Orderings in real rational functions field

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Łódź, December 1, 2006

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Knowledge of orderings of a real field is very important in considerations related to the 17th Hilbert problem. In the paper we consider orderings of the field of real rational functions in several variables (also infinite number of variables). Fields of Nash functions are introduced, and it is shown that these fields are real-closures of the real rational functions field.

1 Introduction

It is known that the algebraic closure of the field $\mathbb{C}(\Lambda_1)$ of complex rational functions in one variable Λ_1 is embedded in the field of Puiseux power series (R. J. Walker, *Algebraic curves.* Dover, New York 1962). So, any algebraic element over $\mathbb{C}(\Lambda_1)$ can be considered as a convergent Puiseux power series

$$\sum \alpha_j \Lambda_1^{j/p}, \quad p \in \mathbb{Z}, \quad p > 0,$$

i.e. as a complex Nash function

$$g = \sum \alpha_j f^j \quad \text{in} \quad (\mathbb{C} \setminus \mathbb{R}_-) \cap U$$

for some branch $f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ of the power $\Lambda_1^{1/p}$,
where $\mathbb{R}_- = \{t \in \mathbb{R} : t \leq 0\}$ and $U \subset \mathbb{C}$ is a neighbourhood of the origin.

This concept has enabled a construction of the algebraic closure of the field $\mathbb{C}(\Lambda)$, $\Lambda = (\Lambda_1, \dots, \Lambda_m)$, in terms of Nash functions.

There exists a family $\mathcal{U}$: U_P, $P \in \mathbb{C}[\Lambda]$, subsets of $\mathbb{C}^m$ such that:

C_0. $U_P \subset \{\lambda \in \mathbb{C}^m : P(\lambda) \neq 0\}$ is a semi-algebraic set,
C_1. $U_P \cap U_Q = U_{PQ}$,
C_2. for $P \neq 0$, U_P is a dense subset of $\mathbb{C}^m$,
C_3. for $P \neq 0$, U_P is an open, connected and simply connected set

(Spodzieja 1996).

For $P \in \mathbb{C}[\Lambda]$, $P \neq 0$, the ring $\mathcal{N}^{\mathbb{C}}(U_P)$ of complex Nash functions on U_P is a domain.

In

$$\bigcup_{P \in \mathbb{C}[\Lambda] \setminus \{0\}} \mathcal{N}^{\mathbb{C}}(U_P)$$

we introduce an equivalence relation

$$"\sim": (f_1 : U_P \to \mathbb{C}) \sim (f_2 : U_Q \to \mathbb{C}) \quad \text{iff} \quad f_1|_{U_{PQ}} = f_2|_{U_{PQ}}.$$

Theorem. (Spodzieja 1996) *The set $\mathcal{N}^{\mathbb{C}}_{\mathcal{U}}$ of equivalence classes of "$\sim$" with the usual operations of addition and multiplication is a field.*

Theorem. (Spodzieja 1996) *$\mathcal{N}^{\mathbb{C}}_{\mathcal{U}}$ is the algebraic closure of $\mathbb{C}(\Lambda)$.*

The purpose of this article is to develop the sketched above idea in the real case.

2 Preliminaries

1. Semi-algebraic sets

Let $\mathbb{K}$ be a field, and let $m \in \mathbb{Z}$, $m > 0$.

By $\Lambda = (\Lambda_1, \dots, \Lambda_m)$ we denote a system of variables, by $\mathbb{K}[\Lambda]$ and $\mathbb{K}(\Lambda)$ – the ring of polynomials in Λ over $\mathbb{K}$ and its quotient field, respectively.

Let T be linearly ordered set (perhaps infinite).

By $\Lambda_T = (\Lambda_t : t \in T)$ we denote a system of variables Λ_t, $t \in T$, by $\mathbb{K}[\Lambda_T]$ and $\mathbb{K}(\Lambda_T)$ – the ring of polynomials in Λ_T over $\mathbb{K}$ and its quotient field, respectively.

Property. *For any $P \in \mathbb{K}(\Lambda_T)$ we have $P \in \mathbb{K}(\Lambda_{t_1}, \dots, \Lambda_{t_m})$ for some finite number of points $t_1, \dots, t_m \in T$.*

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition. By $\mathbb{K}^T$ we denote the set of all functions $T \to \mathbb{K}$ equipped with the unique topology for which projections $\mathbb{K}^T \ni x \mapsto x(t) \in \mathbb{K}$, $t \in T$, are continuous.

Definition. A subset of $\mathbb{K}^T$ is called *algebraic*, when it is defined by a finite system of equations $P = 0$, where P are polynomials in $\mathbb{K}[\Lambda_T]$.

Definition. A subset of $\mathbb{R}^T$ is called *semi-algebraic*, when it is defined by a finite alternative of finite systems of inequalities $P > 0$ or $P \geq 0$, where P are polynomials in $\mathbb{R}[\Lambda_T]$.

Property. *Any algebraic set in $\mathbb{K}^T$ is of the form*

$$\{x \in \mathbb{K}^T : (x(t_1), \ldots, x(t_m)) \in V\},$$

where $t_1, \ldots, t_m \in T$ and V is an algebraic subset of $\mathbb{K}^m$.

Property. *Any semi-algebraic set in $\mathbb{R}^T$ is of the form*

$$\{x \in \mathbb{R}^T : (x(t_1), \ldots, x(t_m)) \in X\},$$

where $t_1, \ldots, t_m \in T$ and X is a semi-algebraic subset of $\mathbb{R}^m$.

From the basic properties of algebraic and semi-algebraic sets in finite dimensional real vector space we obtain:

Property. *The family of algebraic sets in $\mathbb{K}^T$ is closed with respect to union and intersection of finite number of sets.*

Property. *The family of semi-algebraic sets in $\mathbb{R}^T$ is closed with respect to complement, union and intersection of finite number of sets.*

We have a version of the Tarski-Seidenberg Theorem.

Theorem. (Tarski-Seidenberg) *Let $t_1, \ldots, t_m \in T$ and*

$$\pi : \mathbb{R}^T \ni x \mapsto (x(t_1), \ldots, x(t_m)) \in \mathbb{R}^m.$$

If $X \subset \mathbb{R}^T$, $Y \subset \mathbb{R}^m$ are semi-algebraic sets, then $\pi(X)$ and $\pi^{-1}(Y)$ are semi-algebraic sets, too. Moreover, any semi-algebraic set $X \subset \mathbb{R}^T$ is of the form $\pi^{-1}(Y)$, where $Y \subset \mathbb{R}^m$, $m \in \mathbb{Z}$, $m > 0$, is a semi-algebraic set.

Since connected components of semi-algebraic subsets of finite dimensional real vector space are semi-algebraic then we have

Corollary. *Any connected component of a semi-algebraic subset of* $\mathbb{R}^T$ *is semi-algebraic.*

2. Analytic functions in $\mathbb{R}^T$

For any $t_1, \dots, t_m \in T$, we define a projection map

$$\pi^{\mathbb{K}}_{t_1,\dots,t_m} : \mathbb{K}^T \ni x \mapsto (x(t_1), \dots, x(t_m)) \in \mathbb{K}^m.$$

Definition. Let $U \subset \mathbb{R}^T$ be an open set and let $f : U \to \mathbb{R}$. The function f we call *analytic* if there exist $m \in \mathbb{Z}$, $m > 0$, $t_1, \dots, t_m \in T$ and an analytic function $g : \pi^{\mathbb{R}}_{t_1,\dots,t_m}(U) \to \mathbb{R}$ such that

$$f(x) = g(\pi^{\mathbb{R}}_{t_1,\dots,t_m}(x)) \quad \text{for} \quad x \in U. \tag{1}$$

Definition. Let $U \subset \mathbb{C}^T$ be an open set and let $f : U \to \mathbb{C}$. The function f we call *holomorphic* if there exist $m \in \mathbb{Z}$, $m > 0$, $t_1, \dots, t_m \in T$ and a holomorphic function $g : \pi^{\mathbb{R}}_{t_1,\dots,t_m}(U) \to \mathbb{C}$ such that

$$f(z) = g(\pi^{\mathbb{R}}_{t_1,\dots,t_m}(z)) \quad \text{for} \quad z \in U.$$

Remark. Let f be analytic function of form (1), and let $\{t_1, \dots, t_m\} \subset \{u_1, \dots, u_k\} \subset T$. Then $f(x) = h(\pi^{\mathbb{R}}_{u_1,\dots,u_k}(x))$, $x \in U$, for some analytic function $h : \pi^{\mathbb{R}}_{u_1,\dots,u_k}(U) \to \mathbb{R}$. Analogous holds for holomorphic functions.

Since analytic functions in $\mathbb{R}^T$ (resp. holomorphic functions in $\mathbb{C}^T$) are defined by analytic functions (resp. holomorphic functions) in finite number of variables, then we immediately obtain:

Proposition. *Let* $U \subset \mathbb{R}^T$ *be an open set and let* $f, g : U \to \mathbb{R}$ *be analytic functions. Then*
(a) $f + g$, $f - g$, $f \cdot g$ *and* f/g *(under suitable assumption on* g*) are analytic functions.*
(b) *If* $f|_W = g|_W$ *for some nonempty open set* $W \subset U$*, then* $f = g$*.*

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3. Nash functions

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$.

Definition. A function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^m$ is an open set, is called $\mathbb{K}$-*Nash function* if f is analytic and if there exists a nonzero polynomial $P \in \mathbb{K}[\Lambda, Z]$, such that $P(\lambda, f(\lambda)) = 0$ for $\lambda \in U$.

Definition. A function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}^m$ is an open set, is $\mathbb{C}$-*Nash function* if f is holomorphic and if there exists a nonzero polynomial $P \in \mathbb{R}[\Lambda, Z]$, such that $P(\lambda, f(\lambda)) = 0$ for $\lambda \in U$.

For the basic properties of Nash functions and semi-algebraic sets see for instance:

R. Benedetti, J-J. Risler, *Real algebraic and semi-algebraic sets*, Actualités Mathématiques, Hermann, Paris, 1990.

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Definition. A function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}^T$ is an open set, is called $\mathbb{C}$-*Nash function* if f is holomorphic and if there exists a nonzero polynomial $P \in \mathbb{C}[\Lambda_T, Z]$, such that $P(\lambda, f(\lambda)) = 0$ for $\lambda \in U$.

Definition. The *ring of* $\mathbb{K}$-*Nash functions* in U we denote by $\mathcal{N}^{\mathbb{K}}(U)$.

From the basic properties of Nash functions we immediately obtain:

Proposition. *If* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *and* $U \subset \mathbb{K}^T$ *is an open connected set then* $\mathcal{N}^{\mathbb{K}}(U)$ *is a domain.*

Proposition. *If* $U \subset \mathbb{R}^T$ *is an open connected set then* $\mathcal{N}^{\mathbb{Q}}(U)$ *is a domain.*

3 Orderings

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$ and let T be ordered set. Take an open connected semi-algebraic set $X \subset \mathbb{R}^T$.

We introduce orderings in the domain $\mathcal{N}^{\mathbb{K}}(X)$ in the following way.

Definition. A family Ω of nonempty semi-algebraic subsets of X we call *ordering family* if:
(i) For any algebraic set $V \subsetneq \mathbb{R}^T$ there exists $U \in \Omega$ such that $V \cap U = \emptyset$.
(ii) For any $U_1, U_2 \in \Omega$ there exists $U \in \Omega$ such that $U \subset U_1 \cap U_2$.
(iii) Any $U \in \Omega$ is an open, connected and simply connected set.

Definition. We call $f \in \mathcal{N}^{\mathbb{K}}(X)$ *positive*, and write $f \succ_\Omega 0$, if for some $U \in \Omega$ we have $f(x) > 0$ for $x \in U$. We write $f \succ_\Omega g$, if $f - g \succ_\Omega 0$.

Property. *The relation $\succ_\Omega$ determines ordering in $\mathcal{N}^{\mathbb{K}}(X)$, i.e. total ordering satisfying:*
(a) $f \succ_\Omega g \;\Rightarrow\; f + h \succ_\Omega g + h,$
(b) $f \succ_\Omega 0,\; g \succ_\Omega 0 \;\Rightarrow\; fg \succ_\Omega 0.$

The key point in considerations of orderings in $\mathcal{N}^{\mathbb{K}}(X)$ is the following

Theorem. *For any open connected semi-algebraic set $X \subset \mathbb{R}^T$ there exists an ordering family Ω subsets of X.*

At the end of this lecture we give some remarks on the proof of the above Theorem.

Remark. Let $X \subset \mathbb{R}^T$ be an open connected semi-algebraic set and Ω be an ordering family subsets of X. It is easy to see that there exists at most one point $x \in \mathbb{R}^T$ which is an accumulation point of any $U \in \Omega$.

Remark. One can prove that for any $x \in X$ there exists an ordering family Ω_x subsets of X such that x is an accumulation point of any $U \in \Omega_x$.

4 $\mathbb{K}$-Nash fields

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$, let $X \subset \mathbb{R}^T$ be open connected semi-algebraic set, and let Ω be an ordering family subsets of X.

Definition. In

$$\bigcup_{U \in \Omega} \mathcal{N}^{\mathbb{K}}(U)$$

we introduce a relation "$\sim_\Omega$" :

$$(f_1 : U_1 \to \mathbb{R}) \sim_\Omega (f_2 : U_2 \to \mathbb{R}) \quad \text{iff} \quad \exists_{U \in \Omega} (U \subset U_1 \cap U_2 \;\wedge\; f_1|_U = f_2|_U).$$

Property. *"$\sim_\Omega$" is an equivalence relation.*

Definition. The equivalence class of the relation "$\sim_\Omega$" determined by $f : U \to \mathbb{R}$ we denote by $[f]_\Omega$, and the set of all such classes – by $\mathcal{N}_\Omega^{\mathbb{K}}$.

Theorem. *$\mathcal{N}_\Omega^{\mathbb{K}}$ is a real field, together with the operations "$+$", "$\cdot$":*

$$[f_1]_\Omega + [f_2]_\Omega = \left[f_1|_U + f_2|_U\right]_\Omega, \qquad [f_1]_\Omega \cdot [f_2]_\Omega = \left[f_1|_U f_2|_U\right]_\Omega,$$

where $f_1 \in \mathcal{N}^{\mathbb{K}}(U_1)$, $f_2 \in \mathcal{N}^{\mathbb{K}}(U_2)$ and $U \in \Omega$, $U \subset U_1 \cap U_2$.

Definition. The field $\mathcal{N}_\Omega^{\mathbb{K}}$ we call $\mathbb{K}$-*Nash field*.

Let $X \subset \mathbb{R}^T$ be open connected semi-algebraic set, and let Ω be ordering family subsets of X.

Definition. By $\mathcal{F}^{\mathbb{K}}(X)$ we denote the *field of fractions* of $\mathcal{N}^{\mathbb{K}}(X)$.

Property. *For any $U \in \Omega$ the ring $\mathcal{N}^{\mathbb{K}}(X)$ is embedded in $\mathcal{N}^{\mathbb{K}}(U)$. The field $\mathcal{F}^{\mathbb{K}}(X)$ is embedded in $\mathcal{N}_\Omega^{\mathbb{K}}$.*

Main Theorem. *$\mathcal{N}_\Omega^{\mathbb{K}}$ is a real closure of $\mathbb{K}(\Lambda_T)$. In particular, $\mathcal{N}_\Omega^{\mathbb{K}}$ is a real closure of $\mathcal{F}^{\mathbb{K}}(X)$.*

Simply connectedness of sets of family Ω is essential in the proof.

Corollary. *The field $\mathcal{N}_\Omega^{\mathbb{K}}$ is ordered by:*

$$[f]_\Omega \succ 0 \text{ if for some } U \in \Omega,\ f \in \mathcal{N}^{\mathbb{K}}(U) \text{ and } f(x) > 0 \text{ for any } x \in U.$$

Corollary. *Let $x \in X$ and Ω_x be an ordering family of subsets of X such that x is an accumulation point of any $U \in \Omega_x$. Then for any $x, y \in X$, $x \neq y$, the fields $\mathcal{N}_{\Omega_x}^{\mathbb{K}}$, $\mathcal{N}_{\Omega_y}^{\mathbb{K}}$ determine different orders in $\mathbb{K}(\Lambda_T)$. In particular, the fields $\mathcal{N}_{\Omega_x}^{\mathbb{K}}$, $\mathcal{N}_{\Omega_y}^{\mathbb{K}}$ are not $\mathbb{K}(\Lambda_T)$-isomorphic.*

5 $\mathbb{C}$-Nash field

Let T be ordered set. Let Ω be an ordering family subsets of $\mathbb{C}^T$.

Analogously as in previous section we define an equivalence relation $\sim_\Omega$.

Definition. In

$$\bigcup_{U \in \Omega} \mathcal{N}^{\mathbb{C}}(U)$$

we introduce a relation "$\sim_\Omega$" :

$$(f_1 : U_1 \to \mathbb{C}) \sim_\Omega (f_2 : U_2 \to \mathbb{C}) \quad \text{iff} \quad \exists_{U\in\Omega} (U \subset U_1 \cap U_2 \;\wedge\; f_1|_U = f_2|_U).$$

The set of all classes of $\sim_\Omega$ together with the usual operations "$+$", "$\cdot$" we denote by $\mathcal{N}^{\mathbb{C}}_\Omega$.

Theorem. *The field $\mathcal{N}^{\mathbb{C}}_{\Omega_{\mathbb{C}}}$ is an algebraic closure of the field $\mathbb{C}(\Lambda_T)$.*

6 Remarks on the proof

Recall the key lemma of my lecture.

Theorem. *For any open connected semi-algebraic set $X \subset \mathbb{R}^T$ there exists an ordering family Ω subsets of X.*

The essential point in the proof is an analogous result in finite dimensional case. Then by Tarski Seidenberg Theorem we construct ordering families in $\mathbb{R}^T$.

Finite dimensional case

Let $m \in \mathbb{Z}$, $m > 0$ and let $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$ be a system of variables.

Take any $P \in \mathbb{R}[\Lambda]$. Let us defined a set $\Gamma_P \subset \mathbb{R}^m$ by

$$\Gamma_P = \{(\lambda_1, ..., \lambda_m) \in \mathbb{R}^m : P(\lambda_1, ..., \lambda_{m-1}, \lambda_m + \gamma) = 0 \text{ for some } \gamma \in [0, \infty)\}.$$

Property. $\Gamma_{PQ} = \Gamma_P \cup \Gamma_Q$.

We define a polynomial $\omega(P) \in \mathbb{R}[\Lambda_1, ..., \Lambda_{m-1}]$ ($\omega(P) \in \mathbb{R}$ if $m = 1$):

$$\omega(P) = 0 \text{ for } P = 0, \qquad \text{and} \qquad \omega(P) = P_0 \text{ for } P \neq 0,$$

where $P = P_0\Lambda_m^d + P_1\Lambda_m^{d-1} + ... + P_d$, and $P_i \in \mathbb{R}[\Lambda_1, ..., \Lambda_{m-1}]$ ($P_i \in \mathbb{R}$ if $m = 1$) for $i = 0, ..., d$, $P_0 \neq 0$.

Let us define sets $\Omega_P \subset \mathbb{R}^m$. The definition will be inductive with respect to the number of variables $\Lambda_1, ..., \Lambda_m$. For $P \in \mathbb{R}[\Lambda]$, we put

$$\Omega_P = \mathbb{R} \setminus \Gamma_P \subset \mathbb{R}, \quad \text{if} \quad m = 1,$$

$$\Omega_P = (\mathbb{R}^m \setminus \Gamma_P) \cap (\Omega_{\omega(P)} \times \mathbb{R}) \subset \mathbb{R}^m, \quad \text{if} \quad m > 1.$$

Property. *The sets* Ω_P, $P \in \mathbb{R}[\Lambda]$, *are semi-algebraic.*

The finite dimensional version of our theorem is the following.

Proposition. *The family* Ω_P, $P \in \mathbb{R}[\Lambda]$ *satisfies the following conditions:*

R_0. $\Omega_P \subset \{\lambda \in \mathbb{R}^m : P(\lambda) \neq 0\}$,

R_1. $\Omega_P \cap \Omega_Q = \Omega_{PQ}$,

R_2. *for* $P \neq 0$, Ω_P *is an unbounded subset of* $\mathbb{R}^m$,

R_3. *for* $P \neq 0$, Ω_P *is an open, connected and simply connected set.*

Moreover, one can demand that

R_4. $\Omega_P = \mathbb{R}^m$ *for* $P = \text{const}$, $P \neq 0$.

Corollary. $\{\Omega_P$*:* $P \in \mathbb{R}[\Lambda]$*,* $P \neq 0\}$*, is an ordering family of subsets of* $\mathbb{R}^m$*.*

Remark. By a Nash isomorphism we may remove the family Ω_P, $P \in \mathbb{R}[\Lambda]$, $P \neq 0$ into any open semi-algebraic set $X \subset \mathbb{R}^m$.

In the infinite dimensional case very useful is the following

Proposition. *Let* $1 \leq i_1 < \ldots < i_m \leq n$*, and let* $P \in \mathbb{R}[\Lambda_{i_1}, \ldots, \Lambda_{i_m}]$*. Let* $Q \in \mathbb{R}[\Lambda_1, \ldots, \Lambda_n]$ *be a polynomial of the form*

$$Q(x_1, \ldots, x_n) = P(x_{i_1}, \ldots, x_{i_m}), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then $\Omega_P \subset \mathbb{R}^m$, $\Omega_Q \subset \mathbb{R}^n$*, and*

$$\Omega_Q \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_{i_1}, \ldots, x_{i_m}) \in \Omega_P\}.$$

The End