The cone structure theorem

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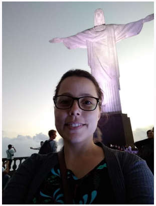
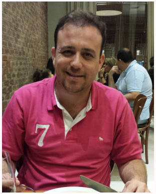
Based on a joint work with E. B. Batista (U. F. Cariri - Brazil) and J.C. F. Costa (UNESP - Brazil)





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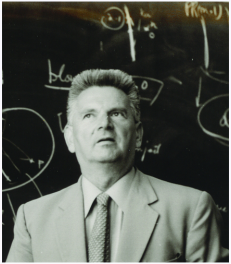
Then it seems natural to investigate the classification of mappings up to weaker equivalence relations. Here we consider the topological $\mathcal{A}$-equivalence (or C^0-$\mathcal{A}$-equivalence), where the change of coordinates are homeomorphisms instead of diffeomorphisms.



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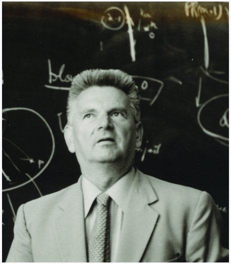

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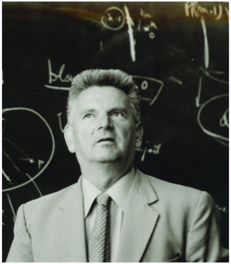

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It is not true again if we remove the finite determinacy assumption. In fact, Thom himself found a 1-parameter family $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that any two distinct members of the family are not topologically equivalent [Enseign. Math. 1962].

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The link is obtained by intersecting the image of f with a small enough sphere centered at the origin in $\mathbb{R}^p$.

The main result is that the link turns out to be a mapping between spheres $\gamma : S^{n-1} \to S^{p-1}$ which is topologically stable (in fact, stable if (n, p) are nice dimensions in Mather's sense).

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We remark that when $n \leq p$, finite determinacy implies that $f^{-1}(0) = \{0\}$.

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The case $f^{-1}(0) \neq \{0\}$ is more complicated: the link is a topologically stable map from a smooth $(n-1)$-manifold with boundary N into S^{p-1}. It is claimed (but without proving it) that the topological type of f can be also determined by its link.

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Recently, in [Geom. Dedicata 2013], J.C. Costa & JJNB proved a version of Fukuda's theorem with respect to the topological contact equivalence (or C^0-$\mathcal{K}$-equivalence), introducing the notions of link diagram and generalized cone.

In this work we show that if $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a finitely determined map germ with $f^{-1}(0) \neq \{0\}$, then f has a link diagram, which is well defined up to topological equivalence, and that f is topologically $\mathcal{A}$-equivalent to the generalized cone of its link diagram (Cone Structure Theorem).

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We define a continuous surjective function r in order to control the flow behavior when arriving to the zero set of f. This function r plays a crucial role to construct the notion of link diagram of f.

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The final step of the proof is to obtain the topological $\mathcal{A}$-equivalence between f and the generalized cone of its link diagram.

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As a consequence of Cone Structure Theorem, the topological type of the link diagram associated to a finitely determined map germ with non isolated zeros determines the topological type of such germ.

The Cone Structure Theorem is a very useful tool to investigate the topological $\mathcal{A}$-classification of finitely determined map germs.

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In other dimensions the problem is either trivial (e.g., for $\mathbb{R}^2 \to \mathbb{R}^n$, with $n \geq 5$) or is too complicated. Sometimes it is possible to handle by adding additional conditions.

Stability and finite determinacy

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Two smooth (i.e. C^∞) multi-germs $f, g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ are said to be $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi : (\mathbb{R}^n, S) \to (\mathbb{R}^n, S)$ and $\psi : (\mathbb{R}^p, y) \to (\mathbb{R}^p, y)$ such that $f = \psi \circ g \circ \phi^{-1}$.

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If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then f and g are said to be topologically $\mathcal{A}$-equivalent (or C^0-$\mathcal{A}$-*equivalent*).

Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ be smooth. A d-parameter unfolding is a smooth germ

$$F : (\mathbb{R}^n \times \mathbb{R}^d, S \times \{0\}) \to (\mathbb{R}^p \times \mathbb{R}^d, (y, 0))$$

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Two unfoldings F, G of f are called $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}^d, S \times \{0\}) \to (\mathbb{R}^n \times \mathbb{R}^d, S \times \{0\})$ and $\Psi : (\mathbb{R}^p \times \mathbb{R}^d, (y, 0)) \to (\mathbb{R}^p \times \mathbb{R}^d, (y, 0))$, which are unfoldings of the identity and such that $G = \Psi \circ F \circ \Phi^{-1}$.

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The Mather-Gaffney criterion of finite determinacy says that $\hat{f}$ is FD if and only if it has isolated instability. However, in the real case, this is only a necessary condition.

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Theorem (Mather-Gaffney criterion)

Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ be a FD map germ. There exists a representative $f : U \to V$, where U and V are open neighborhoods of the S in $\mathbb{R}^n$ and of 0 in $\mathbb{R}^p$, respectively, such that $f^{-1}(0) \cap \Sigma(f) = S$ and the restriction $f : U \smallsetminus f^{-1}(0) \to V \smallsetminus \{0\}$ is locally stable (i.e., any multi-germ $f : (\mathbb{R}^n, S') \to (\mathbb{R}^p, y)$ is stable, where $S' \subset U \smallsetminus f^{-1}(0)$ is finite).

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Hence, if we assume that f is polynomial, then the set of 0-stable singularities of each type is semialgebraic and by the Curve Selection Lemma, they cannot accumulate at the origin.

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Hence, if we assume that f is polynomial, then the set of 0-stable singularities of each type is semialgebraic and by the Curve Selection Lemma, they cannot accumulate at the origin.

Thus, by shrinking again the neighborhoods U and V if necessary, we can assume that f has no 0-stable singularities in $V \setminus \{0\}$.

Definition

A smooth map germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ has isolated instability (abbreviated II) if there exists a representative $f : U \to V$, where U and V are open neighborhoods of the S in $\mathbb{R}^n$ and of 0 in $\mathbb{R}^p$ respectively, such that:

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In such case we say that $f : U \to V$ is a good representative of f.

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A smooth map germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ has isolated instability (abbreviated II) if there exists a representative $f : U \to V$, where U and V are open neighborhoods of the S in $\mathbb{R}^n$ and of 0 in $\mathbb{R}^p$ respectively, such that:

1. $f^{-1}(0) \cap \Sigma(f) = S$,
2. the restriction $f : \Sigma(f) \to V$ is finite,
3. the restriction $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ is locally stable with no 0-stable singularities.

In such case we say that $f : U \to V$ is a good representative of f.

Moreover, if f is a polynomial mapping, we also add the condition that U and V are semialgebraic sets.

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Definition

We say that a smooth map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has discrete stable type (abbreviated DST) if any unfolding F of f presents only a finite number of stable types.

Some cases in which a map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has DST are:

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By definition, this is the set of points $y' \in \Delta(f)$ such that the multi-germ $f : (\mathbb{R}^n, S') \to (\mathbb{R}^p, y')$ is $\mathcal{A}$-equivalent to the multi-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$, where $S' = f^{-1}(y') \cap \Sigma(f)$ and $S = f^{-1}(y) \cap \Sigma(f)$.

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It is well known that $\mathrm{Iso}(f, y)$ is a submanifold of V (when f is stable).

Definition

Let $f : U \to V$ be a good representative of a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with II and DST. We construct a stratification $(\mathcal{A}, \mathcal{B})$ of f defined as follows:

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We call $(\mathcal{A}, \mathcal{B})$ the stratification by stable types.

The fact that f has DST guarantees that the stratification is finite.

If moreover, we add the hypothesis that f is polynomial, then the strata are semialgebraic sets.

Lemma

Let $f : U \to V$ be a good representative of a germ with II and DST. Then the stratification by stable types is a Thom stratification of f.

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We have to show that $\mathcal{A}, \mathcal{B}$ satisfy the Whitney conditions and the Thom A_f condition. This is well known for $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ because of the stability of f.

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Moreover, f is a submersion in a neighborhood of each point of $f^{-1}(0) \setminus \{0\}$, so that we also have the Whitney conditions and the Thom A_f condition outside the origin.

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In the case that f has no DST, we still have a Thom stratification of $f : U \to V$ which is called the canonical Thom stratification of f. It exists provided that f is good representative of a polynomial map germ with II.

The link of a FD map germ

Denote by $J^r(n,p)$ the r-jet space from $(\mathbb{R}^n,0)$ to $(\mathbb{R}^p,0)$. For positive integers r and s with $s \geq r$, let $\pi^s_r : J^s(n,p) \to J^r(n,p)$ be the canonical projection defined by $\pi^s_r(j^s f(0)) = j^r f(0)$. For a positive number $\epsilon > 0$ we set:

$$D^n_\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 \leq \epsilon\},$$
$$B^n_\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 < \epsilon\},$$
$$S^{n-1}_\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 = \epsilon\}.$$

T. Fukuda in [Inv. Math. 1981] and [Tokyo J. Math. 1985] has proved the following cone structure theorem:

Theorem (Fukuda)

For any semialgebraic subset W of $J^r(n,p)$, there exists an integer s $(s \geq r)$ depending only on n, p and r, and there exists a closed semialgebraic subset Σ_W of $(\pi^s_r)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s f(0)$ belonging to $(\pi^s_r)^{-1}(W) \smallsetminus \Sigma_W$ we have the following properties:

(A) *Case $f^{-1}(0) = \{0\}$. There exists $\epsilon_0 > 0$ such that for any number ϵ with $0 < \epsilon \leq \epsilon_0$, we have:*

- (A-i) *$f^{-1}(S^{p-1}_\epsilon)$ is diffeomorphic to the standard unit sphere S^{n-1}.*
- (A-ii) *The restricted mapping $f|_{f^{-1}(S^{p-1}_\epsilon)} : f^{-1}(S^{p-1}_\epsilon) \to S^{p-1}_\epsilon$ is topologically stable (stable if (n,p) is a nice pair) and its topological class is independent of ϵ.*
- (A-iii) *The restricted mapping $f|_{f^{-1}(D^{p-1}_\epsilon)} : f^{-1}(D^{p-1}_\epsilon) \to D^p_\epsilon$ is topologically $\mathcal{A}$-equivalent to the cone of $f|_{f^{-1}(S^{p-1}_\epsilon)}$.*

Theorem (Fukuda Continued)

(B) *Case* $f^{-1}(0) \neq \{0\}$. *There exist* $\epsilon_0 > 0$ *and a strictly increasing smooth function* $\delta : [0, \epsilon_0] \to [0, \infty)$ *with* $\delta(0) = 0$ *such that for any* ϵ, δ *with* $0 < \epsilon \leq \epsilon_0$ *and* $0 < \delta < \delta(\epsilon)$, *we have:*

- (B-i) $f^{-1}(0) \cap S^{n-1}_\epsilon$ *is an* $(n-p-1)$*-dimensional manifold and it is diffeomorphic to* $f^{-1}(0) \cap S^{n-1}_{\epsilon_0}$.
- (B-ii) $D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ *is a smooth manifold with boundary and it is diffeomorphic to* $D^n_{\epsilon_0} \cap f^{-1}(S^{p-1}_{\delta(\epsilon_0)})$.
- (B-iii) *the restriction* $f|_{D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)} : D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta) \to S^{p-1}_\delta$ *is a topologically stable map (stable if* (n, p) *is a nice pair) and its topological class is independent of* ϵ *and* δ.

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Note that when $n \leq p$ we always have $f^{-1}(0) = \{0\}$ by the finite determinacy condition, but when $n > p$ we may have the two possibilities.

The condition that (n, p) is a nice pair in (A-ii) or (B-iii) is not necessary if the map germ f has DST. In fact, the proof of the theorem is based on the stratification by stable types when it is defined or the canonical Thom stratification otherwise.

Definition

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a FD map germ. We take $f : U \to V$ a good representative and ϵ, δ as in Fukuda's Theorem. The link of f is defined as the map:

$$f|_{f^{-1}(S_\epsilon^{p-1})} : f^{-1}(S_\epsilon^{p-1}) \to S_\epsilon^{p-1},$$

when $f^{-1}(0) = \{0\}$, or the map:

$$f|_{D_\epsilon^n \cap f^{-1}(S_\delta^{p-1})} : D_\epsilon^n \cap f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1},$$

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Corollary

Two FD map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $f^{-1}(0) = \{0\} = g^{-1}(0)$ are topologically $\mathcal{A}$-equivalent if their associated links are topologically $\mathcal{A}$-equivalent.

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In fact, the cone of $D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ is not homeomorphic to the closed disk D^n, hence the restriction $f|_{D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)}$ cannot be topologically $\mathcal{A}$-equivalent to the cone of the link.

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To obtain a similar result, it is necessary to use the notion of generalized cone.

The cone structure theorem for $f^{-1}(0) \neq \{0\}$

In [J.C.F. Costa, JJNB Geom.Dedicata 2013], we introduced a generalized notion of cone and we also proved a version of the Fukuda's theorem for topological $\mathcal{K}$-equivalence.

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Definition

A link diagram is a diagram as follows

$$V \xleftarrow{\;r\;} N \xrightarrow{\;\gamma\;} S^{p-1},$$

where N is a manifold with boundary, γ is a continuous mapping, V is a contractible space and r is a continuous and surjective mapping such that the attaching space $(N \times I) \cup_r V$ is homeomorphic to the closed disk D^n (here we set $I = [0,1]$ and we identify $N \cong N \times \{0\} \subset N \times I$, by setting $r(x,0) := r(x)$).

Definition

Given a link diagram

$$V \xleftarrow{\;r\;} N \xrightarrow{\;\gamma\;} S^{p-1},$$

we define the generalized cone as the induced mapping

$$C(\gamma, r) : (N \times I) \cup_r V \to c(S^{p-1})$$

defined as $[x, t] \mapsto [\gamma(x), t]$ if $(x, t) \in N \times I$, where $c(S^{p-1})$ is the usual cone. Since r is surjective, notice that for any $y \in V$, there is $x \in N$ such that $r(x, 0) = y$ and thus $[y] = [r(x, 0)] = [(x, 0)]$.

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Note that in the particular case that $V = \{0\}$, the generalized cone coincides with the usual notion of cone.

Definition

Two link diagrams

$$V_0 \xleftarrow{r_0} N_0 \xrightarrow{\gamma_0} S^{p-1}, \quad V_1 \xleftarrow{r_1} N_1 \xrightarrow{\gamma_1} S^{p-1}$$

are topologically equivalent if there exist homeomorphisms $\alpha : V_0 \to V_1$, $\phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that

$$\begin{array}{ccccc} V_0 & \xleftarrow{r_0} & N_0 & \xrightarrow{\gamma_0} & S^{p-1} \\ \alpha \downarrow & & \phi \downarrow & & \psi \downarrow \\ V_1 & \xleftarrow{r_1} & N_1 & \xrightarrow{\gamma_1} & S^{p-1} \end{array}$$

In other words, $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and $\gamma_1 = \psi \circ \gamma_0 \circ \phi^{-1}$.

We present now the structure cone theorem for map germs with non isolated zeros. Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a smooth map germ, in order to simplify te notation, we put $f_{\epsilon,\delta} := f|_{N_{\epsilon,\delta}} : N_{\epsilon,\delta} \to S_\delta^{p-1}$, where $N_{\epsilon,\delta} = D_\epsilon^n \cap f^{-1}(S_\delta^{p-1})$ and $V_\epsilon = f^{-1}(0) \cap D_\epsilon^n$:

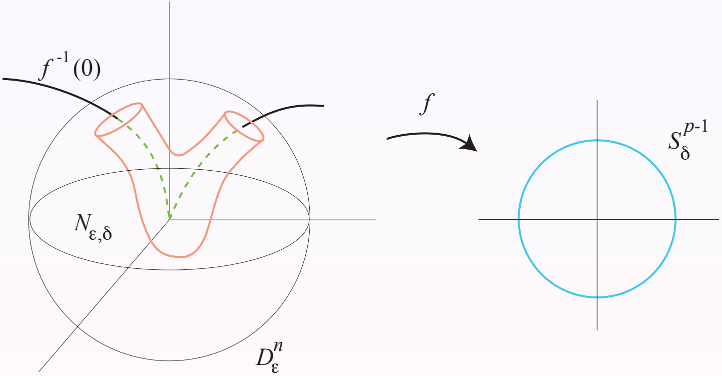

Figure: The Milnor tube and $f|_{N_{\epsilon,\delta}}$

Theorem (Cone Structure Theorem)

Let $f : U \to V$ be a good representative of a polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with II and $f^{-1}(0) \neq \{0\}$. For each $0 < \delta \ll \epsilon \ll 1$ small enough, there exists a continuous and surjective mapping $r_{\epsilon,\delta} : N_{\epsilon,\delta} \to V_\epsilon$, such that:

1. *The link diagram*

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S_\delta^{p-1}$$

   is independent of ϵ, δ up to topological equivalence.

2. *The restriction $f|_{D_\epsilon^n \cap f^{-1}(D_\delta^p)} : D_\epsilon^n \cap f^{-1}(D_\delta^p) \to D_\delta^p$ is topologically $\mathcal{A}$-equivalent to the generalized cone:*

$$C(f_{\epsilon,\delta}, r_{\epsilon,\delta}) : (N_{\epsilon,\delta} \times I) \cup_{r_{\epsilon,\delta}} V_\epsilon \to c(S_\delta^{p-1}),$$

   where $I = [0, \delta]$.

Proof: Let $(\mathcal{A}, \mathcal{B})$ be either the stratification by stable types if f has DST or the canonical Thom stratification otherwise. We choose $0 < \delta_0 \ll \epsilon_0 \ll 1$ small enough such that the following conditions hold:

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1. $f^{-1}(0) \pitchfork S^{n-1}_{\epsilon}$, for all ϵ with $0 < \epsilon \leq \epsilon_0$;
2. $\mathcal{A} \pitchfork S^{n-1}_{\epsilon_0}$ on $S^{n-1}_{\epsilon_0} \cap f^{-1}(D^p_{\delta_0})$.

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3. $\mathcal{B} \pitchfork S^{p-1}_{\delta}$, for all δ with $0 < \delta \leq \delta_0$;

Let $B^n_{\epsilon_0}$ and $B^p_{\delta_0}$ be the interiors of $D^n_{\epsilon_0}$ and $D^p_{\delta_0}$, respectively. We consider the restriction $f : D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0}) \to B^p_{\delta_0}$ and observe the following facts:

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- The restriction of f to the boundary $f : S^{n-1}_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0}) \to B^p_{\delta_0}$ is a submersion.
- The restriction of the stratification $(\mathcal{A}, \mathcal{B})$ to $(D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0}), B^p_{\delta_0})$ provides a Thom stratification of f, taking into account that we must consider, on one hand, the strata of $\mathcal{A}$ in the interior $B^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$ and on the other hand, the strata of $\mathcal{A}$ in the boundary $S^{n-1}_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$.

On the other hand, if in the interval $[0, \delta_0)$ we consider the stratification $\mathcal{C} = \{(0, \delta_0), \{0\}\}$, then the pair $(\mathcal{B}, \mathcal{C})$ is a Thom stratification of the function $\rho : B^p_{\delta_0} \to [0, \delta_0)$, given by $\rho(y) = \|y\|^2$, which is also proper.

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Then, there exists a stratified vector field Y on $B^p_{\delta_0}$ which is a lifting of T through ρ and there exists a stratified vector field X on $D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$ which is a lifting of Y through f.

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Moreover, since T is globally integrable, then Y, X are also globally integrable.

Let $0 < \delta_1 < \delta_0$. We define the mappings Φ, Ψ:

$$\begin{array}{ccc} D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) \setminus V_{\epsilon_0} & \xrightarrow{f} & D^p_{\delta_1} \setminus \{0\} \\ \Phi \big\downarrow & & \Psi \big\downarrow \\ N_{\epsilon_0,\delta_1} \times (0,\delta_1] & \xrightarrow{f_{\epsilon_0,\delta} \times Id} & S^{p-1}_{\delta_1} \times (0,\delta_1] \end{array}$$

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- $\psi(y)$ is the point of $S^{p-1}_{\delta_1}$ where the integral curve of Y passing through y meets $S^{p-1}_{\delta_1}$.

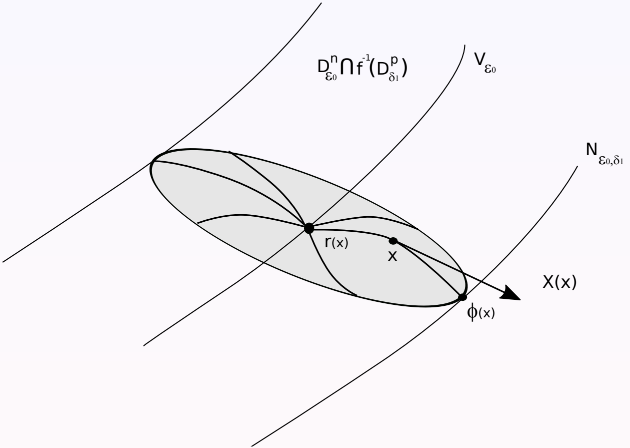

Figure: The maps r and ϕ and the vector field X.

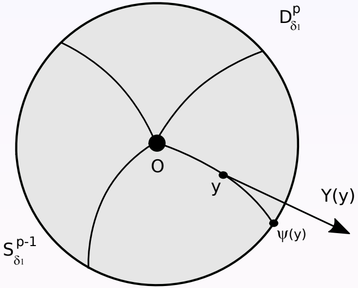

Figure: The map ψ and the vector field Y.

Note that Φ and Ψ are homeomorphisms. In fact, $\Phi^{-1}(x,t)$ is the point where the integral curve of X passing through x meets $N_{\epsilon_0,t}$ and $\Psi^{-1}(y,t)$ is the point where the integral curve of Y passing through y meets S_t^{p-1}.

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Another point is that the above diagram is commutative. We have that $\Psi(f(x)) = (\psi(f(x)), \|f(x)\|^2)$ and $f(\Phi(x)) = (f(\phi(x)), \|f(x)\|^2)$. But since that X is a lifting of Y through f, we have $df \circ X = Y \circ f$ and this implies that f maps integral curves of X into integral curves of Y, from which we deduce $\psi(f(x)) = f(\phi(x))$.

On the other hand, we also define a retraction

$$r : D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) \to V_{\epsilon_0}$$

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We have that r is continuous, surjective and moreover, $r(x) = x$, for all $x \in V_{\epsilon_0}$. We also have

$$\lim_{t \to 0} \Phi^{-1}(x, t) = r(x), \quad \lim_{t \to 0} \Psi^{-1}(y, t) = 0.$$

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This allows us to extend the homeomorphisms Φ and Ψ to homeomorphisms $\overline{\Phi}$ and $\overline{\Psi}$ which make commutative the following diagram:

$$\begin{array}{ccc} D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) & \xrightarrow{f} & D^p_{\delta_1} \\ \overline{\Phi}\downarrow & & \overline{\Psi}\downarrow \\ (N_{\epsilon_0,\delta_1} \times [0,\delta_1]) \cup_r V_{\epsilon_0} & \xrightarrow{C(f_{\epsilon_0,\delta_1}, r)} & C(S^{p-1}_{\delta_1}). \end{array}$$

With this we finish the proof of (2). Let us see now (1). Given $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta(\epsilon)$, by Fukuda's Theorem, there exist homeomorphisms α and β which make commutative the following diagram

$$\begin{array}{ccc} N_{\epsilon_0,\delta_1} & \xrightarrow{f_{\epsilon_0,\delta_1}} & S^{p-1}_{\delta_1} \\ \alpha \downarrow & & \beta \downarrow \\ N_{\epsilon,\delta} & \xrightarrow{f_{\epsilon,\delta}} & S^{p-1}_{\delta}. \end{array}$$

On the other hand, again by Fukuda's Theorem we know there exists a homeomorphism $\sigma : V_{\epsilon_0} \to V_\epsilon$. Then, it is enough to define $r_{\epsilon,\delta} : N_{\epsilon,\delta} \to V_\epsilon$ as $r_{\epsilon,\delta} = \sigma \circ r_{\epsilon_0,\delta_1} \circ \alpha^{-1}$, in such a way that we have a topological equivalence of link diagrams:

$$\begin{array}{ccccc} V_{\epsilon_0} & \xleftarrow{r_{\epsilon_0,\delta_1}} & N_{\epsilon_0,\delta_1} & \xrightarrow{f_{\epsilon_0,\delta_1}} & S^{p-1}_{\delta_1} \\ \sigma\downarrow & & \alpha\downarrow & & \beta\downarrow \\ V_\epsilon & \xleftarrow{r_{\epsilon,\delta}} & N_{\epsilon,\delta} & \xrightarrow{f_{\epsilon,\delta}} & S^{p-1}_{\delta}. \end{array}$$

Definition

Let $f : U \to V$ a good representative of a polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with II and $f^{-1}(0) \neq \{0\}$. The link diagram of f is the link diagram

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S_\delta^{p-1}$$

given in the Cone Structure Theorem for $0 < \delta \ll \epsilon \ll 1$.

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Then, we define the link diagram of f by taking a good representative. It follows that the link diagram is well defined up to topological equivalence and that f is topologically $\mathcal{A}$-equivalent to the generalized cone of its link diagram.

Proposition

If two link diagrams are topologically equivalent, then their generalized cones are topologically $\mathcal{A}$-equivalent.

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Proof.

Suppose that the two link diagrams

$$V_0 \xleftarrow{r_0} N_0 \xrightarrow{\gamma} S^{p-1}, \quad V_1 \xleftarrow{r_1} N_1 \xrightarrow{\gamma} S^{p-1}$$

are topologically equivalent. Then, there are homeomorphisms $\alpha : V_0 \to V_1$, $\phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and $\gamma_1 = \psi \circ \gamma_0 \circ \phi^{-1}$. □

Continued.

Then we have an induced topological equivalence between the generalized cones $C(\gamma_0, r_0)$ and $C(\gamma_1, r_1)$:

$$\begin{array}{ccc} (N_0 \times I) \cup_{r_0} V_0 & \xrightarrow{C(\gamma_0, r_0)} & c(S^{p-1}) \\ \widetilde{\Phi} \downarrow & & \downarrow c(\psi) \\ (N_1 \times I) \cup_{r_1} V_1 & \xrightarrow{C(\gamma_1, r_1)} & c(S^{p-1}) \end{array}$$

where $\widetilde{\Phi}$ is the homeomorphism induced by ϕ and α in the following way: $\widetilde{\Phi}([x,t]) = [\phi(x), t]$ if $x \in N_0$ and $\widetilde{\Phi}([y]) = [\alpha(y)]$ if $y \in V_0$.

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Corollary

Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two FD map germs with non isolated zeros. If their link diagrams are topologically equivalent, then f and g are topologically $\mathcal{A}$-equivalent.

Example: Consider a FD function germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ with $f^{-1}(0) \neq \{0\}$. The finitely determinacy condition implies that f has isolated critical point at the origin.

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$V_\epsilon = f^{-1}(0) \cap D^2_\epsilon$ is made of a finite even number $2r$ of half-branches which intersect transversally the boundary S^1_ϵ and separate the disk D^2_ϵ into $2r$ sectors, so that the sign of f alternates on consecutive sectors:

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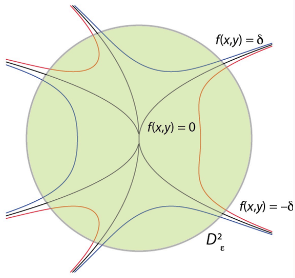

In fact, let σ be a half-brach of $f^{-1}(0) \neq \{0\}$. Since f is FD, $(\sigma - \{0\}) \cap \Sigma(f) = \emptyset$. Then f changes its sign when crossing σ, because otherwise, f would have a critical point in σ, which is a contradiction.

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Thus, the topological $\mathcal{A}$-class of f only depends on the number of half-branches of $f^{-1}(0)$. We deduce that two functions f and g are topologically $\mathcal{A}$-equivalent if and only if the curves $f^{-1}(0)$ and $g^{-1}(0)$ have the same number of half-branches.

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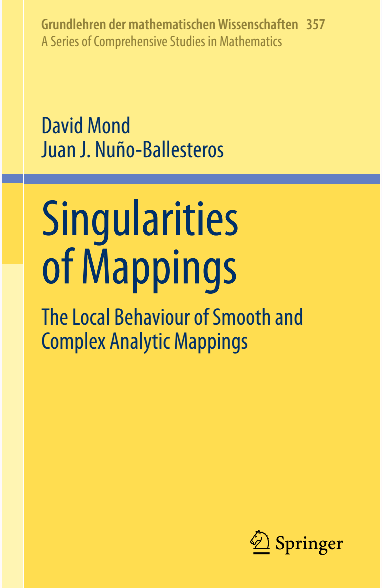
2. if $f^{-1}(0) \neq \{0\}$, the link diagram

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S_\delta^{p-1},$$

is independent of ϵ, δ, up to homotopy $\mathcal{A}$-equivalence and $f|_{D_\epsilon^n \cap f^{-1}(D_\delta^p)}$ is topologically $\mathcal{K}$-equivalent to the generalized cone

$$C(f|_{N_{\epsilon,\delta}, r_{\epsilon,\delta}}) : (N_{\epsilon,\delta} \times I) \cup_{r_{\epsilon,\delta}} V_\epsilon \to c(S_\delta^{p-1})$$

and the map $f_{\epsilon,\delta}$ is not stable.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