April 29, 2022

A CRITERION FOR LIPSCHITZ NORMAL EMBEDDING AMONG DEFINABLE SETS.

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I) Lipschitz normal embeddings

(L. Birbrair – T. Mostowski, 2000)

connected

$X \subset \mathbb{R}^n$

$\longrightarrow d_{out}(x,y) = \|x-y\|$

$\searrow d_{in}(x,y) = \inf_{\gamma_{x,y}} \{ \text{length } \gamma_{x,y} \}$

$d_{in}(x,y) \geq d_{out}(x,y)$

LNE constant.

If $\exists\, C \geq 1$ s.t. $d_{in}(x,y) \leq C\, d_{out}(x,y)$

Then we call X is Lipschitz normally embedded (LNE)

$\Updownarrow$

$Id : (X, d_{in}) \longrightarrow (X, d_{out})$ bi-Lipschitz

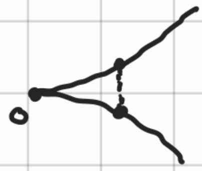
- Hardt's question.

is $d_{in} : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ definable ? (open question)

- (Kurdyka and orro) d' : (definable) $\sim d_{in}$.

(X, x_0) is LNE if $\exists\, U \ni x_0$ st $X \cap U$ is LNE.

Ex :



Problem Find a criterium for LNE.

- Birbair - Mendes : For any pair curve, the contact w.r.t dim, don't are the same.

- (Bolotto, Fantini, Pichon) Complex surface, to check LNE condition, ⇝ check for some pair of curves

- Mendes - Sampaio (2021) It is nough to check the LNE on the link.

$(X,0)$ closed, $\{X \setminus 0\}$ connected Then

$(X,0)$ is LNE $\iff$ $X \cap S_\varepsilon^{n-1}$ is LNE
(subanalytic)

with a LNE constant independent of ε.

(G. Valette) X is sub analytic then $X \cap S_\varepsilon^{n-1}$ is of the same Lipschitz type for all ε.

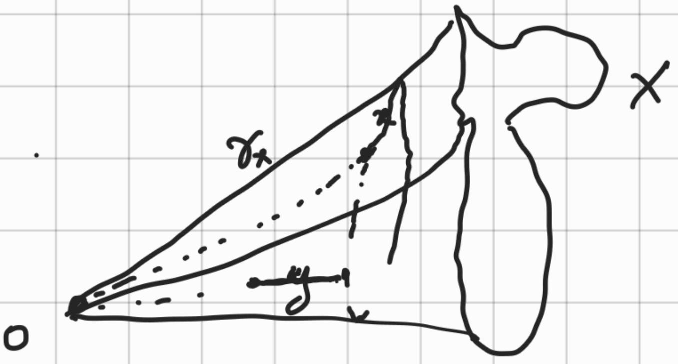
↓ not true in definable case (non poly)

Thm. 1) Mendes - Sampaio' result still holds for definable germs.

2) We may replace the usual link by the general link., ie the fiber of a

Lipschitz function $\rho : (X, 0) \to \mathbb{R}_{\geq 0}$

Property 1:

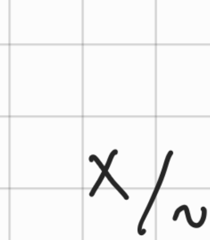


length $\gamma_x \sim \|x\|$

the "shortest" way to go from x to y is either to go through the origin, or follow the link.

Property 2. If $(X, 0)$ is LNE then (X, x) is LNE $\forall\, x$ near 0.

II) Moderately discontinuous homology (Bobadilla et al)



$y^2 = x^3$

$b > 0$

$p \underset{b}{\sim} q$ if $\operatorname{ord}|p - q| > b$.

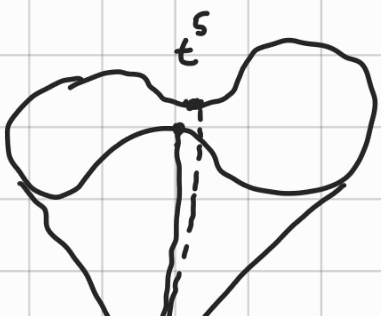
$b \geq \frac{3}{2}$, $p \not\sim q$

$b < \frac{3}{2}$, $p \sim_b q$

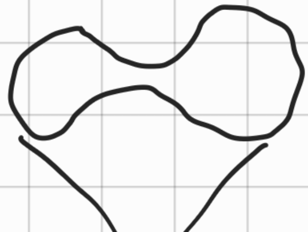
$X/\sim_b$?

$b \geq \frac{3}{2}$, $X/\sim_b$ 

$b < \frac{3}{2}$, $X/\sim_b$ 

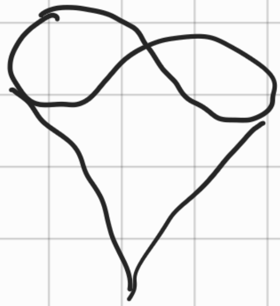


$\to$ $X/\sim_b$



$b \geq 5$

$X/\sim_\delta$



$\delta < \delta$

- $$\Delta_n \subset \mathbb{R}^{n+1} = \{ (p_0, \dots, p_n) \in \mathbb{R}^{n+1}, \ \sum p_i = 1 \}$$

$$\hat{\Delta}_n = \{ (tx, t) \in \Delta_n \times [0,1) \}$$

$$i_n^k : \Delta_{n-1} \longrightarrow \Delta_n$$
$$(p_0, \dots, p_{n-1}) \longmapsto (p_0, \dots, p_{k-1}, 0, p_k, \dots, p_{n-1})$$

$$\hat{i}_n^k : \hat{\Delta}_{n-1} \to \hat{\Delta}_n$$
$$(tx, t) \longmapsto (t\, i_n^k(x), t)$$

<u>Simplex</u>: $\sigma : \hat{\Delta}_{n,0} \longrightarrow (X, o)$ l.v.a

$$\| \sigma(x) \| \sim \| x \|$$

$$\partial \sigma = \sum_{k=0}^{n} (-1)^k \sigma \circ \hat{i}_n^k$$

$\downarrow$

$$MDC^{pre,\infty}(X, x_0, d_X, A) = \{ \sum a_i \sigma_i : a_i \in A. \}$$

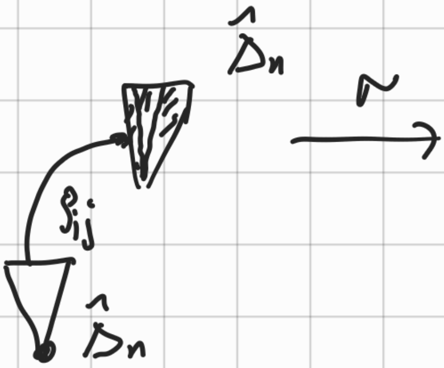
Immediate relation.

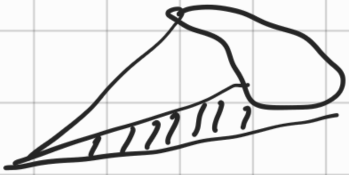
$$z_1 = \sum_{j \in J} a_j \sigma_j \qquad z_2 = \sum_{k \in K} b_k \sigma_k$$

$z_1 \xrightarrow[\infty]{} z_2$ if. $\exists$ subdivision $\{ \rho_{ij} : \hat{\Delta}_n \to \hat{\Delta}_n \}$

$$\sum_{i \in J} \sum_{j \in J_i} a_j \operatorname{sgn}(\rho_{ij}) \sigma_j \circ \rho_{ij}$$

$$= \sum_{\kappa \in K} b_\kappa \sigma_\kappa$$





$z_1 \sim_\infty z_2$ if $\exists\, z$

$$z \to_\infty \to_\infty \cdots z \leftarrow_\infty \cdot \leftarrow \cdots \leftarrow z_2$$

$$MDC^\infty(X, x_0) = MDC^{\infty, pre} \big/ \sim_\infty$$

$$\rightsquigarrow MDH^\infty(X, x_0)$$

$b > 0, \quad \sigma_1, \sigma_2 \in MDC^{\infty, pre}.$

$\sigma_1 \sim_b \sigma_2$ if $\forall \gamma : (0, \varepsilon) \to \hat{\Delta}_n, 0$

l.v.a

$$\lim_{t \to 0} \frac{d_X(\sigma_1 \circ \gamma(t), \sigma_2(\gamma(t)))}{t^b} \to 0$$

$\rightsquigarrow \quad \to_b \quad \rightsquigarrow \quad MDC^b(X, x_0)$

$$\rightsquigarrow MDH^b(X, x_0).$$

- $b = \infty, \quad \rightsquigarrow$ Singular homology

$b = 1 \quad \rightsquigarrow$ homology tangent cone.

$f : (X, x_0, d_X) \longrightarrow (Y, y_0, d_Y)$ l.v.a Lipschitz

$f_\# : MDC^b(X, x_0, d_X) \to MDC^b(Y, y_0, d_Y)$

$f_* : MDH^b(X, x_0, d_X) \to MDH^b(Y, y_0, d_Y)$

if f is subanalytic bi-lip homeomorphism

$\Rightarrow$ f_* is an isomorphism.

Let $(X, 0)$ be subanalytic germ.

$Id_0 : (X, 0, d_{inn}) \longrightarrow (X, 0, d_{out})$

Lipschitz, l.v.a.

If $(X, 0)$ is LNE $\Longrightarrow$ Id is bi-Lipschitz

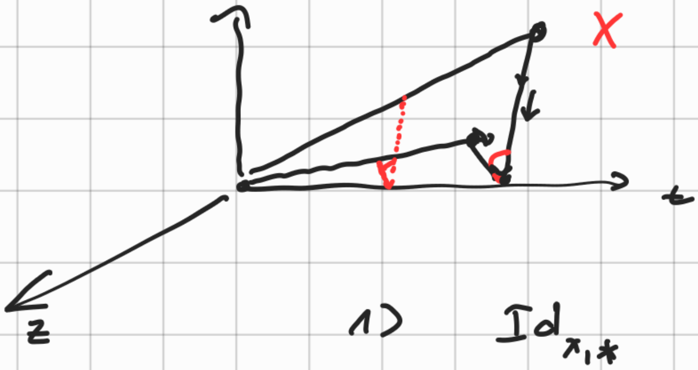
$\Downarrow$

Id_* is an iso.

Suppose $Id_{x,*}$ is isomorphism $\forall\, x \in (X, 0)$

is $(X, 0)$ LNE? No

<u>Example</u>: $X = \{ (x, z, t) \in \mathbb{R}^3,\ z^2 = t^2 x^2,\ 0 \le x \le t \}$



1) $Id_{x,*}$ is an isomorphism

2) $(X, 0)$ is not LNE.

$x \neq 0$, x is Lip regular point. $\to$ LNE $\Downarrow$ isom.

$x = 0$,

$H : X \times [0,1] \to X$. a deform retract
(x, s)

$H(-, s)$ is Lipschitz, with Lipschitz const independent of s.

Id_{0*} = isomorphism.

Open question: X is of isolated singularity?

conjecture. $\to$ real (NO)
$\to$ complex (YES)

$\boxtimes$