delta \in \mathcal{M}_0$. The same toric modification $\hat{\pi} : X \to \mathbb{C}^n$ gives a good resolution of the family f_s for any $s \neq 0$. Let $\zeta^{(s)}(t)$ be the zeta function of f_s, $s \neq 0$. Let $\widetilde{V}_0 = \widetilde{V}$ and $\widetilde{V}_s$ be the strict transform of $f^{-1}(0)$ and $f_s^{-1}(0)$ respectively.

Let $\hat{E}(P)''_s$ be the corresponding factor of the exceptional divisor of $\hat{E}(P)$ which appears in the product expression of $\zeta^{(s)}(t)$ in the formula (AC) of A'Campo and (V) of Varchenko and let $E(P)_s$ and $E(P)_0$ ($= E(P)$ the intersections of $\hat{E}(P)$ and the strict transforms $\widetilde{V}_s$ of $f_s = 0$ and $\widetilde{V}_0$ of $f_0 = 0$ respectively. Note that $\chi(E(P)) = \chi(E(P)_s) + (-1)^{n-1}\sum_{q\in S(\Delta)}\mu_q$. Therefore

$$\begin{aligned}\chi(\hat{E}(P)\setminus(\widetilde{V}_0\cup_{Q\neq P}\hat{E}(Q))) &= \chi(\hat{E}(P)\setminus(\widetilde{V}_s\cup_{Q\neq P}\hat{E}(Q)) \\ &\quad -(-1)^{n-1}\sum_{q\in S(\Delta)}\mu_q.\end{aligned}$$

Thus the factor of zeta function $\zeta'(t)$ coming from the divisor $\hat{E}(P)$ is changed from

$$(1 - t^{d(P)})^{-\chi(\hat{E}(P)''_s)} \quad \to \; (1 - t^{d(P)})^{-\chi(\hat{E}(P)''_s) + (-1)^{n-1}\sum_{q \in S(\Delta)} \mu_q}.$$

For $\Delta \in \mathcal{M}_0$, take $P \in \mathcal{P}_0$ with $\Delta(P) = \Delta$ and we put

$$\zeta_\Delta(t) := \prod_{q \in S(\Delta)} \zeta_q(t),$$

$$\zeta_\Delta^{er}(t) := (1 - t^{d(P)})^{(-1)^{n-1} \sum_{q \in S(\Delta)} \mu_q}$$

$$\zeta^{er}(t) := \prod_{\Delta \in \mathcal{M}_0} \zeta_\Delta^{er}(t)$$

Using the above notations, we now obtain the equality:

$$\zeta'(t) = \prod_{I \subsetneq \{1,\dots,n\}} \zeta_I(t) \prod_{P \in \mathcal{P} \setminus \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(P)}$$
$$\times \prod_{P \in \mathcal{P}_0} (1 - t^{d(P)})^{-\chi(\hat{E}(P)''_s) + (-1)^{n-1} \sum_{q \in S(\Delta)} \mu_q}$$
$$= \zeta^{(s)}(t) \zeta^{er}(t).$$

Second modifications, calculation of $\zeta_q(t)$

For $\Delta \in \mathcal{M}_0$ and $q \in S(\Delta)$, we choose admissible coordinates $\mathbf{w} = (w_1, \dots, w_n)$ centered at q and take an admissible regular simplicial subdivision Σ_q^* of $\Gamma^*(\hat{f}; \mathbf{w})$. As $\hat{f}(\mathbf{w})$ is pseudo-convenient, we assume that the vertices of Σ_q^* are strictly positive except $e_1, \dots, e_n$. Then we take the toric modification $\hat{\omega}_q : Y_q \to X$ with respect to Σ_q^*. Taking the toric modification at each $q \in S(\Delta)$, $\Delta \in \mathcal{M}_0$, let $\hat{\omega} : Y \to X$ is the union of the toric modification. Here Y is the canonical gluing of the union of Y_q, $q \in S(\Delta)$, $\Delta \in \mathcal{M}_0$. The composition

$$\Pi : Y \xrightarrow{\hat{\omega}} X \xrightarrow{\hat{\pi}} \mathbb{C}^n$$

gives a good resolution of f. The exceptional divisors of Π are all compact. The zeta function $\zeta_q(t)$ is described by (AC) or (V).

Thus by Lemma 2, we have the following generalization of Varchenko's formula:

Theorem 3

The zeta function of f is given by

$$\zeta(t) = \zeta^{(s)}(t)\zeta^{er}(t) \prod_{\Delta \in \mathcal{M}_0} \zeta_\Delta(t).$$

where $\zeta^{(s)}(t)$ *is the zeta function of the Newton non-degenerate function* f_s *with* $\Gamma(f_s) = \Gamma(f)$.

Example 2.

Consider $f = z^3 + y^3 + z^3 - 3xyz + z^4$. Note that $x^3 + y^3 + z^3 - 3xyz = 0$ consists of three planes $x + y + z = 0, x + \omega y + \omega^2 z = 0, x + \omega^2 y + \omega z = 0$ with $\omega = \exp(2\pi i/3)$. The dual Newton diagram has a single strictly positive vertex $P = {}^t(1,1,1)$ and it is already regular simplicial cone subdivision. The corresponding toric modification is nothing but the ordinary blowing up. After one blowing up, we work in the toric coordinate chart $\mathbb{C}^3_\sigma$, $\mathbf{u}_\sigma = (u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3})$ where $\sigma = \mathrm{Cone}(P, e_1, e_2)$, $P = {}^t(1,1,1)$ and $x = u_{\sigma 1} u_{\sigma 2}, y = u_{\sigma 1} u_{\sigma 3}, z = u_{\sigma 1}$. We have

$$\pi^* f = u_{\sigma 1}^3 \tilde{f}(\mathbf{u}_\sigma),$$

$$\tilde{f}(\mathbf{u}_\sigma) = (u_{\sigma 2} + u_{\sigma 3} + 1)(u_{\sigma 2} + \omega u_{\sigma 3} + \omega^2)(u_{\sigma 2} + \omega^2 u_{\sigma 3} + \omega) + u_{\sigma 1}$$

and $E(P)$ is defined by $u_{\sigma 1} = 0$, $\tilde{f}(0, u_{\sigma 2}, u_{\sigma 3}) = 0$ which consists of three $\mathbb{P}^1$ and three singular points are $q_0 := (0,1,1)$, $q_1 = (0, \omega, \omega^2)$ and $q_2 = (0, \omega^2, \omega)$.

For example, at q_0, taking the coordinate $w_1 = u_{\sigma 1}, w_2 = u_{\sigma 2} - 1, w_3 = u_{\sigma 3} - 1$, $\hat{\pi}^* f$ is written as

$$\begin{aligned}\hat{\pi}^* f &= u_{\sigma 1}^3 (u_{\sigma 3}^3 + u_{\sigma 2}^3 - 3 u_{\sigma 3} u_{\sigma 2} + u_{\sigma 1} + 1) \\ &= w_1^3 (w_3^3 + w_2^3 + 3 w_3^2 - 3 w_3 w_2 + 3 w_2^2 + w_1)\end{aligned}$$

By the symmetry of the equation, the singularities of $E(P)$ are isomorphic at any q_i. They are A_1 singularity and $\mu_{q_i} = 1$. Thus $\zeta^{(s)}(t) = (1 - t^3)^{-3}$, $\zeta^{er}(t) = (1 - t^3)^3$ and $\zeta'(t) = 1$. $\zeta_{q_i}(t)$ is given as $(1 - t^4)^{-1}$. Thus we get

Assertion 0.2

$\zeta(t) = (1 - t^4)^{-3}$ and $\mu = 11$.

Observe that $11 = 8 + 3 \cdot 1$.

Example 3

. Let $f_d(x, y, z)$ be an irreducible convenient homogeneous polynomial which defines a projective curve of degree d with $k \leq \frac{(n-1)(n-2)}{2}$ nodes. See [10] for an example of such a curve. We consider $f = f_d + x^{d+1}$. As an affine polynomial, f_d has a single maximal dimensional face with weight vector $P = {}^t(1, 1, 1)$ and it is Newton degenerate.
The local zeta function is $\zeta_{q_i}(t) = (1 - t^{d+1})^{-1}$ and we get

$$\zeta(t) = (1 - t^d)^{-d^2+3d-3+k}(1 - t^{d+1})^{-k}, \ \mu(f) = (d-1)^3 + k.$$

Application: A Zariski pair of links

It is well-known that there exists a pair of projective curves $\{C, C'\}$ of degree 6 (so called a Zariski pair) with 6 cusps whose complements have different topologies (Zariski [17]). The first curve C is sextic of torus type. A typical one is defined as follows:

$$\begin{aligned} C :=&\{(x,y,z) \in \mathbb{P}^2 \mid f_6(x,y,z) = 0\} \\ f_6 =& f_2^3 + f_3^2,\ f_2 = x^2 + y^2 + z^2,\ f_3 = x^3 + y^3 + z^3. \end{aligned}$$

Another curve C' is a sextic with 6 cusps such that there exists no conic which contains these 6 points. We use the sextic which is given in [12]. For our purpose, we took the change of coordinates $(x, y) \mapsto (x + 1/2, y + 2)$.

Non-conical 6 cuspidal sextic

$$C' : \quad g_6(x,y,z) = 0$$

$$g_6 = -\frac{215\,z^6}{64} + \frac{51\,xz^5}{16} + \frac{63\,x^2z^4}{16} - \frac{3\,x^3z^3}{2} - \frac{9\,x^4z^2}{4} \tag{6}$$

$$+3\,x^5z + x^6 - \frac{41\,yz^5}{4} + 8\,xyz^4 + 10\,x^2yz^3 - 4\,x^4yz - \frac{571\,y^2z^4}{48}$$

$$+\frac{22\,xy^2z^3}{3} + \frac{47\,x^2y^2z^2}{6} - x^4y^2 - \frac{190\,y^3z^3}{27} + \frac{8\,xy^3z^2}{3}$$

$$+\frac{8\,x^2y^3z}{3} - \frac{85\,y^4z^2}{36} + \frac{xy^4z}{3} + \frac{x^2y^4}{3} - \frac{4\,y^5z}{9} - \frac{y^6}{27}$$

To make the singularity of $f_6^{-1}(0)$ and $g_6^{-1}(0)$ to be isolated at the origin, we put

$$f = f_6 + z^7, \quad g = g_6 + z^7$$

and we consider the corresponding hypersurface $f^{-1}(0)$ and $g^{-1}(0)$ at the origin. We will show that they have the same Newton boundary, the same zeta function (thus the same Milnor number too) and their links M_f and M_g are diffeomorphic where $M_f = V(f) \cap S_\varepsilon^{2n-1}$ and $M_g = V(g) \cap S_\varepsilon^{2n-1}$ and ε is small enough. However the tangent cones are not isomorphic. We call such a pair of links of hypersurfaces $V(f), V(g)$ a Zariski pair of links.

Torus type sextic as the tangent cone

We consider first $V = f^{-1}(0)$. Consider the polynomials:

$$f_2 = x^2 + y^2 + z^2,\ f_3 = x^3 + y^3 + z^3,$$
$$f_6 = f_2^3 + f_3^2,\ \ f = f_6 + z^7.$$

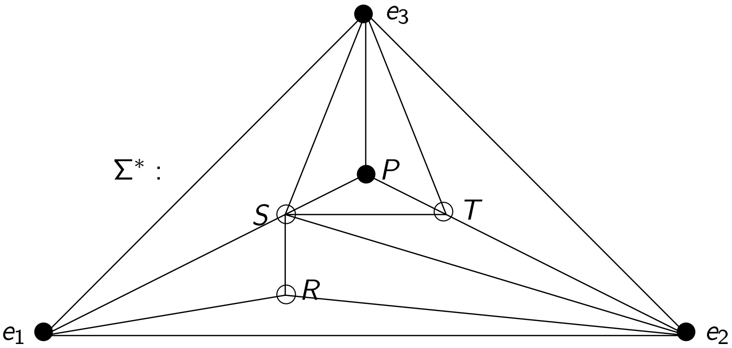
We first take an ordinary blowing up $\hat{\pi} : X \to \mathbb{C}^3$ and we take the chart $(U, \mathbf{u}_\sigma) = (u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3}))$ with $\hat{\pi}(\mathbf{u}_\sigma) = (u_{\sigma 1} u_{\sigma 3}, u_{\sigma 2} u_{\sigma 3}, u_{\sigma 3})$. Denote the exceptional divisor by $\hat{E}_0$ which is $\mathbb{P}^2$ and it is defined by $u_{\sigma 3} = 0$ in U. $E_0 = \hat{E}_0 \cap \widetilde{V}$ is the exceptional divisor of the restriction of $\hat{\pi}$ to $\widetilde{V}$, corresponding to the strict transform of $f_6 = 0$. More precisely we have:

$$\hat{\pi}^* f = u_{\sigma 3}^6 (\bar{f}_6 + u_{\sigma 3}), \ \bar{f}_6 = \bar{f}_2^3 + \bar{f}_3^2,$$
$$\bar{f}_2 = u_{\sigma 1}^2 + u_{\sigma 2}^2 + 1, \quad \bar{f}_3 = u_{\sigma 1}^3 + u_{\sigma 2}^3 + 1.$$

Let $\widetilde{V}$ be the strict transform of V. 6 singular points of E_0 are located at the intesection $\bar{f}_2 = \bar{f}_3 = 0$ and they are A_2 singularities. On each intersection, $\bar{f}_2 = 0$ and $\bar{f}_3 = 0$ are non-singular and intersect transversely. Put them $\rho_1, \dots, \rho_6$. As $V(\bar{f}_2), V(\bar{f}_3), V(u_{\sigma 3})$ intersect transversely at ρ_i, we can take $w_1 = \bar{f}_2$, $w_2 = \bar{f}_3, w_3 = u_{\sigma 3}$ as analytic coordinates in a small neighborhood U_i of ρ_i and then we have

$$\hat{\pi}^* f = w_3^6 (w_1^3 + w_2^2 + w_3).$$

The dual Newton diagram $\Gamma^*(f)$ is as Figure 3 and we take a regular subdivision Σ^*_ρ as in Figure 3.



$$P = {}^t(2,3,6),\ T = {}^t(1,2,3),\ S = {}^t(1,1,2),\ R = {}^t(1,1,1)$$

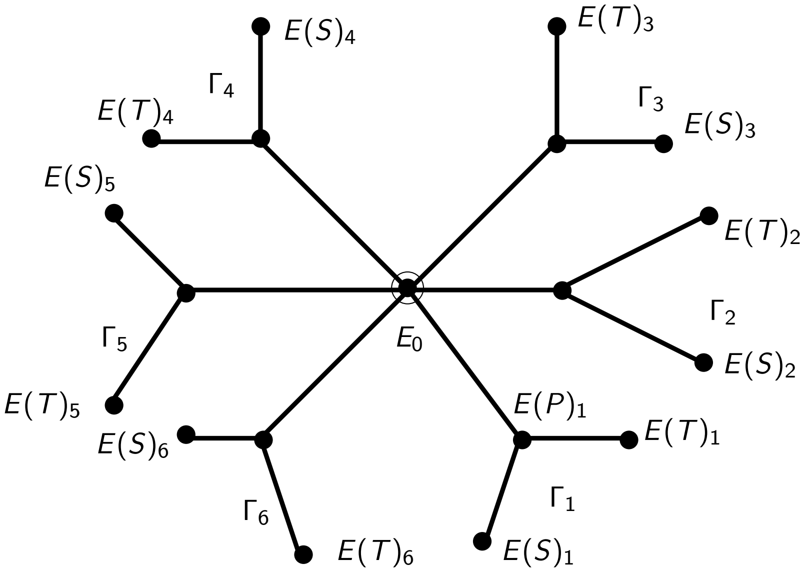
take the associated toric modification $\hat{\omega}_i : Y_i \to U_i$. It has four exceptional divisors $\hat{E}(P)$, $\hat{E}(T)$, $\hat{E}(S)$, $\hat{E}(R)$. The strict transform $\widetilde{V}_i$ of $\widetilde{V}$ is smooth. The restriction $\hat{\omega}_i : \widetilde{V}_i \to \widetilde{V}$ is a resolution of $\widetilde{V}$ at ρ_i and it has three exceptional divisors $E(P) = \hat{E}(P) \cap \widetilde{V}_i$, $E(T) = \hat{E}(T) \cap \widetilde{V}_i$ and $E(S) = \hat{E}(S) \cap \widetilde{V}_i$. On the other hand, $\hat{E}(R) \cap \widetilde{V}_i$ is empty as $\Delta(R) = \{(0, 0, 7)\}$. (See for example Proposition (3.7), page 131, [11].) $E(e_1)$, $E(e_2)$ are strict transforms of the conic $f_2 = 0$ and the cubic $f_3 = 0$. $E(e_3)$ is (the pull-back of) the exceptional divisor E_0. To distinguish divisors over other singular points $\rho_i, 1 \le i \le 6$, we denote them by $E(P)_i$, $E(T)_i$, $E(S)_i$.

Resolution of V_f

We do the same toric modification at $\rho_1, \dots, \rho_6$ and let $\hat{\omega} : Y \to X$ be the union of these modifications and let $\hat{\Pi} = \hat{\pi} \circ \hat{\omega} : Y \to X \to \mathbb{C}^n$ the composition of the modifications. The exceptional divisors of $\hat{\Pi}$ is given as

$$D := \hat{\Pi}^{-1}(0) \cap \widetilde{V} = E_0 + \sum_{i=1}^{6} (E(P)_i + E(T)_i + E(S)_i).$$

19 exceptional divisors of star-shaped.



$E(S)_4$
$E(T)_3$
Γ_4
Γ_3
$E(T)_4$
$E(S)_3$
$E(S)_5$
$E(T)_2$
Γ_2
Γ_5
E_0
$E(S)_2$
$E(P)_1$
$E(T)_5$
$E(T)_1$
$E(S)_6$
Γ_1
Γ_6
$E(T)_6$
$E(S)_1$

Assertion 0.3

E_0 has genus 4.

The assertion follows from the Euler characteristic calculation, $\chi(E_0) = -18 + 6 \cdot 2 = -6$. Here 18 is the Euler characteristic of the smooth sextic $E_{0s} = \hat{E}_0 \cap \widetilde{V}_s$ (genus 10) and -12 is the defect from 6 cusps.

Intersection numbers

The link of V is diffeomorphic to the boundary of the tubular neighborhood of the total exceptional divisor D. To compute the intersection numbers, we use $(\hat{\Pi}^* f_2)$ and the property $(\hat{\Pi}^* f_2) \cdot E = 0$. We use three toric charts $\sigma = \mathrm{Cone}(P, T, e_3)$, $\tau = \mathrm{Cone}(P, S, e_3)$ and $\xi = \mathrm{Cone}(S, e_1, e_3)$ with respective toric coordinates $(u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3})$, $(u_{\tau 1}, u_{\tau 2}, u_{\tau 3})$ and $(u_{\xi 1}, u_{\xi 2}, u_{\xi 3})$. Thus we have

$$(\hat{\Pi}^* f_2) = 2E_0 + \widetilde{V}_2 + \sum_{i=1}^{6} (14E(P)_i + 7E(T)_i + 5E(S)_i).$$

Here $\widetilde{V}_2$ is the strict transform of $V(f_2)$. Let I_f be the 19×19-intersection matrix of the exceptional divisors. Thus we conclude

$$E(P)_i^2 = -1,\ E(T)_i^2 = -2,\ E(S)_i^2 = -3,\ E_0^2 = -42,\ \det I_f = 6.$$

Non Torus type sextic as the tangent cone

We consider $V(g)$ where $g = g_6 + z^7$ and g_6 is given in (6). First we take a blowing up at the origin. Using the same coordinates $(u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3})$,

$$\hat{\pi}^* g = u_{\sigma 3}^6 (\bar{g}_6(u_{\sigma 1}, u_{\sigma 2}) + u_{\sigma 3})$$

$$\bar{g}_6(u_{\sigma 1}, u_{\sigma 2}) = 1728\, {u_{\sigma 1}}^6 + 5184\, {u_{\sigma 1}}^5 + \left(-1728\, {u_{\sigma 2}}^2 - 6912\, u_{\sigma 2} - 3888\right)$$

$$-2592\, {u_{\sigma 1}}^3 + \left(576\, {u_{\sigma 2}}^4 + 4608\, {u_{\sigma 2}}^3 + 13536\, {u_{\sigma 2}}^2 + 17280\, u_{\sigma 2} + 68$$

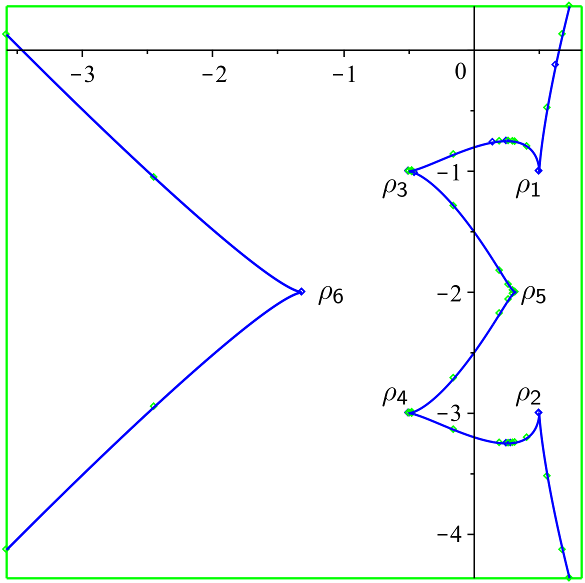
$$+ \left(576\, {u_{\sigma 2}}^4 + 4608\, {u_{\sigma 2}}^3 + 12672\, {u_{\sigma 2}}^2 + 13824\, u_{\sigma 2} + 5508\right)$$

$$- 768\, {u_{\sigma 2}}^5 - 4080\, {u_{\sigma 2}}^4 - 12160\, {u_{\sigma 2}}^3 - 20556\, {u_{\sigma 2}}^2 - 17712$$

The graph of $\bar{g}_6 = 0$ is given in Figure 5. The singular points are

$$\rho_1 = (\frac{1}{2}, -1, 0),\ \rho_2 = (\frac{1}{2}, -3, 0),\ \rho_3 = (-\frac{1}{2}, -1, 0),$$
$$\rho_4 = (-\frac{1}{2}, -3, 0),\ \rho_5 = (-\frac{1}{2} + \frac{\sqrt{6}}{3}, -\frac{1}{2}, 0),\ \rho_6 = (-\frac{1}{2} - \frac{\sqrt{6}}{3}, -\frac{1}{2}, 0).$$

and the tangent cones are vertical for $\{\rho_1, \rho_2\}$ and horizontal for $\{\rho_3, \rho_4, \rho_5, \rho_6\}$. This implies the elementary choice of coordinates are admissible. The resolutions are similar for ρ_1, ρ_2 and also the resolutions for $\rho_3, \ldots, \rho_6$ are similar. So we will see two resolutions at ρ_1 and ρ_3.


−3
−2
−1
0
−1
−2
−3
−4
ρ_1
ρ_2
ρ_3
ρ_4
ρ_5
ρ_6

We do a practical resolutions at $\rho_1, \ldots, \rho_6$. WE omit the detail but found that the resution graph is the same as that of $V(f)$.

Theorem 4

The links of f and g have the same zeta function:

$$\zeta'(t) = (1-t^6)^{-9}, \quad \zeta_{\rho_i}(t) = \frac{(1-t^{21})(1-t^{14})}{(1-t^{42})(1-t^7)}$$

$$\zeta(t) = \frac{1}{(1-t^6)^9}\left(\frac{(1-t^{21})(1-t^{14})}{(1-t^{42})(1-t^7)}\right)^6$$

and the Milnor number is 137*. They also have the same graph of the resolution and the determinants of the intersection matrices are* -6*.*

First homology group

Theorem 5

The surfaces $V(f)$ and $V(g)$ have the same zeta function and the same Milnor number 137 but their links are diffeomorohic and

$$H_1(M_f) = H_(M_g) = \mathbb{Z}^8 \oplus \mathbb{Z}/6.$$

The order of the torsion parts come from the absolute values of the respective determinant of the intersection matrices.

Problem

Is the diffeomorphism of M_f and M_g a restriction of a diffeomorphism of S_ε^5?

Consider a convenient homogeneous polynomial $f_d(\mathbf{z})$ of degree d which defines a projective hypersurface V with s isolated singular points $\rho_1, \dots, \rho_s$. We assume the singular points are in the projective chart $\{z_n \neq 0\}$ with coordinates $\mathbf{x} = (x_1, \dots, x_{n-1})$, $x_i = z_i/z_n$, $i = 1, \dots, n-1$ and we assume that for each ρ_i, there exists a local coordinates $(U_i, \mathbf{w}^{(i)})$ so that the local equation of $V \cap U_{\rho_i}$ is a Newton non-degenerate function $f_i(\mathbf{w}^{(i)})$. Put μ_i the Milnor number of f_i at ρ_i and $\mu_{tot} = \sum_{i=1}^{s} \mu_i$. Consider the modified function

$$f(\mathbf{z}) = f_d(\mathbf{z}) + z_n^{d+1}$$

which makes $V(f)$ has an isolated singularity at the origin. Then we have:

Theorem 6

$f(\mathbf{z})$ is an almost Newton non-degenerate function and the Milnor number $\mu(f)$ of f is given as $\mu(f) = (d-1)^n + \mu_{tot}$.

Thank you for the attention!

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