A variation around Malgrange-Rabier Condition.

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Vincent Grandjean (UFC)

Introduction: critical point/value.

Let $F : U \mapsto \mathbb{K}^s$ smooth mapping (C^∞, $\mathbb{K}$-analytic, $\mathbb{K}$-regular), with U connected smooth manifold of dim = n; $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition. $\mathrm{crit}(F) = \{\mathbf{u} \in U : \text{rank}\, D_{\mathbf{u}}F \leq s-1\}$; $K_0(F) := F(\mathrm{crit}(F)$ and, $\mathbb{K}^s \setminus K_0(F) = \{\text{regular values of } F\}$.

Sard Theorem: *$K_0(F)$ has Lebesgue measure 0.*

G_F = graph of F smooth sub-manifold $\subset U \times \mathbb{K}^s$; $\varphi : G_F \mapsto \mathbb{K}^s$ projection.

Observe:

$\mathbf{x} \in \mathrm{crit}(F) \Longleftrightarrow \mathbf{w} := (\mathbf{x}, F(\mathbf{x})) \in \mathrm{crit}(\varphi) \Longleftrightarrow D_{\mathbf{w}}\varphi$ **NOT surjective at $\mathbf{w}$.**

Introduction: bifurcation value

Definition. *A value $\mathbf{c}$ in $\mathbb{K}^s$ is a **bifurcation value of** F, if F does not induce a C^∞-fibre bundle structure over any nbhd of $\mathbf{c} \in \mathbb{K}^s$.*

$B(F)$:= $\{$bifurcation values of $F\}$. (Can be defined for any $C^{k(\geq 1)}$ regularity)

Known Facts: (1) $K_0(F) \subset B(F)$; (2) U compact $\Longrightarrow$ $K_0(F) = B(F)$;

Bertini-Sard Theorem: *F tame (and $C^{k(\geq 1)}$) $\Longrightarrow$ $B(F)$ contained in a tame subset of $\mathbb{K}^s$ of codim ≥ 1.*

Regular bifurcation values exist: (Broughton polynomial).

$f : \mathbb{K}^2 \to \mathbb{K}$ with $f(x,y) = y(xy-1)$. Then $\mathrm{crit}(f) = \emptyset = K_0(f)$ and $B_0(f) = \{0\}$.

Introduction: Malgrange-Rabier condition - I

$L(n,s) :=$ $\mathbb{K}$-linear maps $\mathbb{K}^n \mapsto \mathbb{K}^s$ with $n \geq s+1$.

Definition. *Rabier function of $A \in L(n,s)$ is*

$$\nu(A) := \inf\{r > 0 : \mathbf{B}_{\mathbb{K}^s}(\mathbf{0}, r) \subset A \cdot \mathbf{B}_{\mathbb{K}^n}(\mathbf{0}, 1)\}$$

In particular: $\mathrm{rank}(A) = s \Longleftrightarrow \nu(A) > 0$.

Definition. *$F : U \mapsto \mathbb{K}^s$ for U closed subset and $\mathbb{K}$-smooth sub-mfd of $\mathbb{K}^N$. A value $\mathbf{c} \in \mathbb{K}^s$ is a* **(MR)-critical value (a.k.a. Asymptotic Critical Value) of** *F if exists $(\mathbf{u}_k)_k \in U$ s.t.*

$$\textit{(i)}\ \ \mathbf{u}_k \to \infty; \quad \textit{(ii)}\ \ F(\mathbf{u}_k) \to \mathbf{c} \quad \textit{and} \quad \textit{(iii)}\ |\mathbf{u}_k| \cdot \nu(D_{\mathbf{u}_k}F) \to 0.$$

$K_\infty(F)$ = set of (MR)-critical values of F = set of A.C.V.of F.

Introduction: Malgrange-Rabier condition - II

$F : U \mapsto \mathbb{K}^s$, with $U \subset \mathbb{K}^N$ connected smooth sub-mfd and **closed subset**.
Recall $\varphi : G_F \mapsto \mathbb{K}^s$. Then $\nu(D_{\mathbf{u}}F) = \nu(D_{(\mathbf{u},F(\mathbf{u}))}\varphi) \implies K_\infty(F) = K_\infty(\varphi)$.

Bertini-Sard type Theorem: [A few authors; some cases about F and U]
Assume F,U as above and **tame**. *Then* $K(F) := K_0(F) \cup K_\infty(f) \subset \mathbb{K}^s$ *is tame of codim* ≥ 1.

Theorem: [Many authors; many cases about F and U] $B(F) \subset K(F)$.

In general: $B(F) \subsetneq K(F)$.
Let $a,b \in \mathbb{N}_{\geq 1}$ and consider $f : \mathbb{C}^3 \mapsto \mathbb{C}$ defined as

$$f_{ab}(x,y,z) = x + yz - 3x^{2a+1}y^{2b} + 2x^{3a+1}y^{3b}.$$

Then $B(f) = \emptyset$, and $a > b \implies K(f) \neq \emptyset$ (Paunescu & Zaharia - 1997).

Definition. *A value* $\mathbf{c} \notin K(F)$ *is called* **(MR)-regular.**

Motivations: Malgrange-Rabier condition and t-regularity

Siersma & Tibăr introduced t**-regularity condition** for $f \in \mathbb{K}[X_1, \ldots, X_N]$.

t-regularity at a value $\mathbf{c} \in \mathbb{K}$ is a co-normal type condition at points of the divisor at infinity of the projective closure of the graph of f.

It allows nearby infinity to stratify f w.r.t. to a Thom's (a_{rel}) type condition.

Theorem: (1) $f \in \mathbb{K}[X_1, \ldots, X_N]$ is t-regular at $\mathbf{c} \iff \mathbf{c} \notin K(f)$ (Tibăr 1996).

(2) $f : \mathbb{R}^{N+s+1} \mapsto \mathbb{R}^s$ C^2 and 1/2-alg is t-regular at $\mathbf{c} \iff \mathbf{c} \notin K(f)$ (Dias & Ruas & Tibăr 2012).

Motivations: Algebraic context

$F : X \mapsto \mathbb{K}^s$ regular mapping, $X \subset \mathbb{K}^N$ non-singular irreducible variety of dimension $\geq s + 1$.

Theorem: [Jelonek & Kurdyka 2005.] *If $\mathbb{K} = \mathbb{C}$, then $K(f)$ is an affine subvariety of $\mathbb{C}^s$ of codim ≥ 1.*

Motivations: Maria Michalska's Statement: *"A (regular) bifurcation value of F should be a critical value of* ***something****".*

Results:Regularity at infinity - OLD

$F, U, \mathbb{K}^s$ as before.

Already exist a few definitions of regular value at ∞. Recall

Definition: *A value* $\mathbf{c} \in \mathbb{K}^s$ *is (MR)-regular (for* F*) if exists* $M_{\mathbf{c}} > 0$ *s.t.* $\forall (\mathbf{u}_k)_k \subset U$

$$\mathbf{u}_k \to \infty \text{ and } F(\mathbf{u}_k) \to \mathbf{c} \implies |\mathbf{u}_k| \cdot \nu(D_{\mathbf{u}_k} F) \geq M_c, \text{ for } k \geq k_0.$$

<u>Results:</u> Regularity at infinity - NEW

$\varphi : X \subset \mathbb{K}^p \times \mathbb{K}^s \mapsto \mathbb{K}^s$ projection onto $\mathbb{K}^s$ for X C^∞ connected submanifold and **closed subset**. We introduce the following **"NEW"-ish**

Definition: *Assume* $\dim X \geq s+1$. **A value $\mathbf{c} \in \mathbb{K}^s$ is regular at ∞** *if* $\forall (\mathbf{x}_k)_k \subset X$

$$\mathbf{x}_k \to \infty \text{ and } \varphi(\mathbf{x}_k) \to \mathbf{c} \text{ and } T_{\mathbf{x}_k}X \to T \implies \varphi(T) = \mathbb{K}^s.$$

Definition for $F : U \mapsto \mathbb{K}^s$ via its graph is straightforward.

This is **JUST** the definition of a **regular point** **BUT** at the boundary of the domain!

Results: A simple geometric characterization of (MR)-regularity.

Definable for sub-analytic = globally sub-analytic.

Theorem: [Dutertre & G. - Michalska & G. 2020] *Let $F : U \subset \mathbb{R}^N \mapsto \mathbb{R}^s$ be C^2 definable mapping over a connected definable sub-manifold U and closed subset of dim. $\geq s+1$. A value $\mathbf{c}$ is (MR)-regular if and only if it is regular at ∞.*

Proof uses spherical compactification of $\mathbb{R}^N$ and definability.

Corollary: *The Theorem provides - AT LAST - a comprehensible and simple definition of t-regularity (for projective varieties) from the affine point of view (as opposed to the boundary $= \infty$ point of view of the usual definition).*

Results: Algebraic Context - Stratification Point of View

$F : X \subset \mathbb{K}^N \mapsto \mathbb{K}^s$ regular, X non-singular, irreducible of dim. $\geq s+1$.

$Y_F := \overline{G_F}^{\text{zar}}$ in $\mathbf{P}^N \times \mathbb{K}^s$ and $Y_{\mathbf{c}} := \overline{F^{-1}(\mathbf{c})}^{\text{zar}} = \varphi_F^{-1}(\mathbf{c})$ for $\varphi_F : Y_F \mapsto \mathbb{K}^s$ projection.

Good point: All fibres $Y_{\mathbf{c}}$ are **disjoint** and **compact**.

Bad point: $\deg F \geq 2 \Longrightarrow$ fibre $Y_{\mathbf{c}}$ may be singular at point(s) of $\mathbf{H}^\infty := \mathbf{P}^N \setminus \mathbb{K}^N$.

Regularity nearby a value $\mathbf{c}$ makes sense **only** from stratification theory point of view: Stratify $Y_F \setminus \mathbf{H}^\infty$ and $Y_F \cap \mathbf{H}^\infty$ to get a Thom (a_{rel}) type condition for φ.

Possible at (MR)-regular values: Tibăr for polynomial $F : \mathbb{K}^N \mapsto \mathbb{K}^s$ (1996-8-9), Gaffney (1997) and Dias & Ruas & Tibăr for polynomial $F : \mathbb{K}^N \mapsto \mathbb{K}^s$ (2012).

Results: Algebraic Context - Bi-rational Point of View - I

$F : X \subset \mathbb{K}^N \mapsto \mathbb{K}^s$ induces a rational mapping

$$F : \overline{X}^{\mathrm{zar}} \dashrightarrow \mathbf{P}^s$$

Has an indeterminacy locus, sub-variety of $X^\infty = \overline{X}^{\mathrm{zar}} \cap \mathbf{H}^\infty$.

Example: $F : \mathbb{K}^N \mapsto \mathbb{K}$ with $\deg F = d$. Let $f : \mathbb{K}^{N+1} \mapsto \mathbb{K}$ homogenization of F.

Rational mapping F is: $\mathbf{P}^n \ni [\mathbf{x} : z] \dashrightarrow [f(\mathbf{x} : z) : z^d] \in \mathbf{P}^1$, equivalently it is $\frac{f(x:z)}{z^d}$.

Definition. *A regular mapping $\widetilde{F} : \widetilde{X} \to \mathbf{P}^s$ over non-singular $\widetilde{X}$, is* **a regularization of** F*, if there exists $\pi : \widetilde{X} \mapsto \overline{X}^{\mathrm{zar}} \subset \mathbf{P}^s$ a regular bi-rational equivalence s.t.*

$$\widetilde{F} = F \circ \pi.$$

Observe: $\widetilde{X}$ is compact $\Longrightarrow$ bifurcation values = critical values !

Results: Algebraic Context - Bi-rational Point of View - II

Exercise: *Every F admits a regularization.*

Theorem: [Michalska & G. 2020] *Let $F : X \mapsto \mathbb{K}^s$ as above with $\dim X \geq s+1$. Let $\widetilde{F}$ any regularization of F. We find*

$$K(F) \subset K_0(\widetilde{F}) \cap \mathbb{C}^s,$$

in other words: Any (MR)-critical value of F is a critical value $\widetilde{F}$.

Corollary 0: *In other words: Any bifurcation value of F is a critical value of $\widetilde{F}$.*

Results: Algebraic Context - Bi-rational Point of View - III

Corollary 1: (Sard Type Theorem) *Since $K(F) \subset K_0(\widetilde{F})$, it is of codim ≥ 1.*

Yields a short proof of

Corollary 2: (Jelonek & Kurdyka) *If $\mathbb{K} = \mathbb{C}$ then $K(F)$ and $K_\infty(F)$ are affine sub-varieties of $\mathbb{C}^s$.*

Corollary 3: Let Reg(F) be the "space" of regularizations $\widetilde{F}$ of F. Then

$$K_{min}(F) := \bigcap_{\widetilde{F} \in \mathrm{Reg}(F)} K_0(\widetilde{F}) \subset \mathbf{P}^s$$

is a projective sub-variety containing $K(F) \subset \mathbb{C}^s$.

Results: Algebraic Context - Bi-rational Point of View - IV

Observe: From any regularization, it is easy to create $\widetilde{F}$ s.t. $K(F)\cap\mathbb{C}^s \subsetneq K_0(\widetilde{F})\cap\mathbb{C}^s$!

Problems at hand: (1) $K(F)=K_{min}(F)\cap\mathbb{C}^s$? I.e., does $K(F)=K_0(\widetilde{F})\cap\mathbb{C}^s$ for some regularization $\widetilde{F}$? (seemingly hopeless!)

(2) $K_{min}(F)=K_0(\widetilde{F})$ (or $K_{min}(F)\cap\mathbb{C}^s=K_0(\widetilde{F})\cap\mathbb{C}^s$) for some regularization $\widetilde{F}$?

Proposition: [Michalska & G. 2020] *Let $F:\mathbb{C}^2\mapsto\mathbb{C}$. Let $\widetilde{F}$ be the minimal regularization of F. Then $K(F)=K_0(\widetilde{F})$.*

This last result can be recovered combining results of Fourrier (1996), Kuo & Parusiński (2000), and Gwoździewicz (2010).

DZIĘKUJĘ

BARDZO