

Properness of polynomial maps with Newton polyhedrons

(joint work with Takeki Tsuchiya)

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Definition

Let $f = (f^1, \dots, f^n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map.

Definition

$y_0 \in \mathbb{C}^n$ is a **proper point** of f , if
for any arc $x(t) : \mathbb{C}^* \rightarrow \mathbb{C}^n$,

$$\lim_{t \rightarrow 0} f(x(t)) = y_0 \implies \lim_{t \rightarrow 0} x(t) \text{ exists in } \mathbb{C}^n.$$

We consider the **non-properness set** of f :

$$\begin{aligned} S_f &= \{y_0 \in \mathbb{C}^n : y_0 \text{ is not a proper point of } f\} \\ &= \left\{ y_0 \in \mathbb{C}^n : \begin{array}{l} \exists x(t) : \mathbb{C}^* \rightarrow \mathbb{C}^n \text{ such that} \\ \lim_{t \rightarrow 0} f(x(t)) = y_0, \quad \lim_{t \rightarrow 0} x(t) = \infty \end{array} \right\} \end{aligned}$$

A preliminary consideration

Compactify $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ as

$$\bar{f} : X \rightarrow Y$$

where X and Y are suitable projective manifolds.

Set $X_\infty = X \setminus \mathbb{C}^n$, $Y_\infty = Y \setminus \mathbb{C}^n$.

Assume that X_∞ and Y_∞ are simple normal crossing divisors. We then have

$$S_f = \bar{f}(X_\infty) \cap \mathbb{C}^n$$

Since \bar{f} is proper, the set $\bar{f}(X_\infty)$ is closed in Y and we obtain that S_f is closed.

Jelonek's results

Theorem [Jelonek 1993]

If a polynomial map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is dominant, then S_f is a \mathbb{C} -uniruled hypersurface or empty.

A variety is a \mathbb{C} -uniruled if it is the image of $X \times \mathbb{C}$, for some X , by a birational map.

Jelonek showed several degree estimates of S_f .

Z. Jelonek, The set of points at which a polynomial map is not proper, *Ann. Polon. Math.* 58 (1993), 259–266.

Z. Jelonek, Testing sets for properness of polynomial mappings, *Math. Ann.*, 315(1), 1–35, 1999.

Z. Jelonek and M. Lason, Quantitative properties of the non-properness set of a polynomial map, *manuscripta math.* 156, 383–397 (2018)

Newton polyhedron

Set $f^j = \sum_{\nu} c_{\nu}^j x^{\nu}$, $j = 1, \dots, n$.

Newton polyhedron of f^j is defined by

$$\Delta(f^j) = \text{convex hull of } \{\nu \in \mathbb{R}^n : c_{\nu}^j \neq 0\}.$$

For $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$

$$d_j(p) = - \min\{\langle p, \nu \rangle : \nu \in \Delta(f^j)\}$$

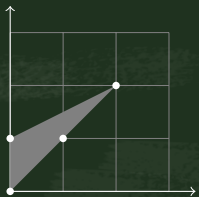
$$\gamma_j(p) = \{\nu \in \Delta(f^j) : \langle p, \nu \rangle = -d_j(p)\}$$

where $\langle p, \nu \rangle = p_1 \nu_1 + \dots + p_n \nu_n$.

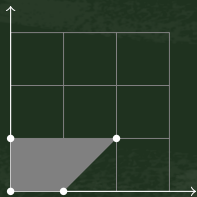
We say $\gamma_j(p)$ the face of $\Delta(f^j)$ supported by p .

Example

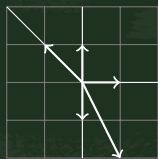
$$(f^1(x, y), f^2(x, y)) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$$



$\Delta(f^1)$



$\Delta(f^2)$



Vectors supporting faces

$$d_1(0, -1) = 2$$

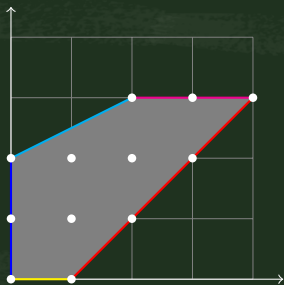
$$d_1(-1, 1) = 0$$

$$d_2(0, -1) = 1$$

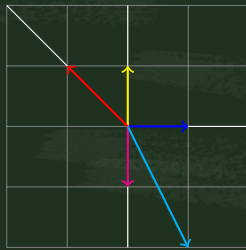
$$d_2(-1, 1) = 1$$

Minkowski sum

- $\Delta(f) = \Delta(f^1) + \dots + \Delta(f^n)$
 - Δ^* is the dual fan of $\Delta(f)$
 - $f_\gamma = (f_{\gamma_1}^1, \dots, f_{\gamma_n}^n)$ where $f_{\gamma_j}^j = \sum_{\nu \in \gamma_j} c_\nu^j x^\nu$
- $(f^1(x, y), f^2(x, y)) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$



$\Delta(f)$



generators of l-cones of Δ^*

Assume that f^j ($j = 1, \dots, n$) are non-constant polynomials with constant terms.

- For $p \in \mathbb{Z}^n$, $\gamma(p) = (\gamma_1(p), \dots, \gamma_n(p))$.
- For $p \in \mathbb{Z}^n$, $J_{\gamma(p)} = \{j \in \{1, \dots, n\} : 0 \notin \gamma_j\}$.
- $\Delta_{nc}(f) = \{\gamma(p) : p \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq 0})^n\}$
the set of non-coordinate faces
- For $J \subset \{1, \dots, n\}$, we denote

$$\mathbb{C}^n = \mathbb{C}^J \times \mathbb{C}^{J^c}, \quad J^c = \{1, \dots, n\} \setminus J$$

- For $J \subset \{1, \dots, n\}$, we denote $f^J = (f^j)_{j \in J}$

$$f^J : \mathbb{C}^n \rightarrow \mathbb{C}^J$$

- f is dominant if $\det \text{Jac}(f)$ is not identically zero.

Main Theorem

Set

$$Z(\mathbf{f}_\gamma^J) = \{x \in (\mathbb{C}^*)^n : \mathbf{f}_\gamma^J(x) = 0\},$$

$$\Sigma(\mathbf{f}_\gamma^J) = \{x \in (\mathbb{C}^*)^n : \text{rank Jac}(\mathbf{f}_\gamma^J) < \#J\}.$$

We say that Z has a **dense nonsingular locus** if $Z \setminus \Sigma$ is dense in Z .

Theorem 1

If f is dominant and $Z(\mathbf{f}_\gamma^J) \setminus \Sigma(\mathbf{f}_\gamma^J)$ is dense in $Z(\mathbf{f}_\gamma^J) \forall \gamma \in \Delta_{\text{nc}}(f)$, $J = J_\gamma$, then

$$S_f = \bigcup_{\gamma \in \Delta_{\text{nc}}(f)} S_\gamma$$

where $S_\gamma = \mathbf{f}^{J^c}(Z(\mathbf{f}_\gamma^J)) \times \mathbb{C}^J$.

Non-degeneracy condition

Y. Chen, L.R.G. Dias, K. Takeuchi and M. Tibar, Invertible polynomial mappings via Newton non-degeneracy, Ann. Inst. Fourier, Grenoble 64, 5 (2014), 1807-1822.
define non-degeneracy condition as

$$Z(f_\gamma^J) \cap \Sigma(f_\gamma^J) = \emptyset, \quad J = J_\gamma \quad \text{for any } \gamma.$$

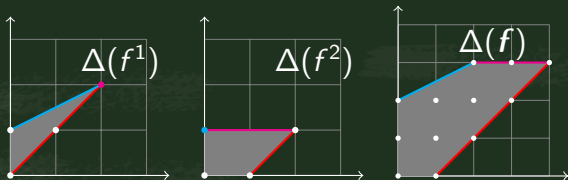
Under this condition, they show that

$$S_f \subset \bigcup_{\gamma \in \Delta_{\text{nc}}(f)} S_\gamma.$$

They actually treated the bifurcation set of a polynomial map $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, $m \geq n$.

Example (continued)

$$(f^1(x, y), f^2(x, y)) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$$



$$f_{\gamma(-1,1)} = (x^2y^2 + xy + 1, x(xy + 1)), \quad Z(f_{\gamma(1,-1)}^{\{2\}}) = \{xy + 1 = 0\}$$

$$f_{\gamma(0,-1)} = (x^2y^2, (x^2 + 1)y), \quad Z(f_{\gamma(0,-1)}^{\{1,2\}}) = \emptyset$$

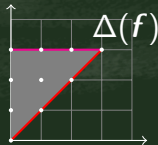
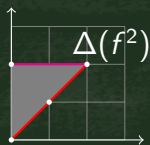
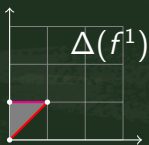
$$f_{\gamma(1,-2)} = (x^2y^2 + y, y), \quad Z(f_{\gamma(1,-2)}^{\{1,2\}}) = \emptyset$$

Therefore

$$\begin{aligned} S_f &= \{x^2y^2 + xy + 1 : xy + 1 = 0\} \times \mathbb{C} \\ &= \{t^2 + t + 1 : t + 1 = 0\} \times \mathbb{C} = \{1\} \times \mathbb{C} \end{aligned}$$

Another example

$$(f^1(x, y), f^2(x, y)) = (xy + y + 1, x^2y^2 + y^2 + xy + 1)$$



$$f_{\gamma(-1,1)} = (xy + 1, x^2y^2 + xy + 1), \quad Z(f_{\gamma(1,-1)}^{\emptyset}) = (\mathbb{C}^*)^2$$

$$f_{\gamma(0,-1)} = ((x+1)y, (x^2+1)y^2), \quad Z(f_{\gamma(0,-1)}^{\{1,2\}}) = \emptyset$$

Therefore

$$\begin{aligned} S_f &= \{(X, Y) : \exists(x, y) \text{ s.t. } (X, Y) = (xy + 1, x^2y^2 + xy + 1)\} \\ &= \{(X, Y) : \exists t \text{ s.t. } (X, Y) = (t + 1, t^2 + t + 1)\} \\ &= \{Y - X^2 + X - 1 = 0\} \end{aligned}$$

Proof of Main Theorem

It is enough to prove the following

Theorem 2.

If f is dominant, then

$$\bigcup_{\gamma \in \Delta_{\text{nc}}(f)} S'_\gamma \subset S_f \subset \bigcup_{\gamma \in \Delta_{\text{nc}}(f)} S_\gamma \quad (1)$$

where

$$S'_\gamma = f^{J_\gamma} \overline{(Z(f_\gamma^J) \setminus \Sigma(f_\gamma^J))} \times \mathbb{C}^J, \quad J = J_\gamma.$$

Here \overline{Z} denotes the closure of Z .

Proof of Main Theorem

For $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$, consider an arc

$$x(t) = (t^{p_1} v^1(t), \dots, t^{p_n} v^n(t)), \text{ where}$$

$$v(t) = (v^1(t), \dots, v^n(t)) = \sum_{i=0}^{\infty} v_i t^i,$$

$$v_i = (v_i^1, \dots, v_i^n), \quad v_0 \in (\mathbb{K}^*)^n.$$

We denote $\mathcal{A}(p)$ the set of such arcs.

$$\mathcal{A}_f(p) = \{x(t) \in \mathcal{A}(p) : \exists y \in S_f \ y = \lim_{t \rightarrow 0} f(x(t))\}$$

Proof of $S_f \subset \bigcup_{\gamma \in \Delta_{\text{nc}}(f)} S_\gamma$

Express $f^j(x(t))$ as

$$f^j(x(t)) = t^{-d_j}(\hat{f}_0^j + \hat{f}_1^j t + \cdots + \hat{f}_{d_j-1}^j t^{d_j-1} + \hat{f}_{d_j}^j t^{d_j} + o(t^{d_j}))$$

where $d_j = d_j(p)$. If $y \in S_f$, then $\exists p \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n$ and

$$\exists x(t) \in \mathcal{A}(p) \text{ such that } \lim_{t \rightarrow 0} f(x(t)) = y,$$

$$\hat{f}_0^j = \hat{f}_1^j = \cdots = \hat{f}_{d_j-1}^j = 0 \quad (j \in J = J_\gamma).$$

In particular, $\hat{f}_0^j = f_{\gamma_j(p)}^j(\mathbf{v}_0) = 0$ ($j \in J$).

As $t \rightarrow 0$, $f^j(x(t)) \rightarrow f_{d_j}^j = f_0^j = f_{\gamma_j(p)}^j(\mathbf{v}_0)$ ($j \notin J$). □

Proof of $\bigcup_{\gamma \in \Delta_{\text{nc}}(\mathbf{f})} S'_\gamma \subset S_f$

We have

$$\hat{f}_i^j = (df_{\gamma_i}^j)_{\mathbf{v}_0}(\mathbf{v}_i) + r_i^j(\mathbf{v}_0, \dots, \mathbf{v}_{i-1}) \quad (i = 1, 2, \dots)$$

where $r_i^j(\mathbf{v}_0, \dots, \mathbf{v}_{i-1})$ is a suitable polynomial of $\mathbf{v}_0, \dots, \mathbf{v}_{i-1}$. Let $(a_k^j)_{j \in J; k \geq 1}$ be any sequence. Assume that we already take $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}$ so that

$$\hat{f}_k^j = a_k^j \quad (1 \leq k < i, j \in J).$$

There exists \mathbf{v}_i so that $\hat{f}_i^j = a_i^j$ for $j \in J$, whenever $(df_{\gamma_i}^j)_{j \in J}$ is of full rank at \mathbf{v}_0 . Thus

$$\begin{aligned} S'_\gamma &= f_\gamma^{J^c} \overline{(Z(f_\gamma^J) \setminus \Sigma(f_\gamma^J))} \times \mathbb{C}^J \\ &\subset \overline{f_\gamma^{J^c} (Z(f_\gamma^J) \setminus \Sigma(f_\gamma^J))} \times \mathbb{C}^J \subset \overline{S_f} = S_f. \end{aligned}$$

□

Generalizations

- to a polynomial map $f : \mathbb{C}^m \rightarrow \mathbb{C}^n$ (our discussion works when f is **generically proper**, that is, there is a nowhere dense subset S of its image so that $f^{-1}(y)$ is compact if $y \notin S$.)
But, when $m > n$, $S_f = \overline{\text{Im } f}$, automatically.

- to relative version $f = (f', f'') : \mathbb{C}^m \rightarrow \mathbb{C}^{n-k} \times \mathbb{C}^k$

$$f : X \rightarrow \mathbb{C}^{n-k} \times \{y_0''\} \subset \mathbb{C}^n$$

where $X = (f'')^{-1}(y_0'')$.

- to degenerate case:

$$Z(f_\gamma^J) \setminus \overline{Z(f_\gamma^J) \setminus \Sigma(f_\gamma^J)} \neq \emptyset \text{ for some } \gamma \in \Delta_{\text{nc}(f)}$$

- to real case

Relative version

Set $f' = (f^1, \dots, f^{n-k})$ and $f'' = (f^{n-k+1}, \dots, f^n)$.

Consider $f'|_X : X \rightarrow \mathbb{C}^{n-k}$ where $X = (f'')^{-1}(y_0'')$.

Assume that the regular locus of X is dense in X . Identify $f'|_X$ with $f|_X$. For a face $\gamma = (\gamma_1, \dots, \gamma_n)$ of $\Delta(f)$, define $\gamma' = (\gamma_1, \dots, \gamma_{n-k})$, $\gamma'' = (\gamma_{n-k+1}, \dots, \gamma_n)$. Set $J = J_{\gamma'} = \{j \in \{1, \dots, n-k\} : 0 \notin \gamma_j\}$, and

$$S_{\gamma'; \gamma''} = f_{\gamma'}^{J^c}(Z(f_{\gamma'}^J, f_{\gamma''}'' - y_0'')) \times \mathbb{C}^J,$$

$$S'_{\gamma'; \gamma''} = \overline{f_{\gamma'}^{J^c}(Z(f_{\gamma'}^J, f_{\gamma''}'' - y_0'') \setminus \Sigma(f_{\gamma'}^J))} \times \mathbb{C}^J.$$

Relative version (Theorem)

Theorem 3

If X does not have a component in $\{x_1 \cdots x_n = 0\}$, then

$$\bigcup_{\gamma \in \Delta_{\text{nc}}(f)} S'_{\gamma'; \gamma''} \subset S_{f|_X} \subset \bigcup_{\gamma \in \Delta_{\text{nc}}(f)} S_{\gamma'; \gamma''}.$$

If $Z(f_{\gamma}^J, f_{\gamma''}'' - y_0'')$ has a dense nonsingular locus for all $\gamma \in \Delta_{\text{nc}}(f)$, we have equalities.

If X has a component X_1 in $\{x_1 \cdots x_n = 0\}$, then we could proceed a similar computation for $f|_{X_1}$ which is a polynomial map with less number of variables and obtain that $S_{f|_{X_1}} \subset S_{f|_X}$.

A trick for degenerate case

Let $h : \mathbb{K}^{m+k} \rightarrow \mathbb{K}^{n+k}$ be a polynomial map with

$$h^{n+i}(x) = \varphi_i(x_1, \dots, x_m) - x_{m+i} \text{ for } i = 1, \dots, k.$$

Let X be a subset defined by

$$x_{m+i} = \varphi_i(x_1, \dots, x_m), \quad i = 1, \dots, k.$$

The set X is isomorphic to \mathbb{K}^m by $x \mapsto (x, \varphi_i(x))$ where $x = (x_1, \dots, x_m)$. If $f(x) = h(x, \varphi_1(x), \dots, \varphi_k(x))$, then we have

$$S_f = S_{h|_X}$$

via the identification of \mathbb{K}^m with $\mathbb{K}^m \times \{0\}$.

Degenerate example

Let $h : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ be the map defined by

$$h(x, y, z) = (1 + x + z^2, 1 + x^2 + z^3, y^2 - x^3 - z).$$

Let X be the subset of \mathbb{K}^3 defined by $z = y^2 - x^3$.

Then the map $h|_X$ is isomorphic to the map

$f : \mathbb{K}^2 \rightarrow \mathbb{K}^2$ defined by

$$f(x, y) = (1 + x + (y^2 - x^3)^2, 1 + x^2 + (y^2 - x^3)^3).$$

Applying Theorem 3, we conclude that h is proper, and thus so is f .

Another idea for deg case

Assume that, for some $\gamma \in \Delta_{\text{nc}}(f)$,

$$V_\gamma := Z(f_\gamma^J) \setminus \overline{Z(f_\gamma^J) \setminus \Sigma(f_\gamma^J)} \neq \emptyset, \quad J = J_\gamma.$$

Remark that

$$S_f = \bigcup_{p \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n} S_f(p), \quad S_f(p) = \left\{ y \in \mathbb{C}^n : \begin{array}{l} \exists x(t) \in \mathcal{A}(p) \\ \lim_{t \rightarrow 0} f(x(t)) = y \end{array} \right\}.$$

Set

$$f^j(x) = f_0^j(x) + f_1^j(x) + \dots, \quad f_m^j(t^{p_1} x_1, \dots, t^{p_n} x_n) = t^{-d_j + m} f_m^j(x).$$

Remark that $f_0^j = f_{\gamma_j}^j(p)$ and thus $f_0 = f_{\gamma}(p)$.

Another idea for deg case (continued)

$$W(\mathbf{p}) = \left\{ \mathbf{y} \in \mathbb{C}^n : \begin{array}{l} \exists \mathbf{x} \in V_{\gamma(\mathbf{p})}, \quad \mathbf{y}^{J^c} = \mathbf{f}_{\gamma(\mathbf{p})}^{J^c}(\mathbf{x}) \\ \text{rank} \begin{bmatrix} \partial_{x_i} \mathbf{f}_0^{J'}(\mathbf{x}) \\ \partial_{x_i} \mathbf{f}_0^{J''}(\mathbf{x}) \end{bmatrix} = \text{rank} \begin{bmatrix} \partial_{x_i} \mathbf{f}_0^{J'}(\mathbf{x}) & \mathbf{f}_1^{J'}(\mathbf{x}) - \mathbf{y}^{J'} \\ \partial_{x_i} \mathbf{f}_0^{J''}(\mathbf{x}) & \mathbf{f}_1^{J''}(\mathbf{x}) \end{bmatrix} \end{array} \right\},$$

$$W'(\mathbf{p}) = \left\{ \mathbf{y} \in W(\mathbf{p}) : \begin{array}{l} \exists \mathbf{x} \in V_{\gamma(\mathbf{p})}, \quad \mathbf{y}^{J^c} = \mathbf{f}_{\gamma(\mathbf{p})}^{J^c}(\mathbf{x}) \\ \text{rank} (\partial_{x_i} \mathbf{f}_0^{J''}(\mathbf{x}))_{i=1, \dots, n} = \#J'' \end{array} \right\}.$$

where $J = J_{\gamma(\mathbf{p})} = \{j; d_j(\mathbf{p}) > 0\}$,

$$J' = \{j \in J : d_j(\mathbf{p}) = 1\}, \quad J'' = \{j \in J : d_j(\mathbf{p}) \geq 2\}.$$

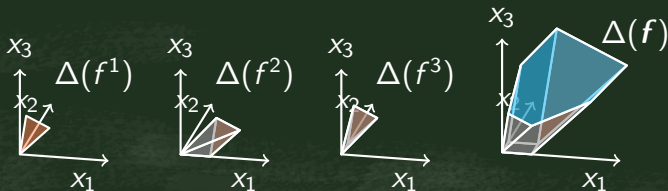
Theorem 4

$$V_{\gamma(\mathbf{p})} \neq \emptyset \implies W'(\mathbf{p}) \subset S_f(\mathbf{p}) \subset W(\mathbf{p})$$

A degenerate example

Consider the map $f : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ defined by

$$f(x_1, x_2, x_3) = \begin{pmatrix} 1 + x_1x_3 + x_2x_3 \\ 1 + x_1(1 - x_1x_3 + x_2x_3) \\ 1 + x_2(1 - x_1x_3 + x_2x_3) \end{pmatrix}.$$



We easily see that

$$\begin{aligned} S_{\gamma(1,1,-1)}(f) &= \{y_1 = y_3 = 1\}, & S_{\gamma(1,1,-2)}(f) &= \{y_2 + y_3 = 2\}, \\ S_{\gamma(-1,0,1)}(f) &= \{y_1 = 2, y_3 = 1\}, & S_{\gamma(0,-1,1)}(f) &= \{y_1 = 0, y_2 = 1\}. \end{aligned}$$

Example (continued)

$f_{\gamma(-1,-1,1)}$ is degenerate. $W(-1,-1,1)$ is given by

$$\left\{ \begin{array}{l} \exists x \ 1 - x_1 x_3 + x_2 x_3 = 0, y_1 = 1 + x_1 x_3 + x_2 x_3, \\ \text{rank} \begin{pmatrix} 1 - 2x_1 x_3 + x_2 x_3 & x_1 x_3 & x_1(x_2 - x_1) & 1 - y_2 \\ x_2 x_3 & 1 - x_1 x_3 + 2x_2 x_3 & x_2(x_2 - x_1) & 1 - y_3 \end{pmatrix} = 1 \end{array} \right\} \\ = \{y_1(y_2 - y_3) - 2y_2 + 2 = 0\}.$$

$$\Phi : \mathbb{K}^3 \times \mathbb{K}^3 \rightarrow \mathbb{K}^3, \quad (x, y) \mapsto z = \left(y_1, y_2, \frac{x_2}{x_1} y_2 + y_3 \right).$$

is well-defined along arc in $\mathcal{A}(-1,-1,1)$ when $t \rightarrow 0$.

$g(x) = \Phi(x, f(x))$ is non-degenerate and we manage to show that

$$W(-1,-1,1) = S_{\gamma(-1,-1,1)}.$$

We conclude that

$$S_f = S_{\gamma(1,1,-2)} \cup S_{\gamma(-1,-1,1)}.$$

Real version

Change \mathbb{C} By \mathbb{R} .
Everything works!

Causion: S_f is semi-algebraic, in general.

For $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $m > n$, S_f may not be $\overline{f(\mathbb{R}^m)}$.

Thank you very much
for your attention!

Dziękuję bardzo za
uwagę!