Łojasiewicz exponent of rational singularities and ideals in their local ring

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Łojasiewicz Inequality

Theorem (Stanislaw Łojasiewicz, 1958)

Let $U \subset \mathbb{R}^N$ be an open set.

Let $F : U \longrightarrow \mathbb{R}$ be a real analytic function.

Assume that $V(F) \neq \emptyset$.

Then, for any compact set K in U there exist $\alpha > 1$ and a constant $c > 0$ such that

$$inf_{z \in V(F)} \mid p - \mathbf{z} \mid^{\alpha} \leq c \cdot \mid F(p) \mid$$

for all $p \in K$.

Łojasiewicz Gradient Inequality

Theorem (Stanislaw Łojasiewicz, 1963)

Let $U \subset \mathbb{R}^N$ be an open set.

Let $F : U \longrightarrow \mathbb{R}$ be a real analytic function.

Assume that $V(F) \neq \emptyset$.

Then, for every $p \in U$ there exists a neighborhood U' of p and constants $\beta, c > 0$ such that

$$\mid F(\mathbf{z}) - F(p) \mid^{\beta} \leq c \cdot \mid \nabla F(\mathbf{z}) \mid$$

for all $\mathbf{z} \in U'$.

Łojasiewicz Gradient Inequality

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$$\mid F(\mathbf{z}) - F(p) \mid^{\beta} \leq c \cdot \mid \nabla F(\mathbf{z}) \mid$$

for all $\mathbf{z} \in U'$.

Remark

The first inequality implies the second inequality.

Łojasiewicz Inequalities

The aim is to find the smallest possible exponents α, β, θ such that

$$\mid f(\mathbf{x}) \mid \geq c \cdot \mid \mathbf{x} \mid^{\alpha}$$

$$\mid \nabla f(\mathbf{x}) \mid \geq c \cdot \mid \mathbf{x} \mid^{\beta}$$

$$\mid \nabla f(\mathbf{x}) \mid \geq c \cdot \mid f(\mathbf{x}) \mid^{\theta}$$

for an analytic function f defined in a neighborhood of 0 in k^n such that $f^{-1}(0) = 0$.

Łojasiewicz Inequality

Theorem (B. Teissier, 1977)

We have $\theta = \frac{\beta}{\beta+1}$.

Łojasiewicz Inequality

Theorem (J. Gwodziewicz, 1999)

We have:

$\alpha = \beta + 1,$

$\theta = \frac{\beta}{\alpha},$

$\beta = N + \frac{a}{b}$ where $0 < a < b < N^{n-1}$.

In complex case

Łojasiewicz Exponent

Let $f(\mathbf{z}) = f(z_1, \ldots, z_N) \in \mathbb{C}\{z_1, \ldots, z_N\}$ with $f(0) = 0$.

Then there exists a neighborhood U of 0 in $\mathbb{C}^N$ and constants $\theta, c > 0$ such that

$$\mid \mathbf{z} \mid^{\theta} \leq c \cdot \mid \nabla f(\mathbf{z}) \mid$$

for all $\mathbf{z} \in U$.

The infimum of all possible θ is called the Łojasiewicz exponent $\mathcal{L}_0(f)$ of f.

Łojasiewicz Exponent of an Hypersurface

Let $f : \mathbb{C}^N \to \mathbb{C}$ be an analytic function germ.

Consider the hypersurface

$$X := \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid f(z_1, \ldots, z_N) = 0\}$$

with an isolated singularity at the origin.

Definition

The Łojasiewicz exponent $\mathcal{L}_0(X)$ of X is the Łojasiewicz exponent $\mathcal{L}_0(f)$.

Łojasiewicz Exponent of an Hypersurface

Question

Is $\mathcal{L}_0(X)$ a topological invariant?

Łojasiewicz Exponent of an Hypersurface

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Łojasiewicz Exponent of an Hypersurface

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Is $\mathcal{L}_0(X)$ a topological invariant?

Question
Find a formula to compute $\mathcal{L}_0(X)$ using other invariants of X?

Question
What is the best estimation of $\mathcal{L}_0(X)$ for a given X?

Question
Is there any relation between the multiplicity $m_0(X)$ and $\mathcal{L}_0(X)$ for a given X?

Łojasiewicz Exponent of an Hypersurface

Theorem (A. Ploski, 1990)

Let $C := \{(z_1, z_2) \mid f(z_1, z_2) = 0\} \subset \mathbb{C}^2$.

Consider

$$f'_{z_1} = \frac{\partial F}{\partial z_1} = g_1 \cdots g_r, \quad f'_{z_2} = \frac{\partial F}{\partial z_2} = h_1 \cdots h_s$$

where g_i and h_j are irreducible for each i, j.

Then the Łojasiewicz exponent of the curve C is given by

$$\mathcal{L}_0(C) = \max_{i,j} \left\{ \frac{(f'_{z_1}, h_i)_0}{ord(h_i)}, \frac{(f'_{z_2}, g_j)_0}{ord(g_j)} \right\}.$$

Here $(f, g)_0$ denotes the intersection multiplicity at the origin.

Łojasiewicz Exponent of an Hypersurface

Theorem (T.Krasinski, G.Oleksik, A.Ploski, 2009)

Let f be a weighted homogeneous polynomial with an isolated singularity at 0 with weights

$(w_1, \ldots, w_N)$ and degree d.

Assume that $d \geq 2w_i$ for all i.

Then

$$\mathcal{L}_0(X) = \frac{d - min\{w_i\}}{min\{w_i\}}$$

Without the assumption $d \geq 2w_i$, we have:

$$\mathcal{L}_0(X) = min\Big\{\prod_{i=1}^{3}(\frac{d}{w_i} - 1), \frac{d - min\{w_i\}}{min\{w_i\}}\Big\}$$

Example - $\mathcal{L}_0(X)$ of ADE-singularities

Singularity $(X,0)$	(w_1, w_2, w_3)	d	$\mathcal{L}_0(X)$
A_{2k}: $z_3^2 + z_2^2 + z_1^{n+1} = 0$	$(2, 2k+1, 2k+1)$	$4k+2$	n
A_{2k+1}: $z_3^2 + z_2^2 + z_1^{n+1} = 0$	$(1, k+1, k+1)$	$2k+2$	n
D_n: $z_3^2 + z_1 z_2^2 + z_1^{n-1} = 0$	$(2, n-2, n-1)$	$2(n-1)$	$n-2$
E_6: $z_3^2 + z_2^3 + z_1^4 = 0$	$(3,4,6)$	12	3
E_7: $z_3^2 + z_2^3 + z_1^3 z_2 = 0$	$(4,6,9)$	18	$\frac{7}{2}$
E_8: $z_3^2 + z_2^3 + z_1^5 = 0$	$(6,10,15)$	30	4

Łojasiewicz Exponent of a Surface

Remark

We can define the Łojasiewicz exponent $\mathcal{L}_0(f)$ of any holomorphic map

$$F : (\mathbb{C}^N, 0) \to (\mathbb{C}^m, 0)$$

having an isolated zero at the origin.

Rational Singularities of Surfaces

Definition

Let $X \subset \mathbb{C}^N$ be a surface with an isolated singularity at the origin.

Let $\pi : (\tilde{X}, E) \to (X, 0)$ be a resolution of $(X, 0)$.

Let $\pi^{-1}(0) := \cup E_i$ be the exceptional curve.

$(X, 0)$ is a rational singularity if $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$.

Local Ring of a Rational Singularity

Let $g \in \mathcal{O}_{X,0}$.

We have $\pi^*(g) = D_g + T_g$ where $D_g = \sum_{i=1}^{n} \nu_{E_i}(g) E_i$ and T_g is the strict transform of f by π.

Let $\mathcal{S}(\pi)$ be the set of such positive divisors D_g.

Łojasiewicz Exponent of Rational Singularities

partial ordering

Rational Singularities of Surfaces

Theorem (M.Artin, 1964)

Let $X := (X, 0)$ be a surface with a rational singularity at 0 in $\mathbb{C}^N$.

Let Z be the Artin cycle of π. Then $mult_0(X) = -(Z \cdot Z)$.

Local Ring of a Rational Singularity

Let $\mathcal{S}(\mathbf{I})$ be the set of $\mathcal{M}$-primary integrally closed ideals I in $\mathcal{O}_{X,0}$ such that

$I\mathcal{O}_{\tilde{X}}$ is invertible.

Theorem (J.Lipman,1969)

The product of integrally closed ideals in $\mathcal{O}_{X,0}$ is integrally closed.

Corollary

The set $\mathcal{S}(\mathbf{I})$ is a semigroup with respect to the product.

Local Ring of a Rational Singularity

Theorem (J.Lipman,1969)

For a rational singularity, we have a 1-1 correspondence between $\mathcal{S}(\mathbf{I})$ and $\mathcal{S}(\pi)$.

Łojasiewicz Exponent of an Ideal

Definition

Let $X \subset \mathbb{C}^N$ be a germ of surface with an isolated singularity at 0.

Let $I =< f_1, \ldots, f_k > \subset \mathcal{O}_{X,0}$ and $g \in \mathcal{O}_{X,0}$ with $g \in \sqrt{I}$.

If there is an open neighbourhood U of 0 in X and $c \in \mathbb{R}_+$ with

$$\mid g(z) \mid^{\theta} \leq c \cdot \sup_{i=1,\ldots,k} \mid f_i(z) \mid, \quad \forall z \in U$$

then the greatest lower bound of θ's is called the Łojasiewicz exponent of g w.r.t. I.

We denote it by $\mathcal{L}_I(g)$.

This definition does not depend on the generators of I.

Łojasiewicz Exponent of an Ideal

Theorem (B.Teissier, M.Lejeune-Jalabert, 1974)

Let $I =< f_1, \ldots, f_k > \subset \mathcal{O}_{X,0}$ and $g \in \mathcal{O}_{X,0}$ with $g \in \sqrt{I}$.

Let $\nu_{E_i}(g)$ be the vanishing order of $g \circ \pi$ along E_i, the largest integer p such that $g \in I^p$.

$$\mathcal{L}_I(g) = max_{i=1}^{k}\{\frac{\nu_{E_i}(I)}{\nu_{E_i}(g)}\}$$

Łojasiewicz Exponent of an Ideal

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Let $\nu_{E_i}(g)$ be the vanishing order of $g \circ \pi$ along E_i, the largest integer p such that $g \in I^p$.

$$\mathcal{L}_I(g) = max_{i=1}^{k}\{\frac{\nu_{E_i}(I)}{\nu_{E_i}(g)}\}$$

Corollary

$\mathcal{L}_I(g) \in \mathbb{Q}_+$.

Łojasiewicz Exponent of an Ideal

More generally:

Definition

Let I, J be two ideals in $\mathcal{O}_{X,0}$ with $J \subset \sqrt{I}$.

The Łojasiewicz exponent of the ideal $J =< h_1, \ldots, h_r >\subset \mathcal{O}_{X,0}$ with respect to I is

$$\mathcal{L}_I(J) = max_{i=1,\ldots,r}\mathcal{L}_I(h_i)$$

Łojasiewicz Exponent of an Ideal

Theorem (B.Teissier, M.Lejeune-Jalabert, 1974)

Let $X \subset \mathbb{C}^N$ be a germ of surface with an isolated singularity at 0.

Let $I, J \subset \mathcal{O}_{X,0}$ be two ideals.

Then

$$\mathcal{L}_I(J) = inf\{\frac{a}{b} \mid a, b \in \mathbb{N}^*,\ I^a \subseteq \overline{J^b}\}$$

Local Ring of a Rational Singularity

Definition

Let $I \in \mathcal{S}(\mathbf{I})$. An element $f \in I$ is called generic for I if

$$\nu_{E_i}(f) \leq \nu_{E_i}(h)$$

for all $h \in I$.

Łojasiewicz Exponent of Rational Singularities

Proposition

Let $I \in \mathcal{S}(\mathbf{I})$ and g be the generic element of I.

Let Z be the Artin divisor of π.

Then

$$\mathcal{L}_{\mathcal{M}}(I) = max\{\frac{a}{b} \mid a \cdot Z \geq b \cdot D_g \text{ with } a, b \in \mathbb{N}^*\}$$

where g is the generic element of I and $\mathcal{M}$ is the maximal ideal in $\mathcal{O}_{X,0}$.

Łojasiewicz Exponent of Rational Singularities

Proposition

Let $I \in \mathcal{S}(\mathbf{I})$.

The Łojasiewicz exponent $\mathcal{L}_0(I)$ is given by

$$\mathcal{L}_0(I) := \max_{i=1}^{n} \left\{ \frac{\nu_{E_i}(D_I)}{\nu_{E_i}(Z)} \right\}$$

In particular, we have $\mathcal{L}_0(\mathcal{M}) = 1$.

Łojasiewicz Exponent of Rational Singularities

$\mathbb{Q}$-gen. of E_6	$\ell(I)$	$\mathcal{L}_0(I)$	$\mathbb{Q}$-gen. of E_7	$\ell(I)$	$\mathcal{L}_0(I)$	$\mathbb{Q}$-gen. of E_8	$\ell(I)$	$\mathcal{L}_0(I)$
$(1,2,3,2,1,2)*$	1	1	$(2,3,4,3,2,1,2)*$	1	1	$(2,4,6,5,4,3,2,3)*$	1	1
$(2,3,4,3,2,2)$	2	2	$(2,4,6,5,4,2,3)*$	2	2	$(4,7,10,8,6,4,2,5)*$	2	2
$(2,4,6,4,2,3)*$	3	2	$(2,4,6,5,4,3,3)*$	3	$3/2$	$(4,8,12,10,8,6,3,6)*$	3	2
$(4,5,6,4,2,3)*$	6	4	$(3,6,8,6,4,2,4)*$	3	2	$(3,6,9,12,15,10,5,8)*$	4	$8/3$
$(2,4,6,5,4,3)*$	6	4	$(3,6,9,7,5,3,5)$	4	3	$(6,12,18,15,12,8,4,9)*$	6	3
$(5,10,12,8,4,6)*$	15	5	$(4,8,12,9,6,3,6)*$	6	3	$(7,14,20,16,12,8,4,10)*$	7	$7/2$
$(4,8,12,10,5,6)*$	15	5	$(4,8,12,9,6,3,7)*$	7	$7/2$	$(7,14,21,17,13,9,5,11)$	8	$11/3$
			$(6,12,18,15,10,5,9)*$	15	5	$(8,16,24,20,15,10,5,12)*$	10	4
						$(10,20,30,24,18,12,6,15)*$	15	5

Łojasiewicz Exponent of Rational Singularities

Recall

The length of an ideal I in a ring R is the dimension of R/I over k.

Łojasiewicz Exponent of Rational Singularities

Theorem

The length of $I \in \mathcal{S}(\mathbf{I})$ is given by

$$\ell(I) = \frac{-(D_I \cdot D_I) - \sum_{i=1}^{n} \nu_{E_i}(D_I)(w_i - 2)}{2}$$

where $w_i = -E_i^2$ for all i.

Remark

For an ideal I with $\ell(I) = p$ we have $\mathcal{M}^p \subseteq I$.

Łojasiewicz Exponent of Rational Singularities

$\mathbb{Q}$-gen. of E_6	$\ell(I)$	$\mathcal{L}_0(I)$	$\mathbb{Q}$-gen. of E_7	$\ell(I)$	$\mathcal{L}_0(I)$	$\mathbb{Q}$-gen. of E_8	$\ell(I)$	$\mathcal{L}_0(I)$
$(1,2,3,2,1,2)*$	1	1	$(2,3,4,3,2,1,2)*$	1	1	$(2,4,6,5,4,3,2,3)*$	1	1
$(2,3,4,3,2,2)$	2	2	$(2,4,6,5,4,2,3)*$	2	2	$(4,7,10,8,6,4,2,5)*$	2	2
$D_p = (2,4,6,4,2,3)*$	3	2	$D_p = (2,4,6,5,4,3,3)*$	3	$3/2$	$(4,8,12,10,8,6,3,6)*$	3	2
$(4,5,6,4,2,3)*$	6	4	$(3,6,8,6,4,2,4)*$	3	2	$D_p = (3,6,9,12,15,10,5,8)*$	4	$8/3$
$(2,4,6,5,4,3)*$	6	4	$(3,6,9,7,5,3,5)$	4	3	$(6,12,18,15,12,8,4,9)*$	6	3
$(5,10,12,8,4,6)*$	15	5	$(4,8,12,9,6,3,6)*$	6	3	$(7,14,20,16,12,8,4,10)*$	7	$7/2$
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			$(6,12,18,15,10,5,9)*$	15	5	$(8,16,24,20,15,10,5,12)*$	10	4
						$(10,20,30,24,18,12,6,15)*$	15	5

Łojasiewicz Exponent of Rational Singularities

Observations

Let X be a surface with an ADE-type singularity. Then

$$\mathcal{L}_0(X) \leq m_0(X) + \mathcal{L}_0(D_p) \leq \ell(D_p) + 1$$

where D_p is a special divisor in $S(\pi)$.

$$m_0(X) < \mathcal{L}_0(X) \leq \tau(X)$$

where $\tau(X)$ equals $dim(\frac{\mathcal{O}_{\mathbb{C}^N,0}}{<f,J(f)>})$, called the Tjurina number of X.

Rational Singularities of Surfaces

Theorem (M.Artin, 1964)

Let $X := (X, 0)$ be a surface with a rational singularity at 0 in $\mathbb{C}^N$.

Let Z be the Artin cycle of π. Then

(i) $p_a(Z) = 0$

(ii) $mult_0(X) = -(Z \cdot Z)$

(iii) $emb.dim.(X) = -(Z \cdot Z) + 1$

Rational Singularities of Surfaces

Corollary

A rational singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ has multiplicity $N-1$ and is defined by

$$k := \frac{(N-1)(N-2)}{2} \text{ equations.}$$

Tjurina equations

RTP	Tjurina's equations	RTP	Tjurina's equations
$A_{k-1,\ell-1,m-1}$ $k,\ell,m \geq 1$	$xw - y^m w - y^{\ell+m} = 0$ $zw + y^\ell z - y^k w = 0$ $xz - y^{m+k} = 0$	$C_{k-1,\ell+1}$ $k \geq 1,\ \ell \geq 2$	$xz - y^k w = 0$ $w^2 - x^{\ell+1} - xy^2 = 0$ $zw - x^\ell y^k - y^{k+2} = 0$
$B_{k-1,n}$ $n = 2\ell > 3$	$xz - y^{k+\ell} - y^k w = 0$ $w^2 + y^\ell w - x^2 y = 0$ $zw - xy^{k+1} = 0$	$B_{k-1,n}$ $n = 2\ell - 1 \geq 3$	$xz - y^k w = 0$ $zw - xy^{k+1} - y^{k+\ell} = 0$ $w^2 - x^2 y - xy^\ell = 0$
D_{k-1} $k \geq 1$	$xz - y^{k+2} - y^k w = 0$ $zw - x^2 y^k = 0$ $w^2 + y^2 w - x^3 = 0$	F_{k-1} $k \geq 1$	$xz - y^k w = 0$ $zw - x^2 y^k - y^{k+3} = 0$ $w^2 - x^3 - xy^3 = 0$
H_n $n = 3k$	$z^2 - xw = 0$ $zw + y^k z - x^2 y = 0$ $w^2 + y^k w - xyz = 0$	H_n $n = 3k+1$	$z^2 - xy^{k+1} - xyw = 0$ $zw - x^2 y = 0$ $w^2 + y^k w - xz = 0$
H_n $n = 3k - 1$	$z^2 - xw = 0$ $zw - x^2 y - xy^k = 0$ $w^2 - y^k z - xyz = 0$		
$E_{6,0}$	$z^2 - yw = 0$ $zw + y^2 z - x^2 y = 0$ $w^2 + y^2 w - x^2 z = 0$		
$E_{0,7}$	$z^2 - yw = 0$ $zw - x^2 y - y^4 = 0$ $w^2 - x^2 z - y^3 z = 0$		
$E_{7,0}$	$z^2 - yw = 0$ $zw + x^2 z - y^3 = 0$ $w^2 + x^2 w - y^2 z = 0$		

Łojasiewicz Exponent of Rational Singularities

Consider the analytic map germs $f_i : \mathbb{C}^N \to \mathbb{C}$ so that

$$F = (f_1, f_2, \dots, f_k) : \mathbb{C}^N \to \mathbb{C}^k$$

defines the rational singularity $(X, 0)$.

The Łojasiewicz exponent $\mathcal{L}_0(F)$ of F at the origin in $\mathbb{C}^N$ is the infimum of the set of all real numbers $\theta > 0$ such that there exists a positive constant c such that

$$c\|z\|^{\theta} \leq \|F(z)\| \quad as \quad \|z\| << 1$$

Quasi-Homogeneous Ideals

Definition

A map $F = (f_1, \ldots, f_k) : \mathbb{C}^N \longrightarrow \mathbb{C}^k$ is called quasi-homogeneous if

$$f_i(\lambda^{w_1} z_1, \lambda^{w_2} z_2, \ldots, \lambda^{w_N} z_N) = \lambda^{d_i} f_i(z_1, z_2, \ldots, z_N)$$

where

$w = (w_1, \ldots, w_N) \in (\mathbb{R}_+ - \{0\})^N$ and $d = (d_1, \ldots, d_k) \in (\mathbb{R}_+ - \{0\})^k$.

Quasi-Homogeneous Ideals

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where

$w = (w_1, \ldots, w_N) \in (\mathbb{R}_+ - \{0\})^N$ and $d = (d_1, \ldots, d_k) \in (\mathbb{R}_+ - \{0\})^k$.

Remark

The RTP-singularities are quasi-homogeneous.

Łojasiewicz Exponent of Quasi-Homogeneous Ideal

Theorem (A.Haraux and T.S.Pham, 2015)

$F = (f_1, \ldots, f_k) : \mathbb{C}^N \longrightarrow \mathbb{C}^k$ be a quasi-homogeneous map germ with the

weight $w = (w_1, \ldots, w_N) \in \mathbb{Z}^N_{>0}$ and the quasi-degree $d = (d_1, \ldots, d_k) \in \mathbb{Z}^k_{>0}$.

Assume that $F^{-1}(0) = \{0\}$. Then

$$\frac{min\{d_1, \ldots, d_k\}}{min\{w_1, \ldots, w_N\}} \leq \mathcal{L}_0(F) \leq \frac{max\{d_1, \ldots, d_k\}}{min\{w_1, \ldots, w_N\}}$$

RTP-Singularities as Quasi-homogeneous Functions

RTP	**weights**	$min\{\mathbf{d}\}$	$max\{\mathbf{d}\}$
$A_{k,\ell,m}$	$(m,1,k,\ell)$	$2m$	$2k+\ell-1$
$B_{k-1,2\ell}$	$(2\ell-1,2,2k+1,2\ell)$ for $l \geq k+1$, $(k+1,2,k+\ell,2\ell)$ for $l < k+1$	4ℓ or $2k\ell+2\ell-1$	4ℓ or $2k\ell+2\ell+1$
$B_{k-1,2\ell-1}$	$(2\ell-2,2,2k+1,2\ell-1)$	$2k+2\ell-1$ or $4\ell-2$	$4\ell-2$ or $2k+2\ell$
$C_{k-1,\ell+1}$	$(2,\ell,k.\ell+\ell-2,\ell+1)$	$2\ell+2$ or $k\ell+\ell^2$	$k.\ell+\ell+1$
D_{k-1}	$(4,3,3k+2,6)$	9,12 $k \geq 2$	12,18, $3k+8$ $k \geq 3$
F_{k-1}	$(6,4,4k+3,9)$	13,17,18 $k \geq 3$	18,4k+12 $k \geq 2$
H_{3k-1}	$(3k-3,3,3k-2,3k-1)$	6k-4	6k-2
H_{3k}	$(3k-2,3,3k-1,3k)$	6k-2	6k
H_{3k+1}	$(3k-1,3,3k+1,3k)$	6k	6k+2
$E_{6,0}$	(5,4,6,8)	12	16
$E_{0,7}$	(9, 6, 10, 14)	18	28
$E_{7,0}$	(5, 6, 8, 10)	16	20

Łojasiewicz Exponent of Rational Singularities

Let $F = (f_1, f_2, \ldots, f_k) : \mathbb{C}^N \to \mathbb{C}^k$ defines the rational singularity $(X, 0)$.

Let $g_1, \ldots, g_s$ be the 2×2 minors of $\left(\frac{\partial f_i}{\partial z_j}\right)$ where $s := \binom{N}{2}\binom{k}{2}$.

Consider $F = (f_1, \ldots, f_k, g_1, \ldots, g_s) : \mathbb{C}^N \to \mathbb{C}^{k+s}$.

The Łojasiewicz exponent $\mathcal{L}_0(F)$ of F at the origin in $\mathbb{C}^N$ is the infimum of the set of all real

numbers $\theta > 0$ such that there exists a positive constant c such that

$$c\|z\|^{\theta} \leq \|F(z)\| \quad as \quad \|z\| << 1$$

RTP-Singularities as Quasi-homogeneous Functions

RTP	weights	$min\{\mathbf{d}\}$	$max\{\mathbf{d}\}$	$\ell(Jac)$
$A_{k,\ell,m}$	$(m,1,k,\ell)$	$2m$	$2k+\ell-1$	$k+\ell+m+5$
$B_{k-1,2\ell}$	$(2\ell-1,2,2k+1,2\ell)$ for $l \geq k+1$, $(k+1,2,k+\ell,2\ell)$ for $l < k+1$	$4k+2$	$6\ell-3$	$3k+2l+3$ for $l \geq k+1$, $k+4l+2$ for $l < k+1$
$B_{k-1,2\ell-1}$	$(2\ell-2,2,2k+1,2\ell-1)$	$4k+2$	$6\ell-3$	$k+4\ell$ for $l \leq k+1$, $3k+2\ell+2$ for $\ell > k+1$
$C_{k-1,\ell+1}$	$(2,\ell,k.\ell+\ell-2,\ell+1)$	$k.\ell+\ell-4$	$\ell+3$	$k+\ell+7$
D_{2t-1}	$(4,3,3k+2,6)$	10	$6k+7$	$k+11$
F_{k-1}	$(6,4,4k+3,9)$	15	$4k+26$	$k+14$
H_{3k-1}	$(3k-3,3,3k-2,3k-1)$	$6k-4$	$9k-7$	$5k+2$
H_{3k}	$(3k-2,3,3k-1,3k)$	$6k-2$	$9k-4$	$5k+3$
H_{3k+1}	$(3k-1,3,3k+1,3k)$	$6k$	$9k-1$	$5k+5$
$E_{6,0}$	$(5,4,6,8)$	12	21	13
$E_{0,7}$	$(9,6,10,14)$	20	37	14
$E_{7,0}$	$(5,6,8,10)$	16	27	14

Łojasiewicz Exponent of Rational Singularities

Conjecture

Let X be a surface with a rational singularity. Then

$$\mathcal{L}_0(X) \leq \mathcal{L}_0(Jac) + m_0(X) \leq \ell(Jac) + 1$$

Łojasiewicz Exponent of Rational Singularities

Proposition

Let $G_1, \ldots, G_n$ be the $\mathbb{Q}$-generators in $\mathcal{S}(\pi)$.

$$\mathcal{L}_0(X) \leq \min_{i=1}^{n}\{k \in \mathbb{Q}_{>0} \mid G_i \leq k \cdot Z, \forall i = 1, \ldots, r\}$$

Łojasiewicz Exponent of Rational Singularities

Łojasiewicz Exponent of Rational Singularities