On the Łojasiewicz exponent at infinity of polynomial mappings

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Definitions

Definition 1. *Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping. The Łojasiewicz exponent of F at infinity is defined as the biggest exponent ν for which the following inequality*

$$|F(z)| \geqslant C\,|z|^{\nu} \tag{1.1}$$

holds for some constant C and sufficiently large $|z|$. Precisely

$$\mathcal{L}_\infty(F) := \sup\left\{ \nu \in \mathbb{R} : \underset{C>0}{\exists} \quad \underset{R>0}{\exists} \quad \underset{\substack{z\in\mathbb{C}^n \\ |z|\geq R}}{\forall} \quad |F(z)| \geqslant C\,|z|^{\nu} \right\}$$

We denote this exponent by $\mathcal{L}_\infty(F)$.

Definition 2. *Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping and $S \subset \mathbb{C}^n$ be an unbounded set. The Łojasiewicz exponent of F at infinity on S is defined as the best exponent ν for which the following inequality holds*

$$|F(z)| \geqslant C\,|z|^{\nu}$$

for some constant C and sufficiently large $|z|$ in S. Precisely

$$\mathcal{L}_\infty(F|S) := \sup\left\{ \nu \in \mathbb{R} : \underset{C>0}{\exists} \quad \underset{R>0}{\exists} \quad \underset{\substack{z\in S \\ |z|\geq R}}{\forall} \quad |F(z)| \geqslant C\,|z|^{\nu} \right\}$$

We denote this exponent by $\mathcal{L}_\infty(F|S)$.

Examples

Example 1. *Let*

$$F(x,y) = (x, xy-1) : \mathbb{C}^2 \to \mathbb{C}^2.$$

Then

$$\mathcal{L}_\infty(F) = -1,$$

If $S := \{y = 0\}$, then

$$\mathcal{L}_\infty(F|S) = 1.$$

Example 2. *Take $p, q \in \mathbb{N}$, $0 < q < p$ and define*

$$\begin{aligned} f(x,y) \quad &: \quad = y + y^{1+q}x^{p-q} : \mathbb{C}^2 \to \mathbb{C} \\ F(x,y) \quad &: \quad = \operatorname{grad} f = ((p-q)y^{1+q}x^{p-q-1}, 1 + (1+q)y^q x^{p-q}) : \mathbb{C}^2 \to \mathbb{C}^2. \end{aligned}$$

Then

$$\mathcal{L}_\infty(F) = -\frac{p}{q}.$$

Basic properties of the Łojasiewicz exponent at infinity

Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping.

1. $\mathcal{L}_\infty(F)$ is an invariant of linear automorphisms of C^n and C^m.

2. $\mathcal{L}_\infty(F)$ is not an invariant of polynomial automorphisms of C^n and C^m.

$$\mathcal{L}_\infty(F) = \begin{cases} 1 & \text{if} \quad F(x,y) = (x,y) \\ \frac{1}{2} & \text{if} \quad F(x,y) = (x+y^2, y) \end{cases}$$

$$\mathcal{L}_\infty(F) = \begin{cases} 1 & \text{if} \quad F(x,y) = (x, xy-1) \\ -\infty & \text{if} \quad F(x,y) = (x, xy) \end{cases}$$

3. $\mathcal{L}_\infty(F) > -\infty$ iff the zero-set $V(F)$ of F is isolated (i.e. $\#F^{-1}(0) < +\infty$).

4. $\mathcal{L}_\infty(F) \leqslant \max(\deg F_1, ..., \deg F_m)$ for any n, m.

5. $\mathcal{L}_\infty(F)$ is attained on a meromorphic curve i.e.

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|\Gamma),$$

where Γ is a meromorphic curve.

6. $\mathcal{L}_\infty(F)$ is a rational number.

7. $\mathcal{L}_\infty(F)$ is attained i.e.

$$|F(z)| \geqslant C\,|z|^{\mathcal{L}_\infty(F)} \qquad \text{for sufficiently large } z.$$

Applications of the Łojasiewicz exponent at infinity

I. Jacobian Conjecture

Proposition 1.1. *Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping. Then F is a proper mapping if and only if $\mathcal{L}_\infty(F) > 0$.*

Theorem 1.2. *Jacobian Conjecture holds for $F : \mathbb{C}^n \to \mathbb{C}^n$, $\operatorname{Jac} F = 1$, if and only if $\mathcal{L}_\infty(F) > 0$.*

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial. The gradient $\operatorname{grad} f := (\frac{\partial f}{\partial z_1}, ..., \frac{\partial f}{\partial z_n}) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial mapping.

Theorem 1.3 (Chądzyński, Krasiński 1992)**.** *A polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ is a coordinate of a polynomial automorphism of $\mathbb{C}^2$ if and only if $V(\operatorname{grad} f) = \emptyset$ and $\mathcal{L}_\infty(\operatorname{grad} f) > -1$.*

Theorem 1.4 (Płoski 1985)**.** *A polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$, is a polynomial automorphism of $\mathbb{C}^2$ if and only if and $\mathcal{L}_\infty(F) = 1/\deg F$.*

II. Bifurcation points of polynomials

Definition 3. . *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial. A point $\lambda \in \mathbb{C}$ is a critical value of f at infinity (or an atypical value of f) if there exists no neighbourhood U of λ such that*

$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$

is a trivial C^∞-bundle outside a compact set K. The set of critical values of f at infinity we will denote by $\Lambda(f)$.

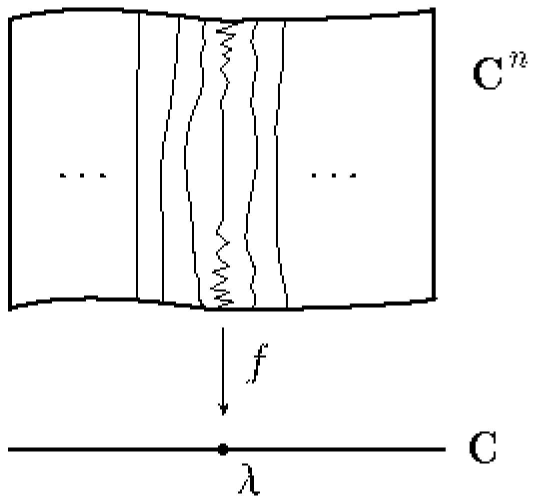

Fig. 1. λ is a critical value of f at infinity

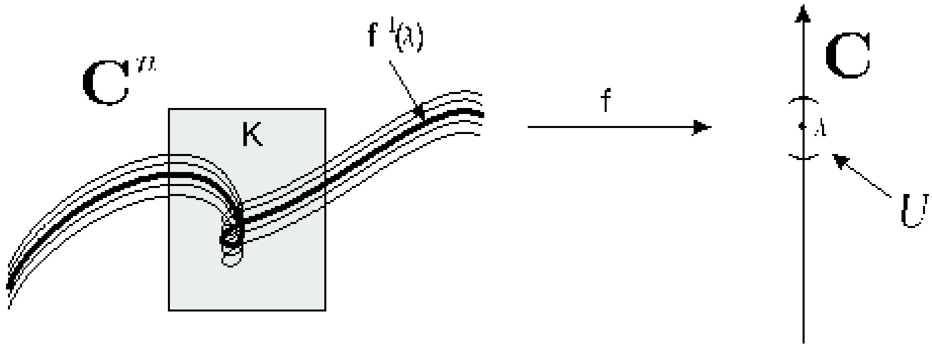

Fig. 2. λ is not a critical value of f at infinity

Example 3. *The point* 0 *is a critical value of* $f(x,y) := xy^2 - y$ *at infinity.*

Theorem 1.5 (Pham 1983)**.** $\#\Lambda(f) < +\infty$.

Definition 4. *Let* $\lambda \in \mathbb{C}$*. The Łojasiewicz exponent at infinity near a fibre* $f^{-1}(\lambda)$ *of* f *is defined by*

$$\mathcal{L}_{\infty,\lambda}(f) := \lim_{\delta\to 0^+} \mathcal{L}_\infty(\operatorname{grad} f|f^{-1}(K_\delta)) \tag{1.2}$$

where K_δ *is a disc of radius* δ *with the centre at* λ*. Equivalently*

$$\mathcal{L}_{\infty,\lambda}(f) := \inf_\Gamma \mathcal{L}_\infty(\operatorname{grad} f|\Gamma)$$

where Γ *is a meromorphic curve approximating* $f^{-1}(\lambda)$ *at* ∞*.*

A complete characterization of $\Lambda(f)$ for $n = 2$.

Theorem 1.6 (Ha Huy Vui 1990)**.** *Let* $f : \mathbb{C}^2 \to \mathbb{C}$ *be a non-constant polynomial. Then* $\lambda \in \Lambda(f)$ *if and only if* $\mathcal{L}_{\infty,\lambda}(f) < -1$ *(an equivalent condition* $\mathcal{L}_{\infty,\lambda}(f) < 0$*).*

A partial characterization of $\Lambda(f)$ for $n \geq 2$.

Theorem 1.7 (Parusiński 1995)**.** *Let* $f : \mathbb{C}^n \to \mathbb{C}$ *be a non-constant polynomial.*
1. If $\lambda \in \Lambda(f)$ *then* $\mathcal{L}_{\infty,\lambda}(f) \leq -1$*.*
2. If $\mathcal{L}_{\infty,\lambda}(f) \leq -1$ *and* f *has only isolated singularities at infinity, then* $\lambda \in \Lambda(f)$*.*

The inverse implication in 1 of the above theorem does not hold in general.

Example 4 (Paunescu-Zaharia 1999)**.** *Define*

$$f(x,y,z) := x - 3x^3y^2 + 2x^4y^3 + yz.$$

Then
1. $\Lambda(f) = \emptyset$ *(because* f *is a coordinate i.e. a component of a polynomial automorphism),*
2. $\mathcal{L}_{\infty,0}(f) = -1$*.*

Formulas and estimations of the Łojasiewicz exponent at infinity

Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping.

Almost all is known on the Łojasiewicz exponent at infinity in two-dimensional case $(n = m = 2)$

Two-dimensional case

Theorem 1.8. *Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping. Then*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|V(f) \cup V(g)).$$

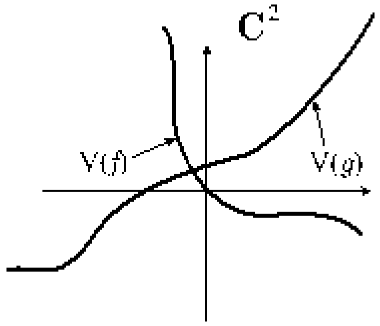

Fig. 3.

Remark 1 (Chądzyński, Krasiński 1997)**.** *The same theorem is true in general case $F : \mathbb{C}^n \to \mathbb{C}^m$.*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|V(F_1) \cup \ldots \cup V(F_n)).$$

An effective formula for $\mathcal{L}_\infty(F)$

Theorem 1.9 (Ha Huy Vui 1990, Chądzyński, Krasiński 1992)**.** *Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping $0 < \deg f = \deg_y f$, $0 < \deg g = \deg_y g$ and $\mathrm{Res}_y(f(x,y) - u, g(x,y) - v) = Q_0(u,v)x^N + \cdots + Q_N(u,v)$, $Q_0 \neq 0$, be the resultant of $f(x,y) - u$ and $g(x,y) - v$ with respect to y (u, v are new variables). Then*

$$\mathcal{L}_\infty(F) = \begin{cases} \left[\max\limits_{1 \leqslant i \leqslant N} \frac{\deg Q_i}{i}\right]^{-1} & \text{if } Q_0 = const. \\ 0 & \text{if } Q_0 \neq const.,\ Q_0(0,0) \neq 0 \\ -\left[\min\limits_{0 \leqslant i \leqslant r} \frac{\mathrm{ord}_{(0,0)} Q_i}{r+1-i}\right]^{-1} & \text{if } \begin{array}{l} Q_0(0,0) = ... = Q_r(0,0) = 0, \\ Q_{r+1}(0,0) \neq 0 \end{array} \\ -\infty & \text{if } Q_0(0,0) = ... = Q_N(0,0) = 0. \end{cases}$$

Other formulas for $\mathcal{L}_\infty(F)$:

1.**Ha Huy Vui, Cassou-Noguès 1995.** A formula for $\mathcal{L}_\infty(\operatorname{grad} f)$ in terms of the Eisenbud-Neumann diagrams of f at infinity.

2. **Gwoździewicz, Płoski 2005.** A formula for $\mathcal{L}_\infty(\operatorname{grad} f)$ in terms of the polar quotients at points of f at infinity.

3. **Lenarcik 1999.** A formula for $\mathcal{L}_\infty(\operatorname{grad} f)$ in terms of the Newton polygon of f at infinity (in non-degenerate case).

4. **Chądzyński, Krasiński. 2003.** A formula for $\mathcal{L}_{\infty,\lambda}(f)$ in terms of the resultant $\operatorname{Res}_y(f(x,y)-\lambda, f'_y(x,y)-u)$.

General case

In this case we have only estimations of $\mathcal{L}_\infty(F)$

Theorem 1.10. *Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping. Then*

$$\mathcal{L}_\infty(F) \geq \min_i(\deg F_i) + d(F) - \prod_{i=1}^m \deg(F_i),$$

where

$$d(F) = \textit{geometric degree of } F := \sum_{P \in F^{-1}(0)} \mu_P(F)$$

($\mu_P(F)$ - multiplicity of F at P).

It was proved by:
Chądzyński 1983 for $n = m = 2$.

Płoski 1985 for $n = m$ and F proper

Cygan, Tworzewski and Krasiński 1999 in general.

Other estimations:
Brownawell (1987), Kollar (1988, 1999), Płoski (1992), Płoski, Tworzewski (1996), Spodzieja (2002), Bivia-Ausina (2006).

Remarks

Remark 1. The above facts have local counterparts i.e. properties of the local Łojasiewicz exponent $\mathcal{L}_0(F)$ for holomorphic mappings. A particular case of it i.e. $\mathcal{L}_0(\operatorname{grad} f)$ is a very important invariant of a singularity f.

Remark 2. There are also results on the Łojasiewicz exponent at infinity in real case i.e. for polynomial mappings $F = (F_1, ..., F_m) : \mathbb{R}^n \to \mathbb{R}^m$

(**Gwoździewicz, Kollár , Bivia-Ausina**).

Problems

Problem 1. Find effective formulas for the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(F)$ for polynomial mappings F in n-dimensional case.

Problem 2. Find effective formulas for the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(\operatorname{grad} f)$ of the gradient of a polynomial f in the nondegenerate case in terms of its Newton polyhedron at infinity in n-dimensional case.

Problem 3. Find effective formulas for the Łojasiewicz exponent at infinity of the gradient $\mathcal{L}_\infty(\operatorname{grad} f)$ of a quasi homogeneous polynomial f.

Problem 4. Characterize critical values at infinity of a polynomial f in terms of the Łojasiewicz exponent in n-dimensional case.