Simple elliptic singularities and Generalized Slodowy Slices

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Plan and why

Lie Algebras

Definition

A complex Lie algebra is a vector space $\mathfrak{g}$ over $\mathbb{C}$ together with a map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

such that:

(i) $[-,-]$ is a $\mathbb{C}$-bilinear map,

(ii) $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$,

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Lie Algebras

Example

The set of $n \times n$ matrices can be made into a Lie algebra by

$$[X, Y] = XY - YX$$

It is denoted by $\mathfrak{gl}(n, \mathbb{C})$.

Lie Algebras

Example

$$\mathfrak{g} := \mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid trace(X) = 0\}$$

is a Lie algebra.

Nilpotent Elements of a Lie Algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra.

Definition

The nilpotent variety of $\mathfrak{g}$ is defined as

$$\mathcal{N}(\mathfrak{g}) := \{X \in \mathfrak{g} \mid ad(X) : \mathfrak{g} \longrightarrow \mathfrak{g} \text{ is nilpotent}\}$$

Nilpotent Variety

Example

Let $\mathfrak{g} = \mathfrak{s}l(2, \mathbb{C})$.

The nilpotent variety is

$$\mathcal{N}(\mathfrak{g}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g} \,\middle|\, a^2 + bc = 0 \; with \; a, b, c \in \mathbb{C} \right\}$$

Nilpotent Variety

Example

Let $\mathfrak{g} = \mathfrak{s}l(3, \mathbb{C})$.

The nilpotent variety is

$$\mathcal{N}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid tr(X^2) = tr(X^3) = 0\}$$

Nilpotent Variety

Definition

Let $\mathfrak{g}$ be a Lie algebra.

The nilpotent variety $\mathcal{N}(\mathfrak{g})$ is called trivial if $\mathcal{N}(\mathfrak{g}) \cong \mathbb{C}^k$ for some k.

Nilpotent Variety

Proposition

If $\mathfrak{g}$ is a solvable Lie algebra, then $\mathcal{N}(\mathfrak{g})$ is trivial.

The sequence $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}'' = [\mathfrak{g}', \mathfrak{g}']$,…,$\mathfrak{g}^{(r)} := [\mathfrak{g}^{(r-1)}, \mathfrak{g}^{(r-1)}]$,….

We have $\mathfrak{g} \supset \mathfrak{g}' \supset \cdots \supset \mathfrak{g}^{(n)} \supset \ldots$.

We say that $\mathfrak{g}$ is a *solvable* Lie algebra if there exists a positive integer n such that $\mathfrak{g}^{(n)} = 0$.

Nilpotent Variety

Example

Let $\mathfrak{g} = \mathfrak{s}l(3, \mathbb{C})$.

The nilpotent variety is

$$\mathcal{N}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid tr(X^2) = tr(X^3) = 0\}$$

Transversal Slices

Definition

A local subspace $\mathcal{S}$ of $\mathfrak{g}$ is called a transversal slice if $(\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}, \mathbf{0})$ is a surface with an isolated singularity.

Slodowy Slices

Theorem (Morozov-Jacobson, 1942-1951 resp.)

Let $\mathfrak{g}$ be a semisimple Lie algebra.

Let $X \in \mathcal{N}(\mathfrak{g})$.

Then there exists $Y, H \in \mathfrak{g}$ such that

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = H \quad .$$

Such (X, Y, H) in $\mathfrak{g}$ is called an $\mathfrak{s}l_2$-triple in $\mathfrak{g}$.

Slodowy Slices

Definition

Let (X, Y, H) be an $\mathfrak{s}l_2$-triple.

Let $C_{\mathfrak{g}}(Y) = \{Z \in \mathfrak{g} \mid [Z, Y] = 0\}$.

The set $\mathcal{S}_X := X + C_{\mathfrak{g}}(Y)$ is called a Slodowy slice at X.

Slodowy Slices

Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. The nilpotent variety is

$$\mathcal{N}(\mathfrak{g}) = \left\{ X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \middle| \begin{matrix} abd-a^2e+bde-ae^2-ceg+bfg+cdh-afh=0 \\ a^2+bd+ae+e^2+cg+fh=0 \end{matrix} \right\}$$

Slodowy Slices

An $\mathfrak{sl}_2$-triple (X, Y, H) with

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding transverse Slodowy slice $\mathcal{S}_X$ is

$$\mathcal{S}_X = X + Z_{\mathfrak{g}}(Y), Z_{\mathfrak{g}}(Y) = \left\{ \begin{pmatrix} x & 0 & 0 \\ w & x & y \\ z & 0 & -2x \end{pmatrix} \right\}, \mathcal{S}_X = \left\{ \begin{pmatrix} x & 1 & 0 \\ w & x & y \\ z & 0 & -2x \end{pmatrix} \right\}$$

Slodowy Slices

The intersection of $\mathcal{N}(\mathfrak{g})$ with the slice $\mathcal{S}_X$ is given by the equations

$$-2x^3 + 2xw + yz = 0 \tag{1}$$

$$3x^2 + w = 0, \tag{2}$$

$$8x^3 - yz = 0 \tag{3}$$

which is the equation defining an A_2-singularity.

Grothendieck's Conjecture

Theorem (E. Brieskorn, 1970)

Let $\mathfrak{g}$ be a Lie algebra of one of ADE-type. Then:

(i) $(\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}_X)$ is a surface with the same singularity as $\mathfrak{g}$.

(ii) The restriction to $\mathcal{S}_X$ of the map

$$f : \mathfrak{g} \longrightarrow \mathfrak{h}/W$$

is a semi-universal deformation of the singularity.

Slodowy Slices

Restricting the adjoint quotient to the Slodowy slice

$$\gamma \colon \mathcal{S}_X \longrightarrow \mathbb{C}^2$$

$$X = \begin{pmatrix} x & 1 & 0 \\ w & x & y \\ z & 0 & -2x \end{pmatrix} \mapsto (2x^3 - yz - 2wx, -w - 3x^2), \quad (4)$$

setting $t = -w - 3x^2$ we get $w = 3x^2 - t$.

We get $(x, y, z, t) \mapsto (8x^3 - yz + 2tx, t)$ the semi-universal deformation of the A_2-singularity.

Lie Algebra of a Simple Elliptic Singularity

Let $\mathfrak{g} := \mathfrak{s}l(2,\mathbb{C}) \oplus \mathfrak{s}l(2,\mathbb{C})$.

Its nilpotent variety is:

$$\mathcal{N}(\mathfrak{g}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \, a^2 + bc = 0 \right\} \times \left\{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \middle| \, d^2 + ef = 0 \right\}$$

Lie Algebra of a Simple Elliptic Singularity

Definition

A 4-dimensional transversal slice in $\mathfrak{g}$ is called good slice if $(\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}, \mathbf{0})$ is a surface with an isolated singularity.

Lie Algebra of a Simple Elliptic Singularity

Definition

A 4-dimensional transversal slice in $\mathfrak{g}$ is called good slice if $(\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}, \mathbf{0})$ is a surface with an isolated singularity.

Proposition

If $\mathcal{S}$ be a good slice then $(\mathcal{N}(\mathfrak{g}) \cap \mathcal{S})$ is a surface with a simple elliptic singularity of type $\tilde{D}_5$.

Lie Algebra of a Simple Elliptic Singularity

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Proposition

If $\mathcal{S}$ be a good slice then $(\mathcal{N}(\mathfrak{g}) \cap \mathcal{S})$ is a surface with a simple elliptic singularity of type $\tilde{D}_5$.

For $\mathcal{S} := \{c = d + e,\ f = a + b\}$, we get:

$$(\mathcal{N} \cap \mathcal{S}, 0) = \{(a, b, d, e) \in \mathbb{C}^4 \mid a^2 + bd + be = 0, d^2 + ae + be = 0\}$$

Simple Elliptic Singularities (K.Saito,1974)

A surface S with a simple elliptic singularity defined by one of:

$$\tilde{E}_6 \quad x^6 + y^3 + z^2 + \lambda xyz = 0$$

$$\tilde{E}_7 \quad x^4 + y^4 + z^2 + \lambda xyz = 0$$

$$\tilde{E}_8 \quad x^3 + y^2 + \lambda xyz = 0$$

$$\tilde{D}_5 \quad x^2 + y^2 + \lambda zw = 0, xy + z^2 + w^2 = 0$$

Minimal Resolution of a Simple Elliptic Singularity

The minimal resolution graph of a simple elliptic singularity is a nonsingular curve **E** with genus 1 and self-intersection

$$\tilde{E}_6 \quad x^6 + y^3 + z^2 + \lambda xyz = 0, \mathbf{E}^2 = -\mathbf{1}$$

$$\tilde{E}_7 \quad x^4 + y^4 + z^2 + \lambda xyz = 0, \mathbf{E}^2 = -\mathbf{2}$$

$$\tilde{E}_8 \quad x^3 + y^2 + \lambda xyz = 0, \mathbf{E}^2 = -\mathbf{3}$$

$$\tilde{D}_5 \quad x^2 + y^2 + \lambda zw = 0, xy + z^2 + w^2 = 0, \mathbf{E}^2 = -\mathbf{4}$$

Generic Transversal Slices

Proposition

Let $(A, B) \in Sym_4(\mathbb{C}) \times Sym_4(\mathbb{C})$. Assume that $A \in GL(4, \mathbb{C})$.

Then $(S_{AB}, 0)$ with

$$S_{AB} := \{\mathbf{v} = (x, y, z, w) \in \mathbb{C}^4 \mid \mathbf{v}^t A\mathbf{v} = 0, \mathbf{v}^t B\mathbf{v} = 0\}$$

is a simple elliptic singularity of type $\tilde{D}_5$ if and only if the

characteristic polynomial of $A^{-1}B$ has no multiple root.

Special elements in $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$

Definition

A nonzero element $X_s \in \mathfrak{sl}(2,\mathbb{C})$ is called semisimple if there exists $P \in SL(2,\mathbb{C})$ such that $P^{-1}X_sP = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$.

Definition

A nonzero element $X_n \in \mathfrak{sl}(2,\mathbb{C})$ is called nilpotent if $X_n^2 = 0$.

Good subspaces in $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$

Definition

A subspace $V \subset \mathfrak{g}$ is called good if V has a basis $X, Y \in V$ such that $X = (X_s, X_n)$ and $Y = (Y_n, Y_s)$.

Generalized Slodowy slices for $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$

Definition

A 4-dimensional subspace $\mathcal{S} \subset \mathfrak{g}$ is called good slice if there exists a good subspace $V \subset \mathfrak{g}$ such that

$$\mathcal{S} := \{Z \in \mathfrak{g} \mid \langle Z, Y \rangle = 0 \text{ for each } Y \in V\}$$

Here $\langle Z, Y \rangle$ is the Killing form of $Z = (Z_1, Z_2) \in \mathfrak{g}$ and $Y = (Y_1, Y_2) \in \mathfrak{g}$:

$$\langle Z, Y \rangle = 4(\mathrm{tr}(Z_1 Y_1) + \mathrm{tr}(Z_2 Y_2))$$

Slodowy slices for Simple Elliptic Singularities

Theorem

Let $\mathfrak{g} =:= \mathfrak{s}l(2,\mathbb{C}) \oplus \mathfrak{s}l(2,\mathbb{C})$

Let $x = (x_s, x_n)$, $y = (y_n, y_s)$ be general and generates $V_{(x,y)}$.

Let $\mathcal{S}_{(x,y)}$ be a good slice. Then

$$(X_{(x,y)}, 0) := (\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}_{(x,y)}, 0)$$

is a surface with $\tilde{D}_5$-singularity.

Simple Elliptic Singularities of type $\tilde{D}_5$

Lemma (x, y general)

A good subspace $V \subset \mathfrak{g}$ has a basis $x = (x_s, x_n)$ and $y = (y_n, y_s)$ with:

(1) $x_s = y_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

(2) $x_n = \begin{pmatrix} p & 1 \\ -p^2 & -p \end{pmatrix}$ for some $p \in \mathbb{C}$, or $x_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(3) $y_n = \begin{pmatrix} q & 1 \\ -q^2 & -q \end{pmatrix}$ for some $q \in \mathbb{C}$, or $y_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Simple Elliptic Singularities of type $\tilde{D}_5$

An element

$$\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \right) \in \mathcal{S}_{(x,y)}$$

if and only if $2a + 2pd - p^2e + f = 0$ and $2qa - q^2b + c + 2d = 0$.

Simple Elliptic Singularities of type $\tilde{D}_5$

An element

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if and only if $2a + 2pd - p^2 e + f = 0$ and $2qa - q^2 b + c + 2d = 0$.

$$X_{(x,y)} = (\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}_{(x,y)}, 0) = \left\{ (a, b, c, d, e, f) \in \mathbb{C}^6 \;\middle|\; \begin{array}{l} g_1 := a^2 - 2qab + q^2 b^2 - 2bd = 0 \\ g_2 := -2ae + d^2 - 2pde + p^2 e^2 = 0 \end{array} \right\}$$

Deformations of $X_{(x,y)}$

Deform the map f by $(\alpha, \beta) \in \mathbb{C}^2$ as:

$$
\begin{aligned}
f_{(\alpha,\beta)} : \quad & \mathfrak{g} \quad \to \mathfrak{h}/W \cong \mathbb{C}^2 \\
& z = (z_1, z_2) \mapsto (\det z_1 + \alpha \operatorname{tr}(z_2 x_\infty), \det z_2 + \beta \operatorname{tr}(z_1 y_\infty))
\end{aligned}
$$

where $x_\infty := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $y_\infty := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Deformations of $X_{(x,y)}$

Deform the good slice

$$\mathcal{S}_{(x,y)} = \{z \in \mathfrak{g} \mid \langle z, x\rangle = \langle z, y\rangle = 0\}$$

by $(\gamma, \delta, \varepsilon) \in \mathbb{C}^3$ as:

$$\mathcal{S}_{(x,y)}(\gamma, \delta, \varepsilon) := \{z \in \mathfrak{g} \mid \langle z, x\rangle + \gamma\langle z, (x_s, 0)\rangle = \delta,\ \langle z, y\rangle = \varepsilon\}.$$

Deformations of $X_{(x,y)}$

Theorem

Let $B := \mathbb{C}^2 \times \mathbb{C}^3 \times \mathfrak{h}/W \cong \mathbb{C}^7$ be the base space.

Let $\mathcal{X}$ be the total space:

$\mathcal{X} := \{(z, \alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \mu) \in \mathfrak{g} \times B \mid f_{(\alpha,\beta)}(z) = (\lambda, \mu), z \in \mathcal{S}_{(x,y)}(\gamma, \delta, \varepsilon)\}$

Then $\pi : (\mathcal{X}, 0) \to (B, 0)$ gives a semi-universal deformation of $(X_{(x,y)}, 0)$ for general x, y (for $pq \neq 0, 1/4$).

Remark

Let $\mathfrak{g}_m = \mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C}) \oplus \cdots \oplus \mathrm{sl}(2,\mathbb{C})$.

Take $\mathcal{S}$ such that $(\mathcal{N}(\mathfrak{g}_m) \cap \mathcal{S}, 0)$ is an isolated surface singularity.

$$BW(m) := dim T^1(\mathcal{N}(\mathfrak{g}_m) \cap \mathcal{S}) = \sum_{i=3}^{m+2} \binom{m+2}{i} \binom{i-1}{2}$$
$$= 2^{m-1}(m^2 - m + 2) - 1$$

The sequence $BW(m)$ is called Björner-Welker sequence.

We can't construct deformations by our method!

Remark

Let $\mathfrak{g} = \mathrm{sl}(n_1, \mathbb{C}) \oplus \mathrm{sl}(n_2, \mathbb{C}) \oplus \cdots \oplus \mathrm{sl}(n_r, \mathbb{C})$.

For a transversal slice, the minimal resolution of the corresponding

singularity consists of an exceptional curve **E** with genus

$$g = [\{\sum_{i=1}^{r} n_i(n_i - 1)/2\} - 2](n_1)!(n_2)! \cdots (n_r)!/2 + 1$$

and with the self-intersection $\mathbf{E}^2 = -(\mathbf{n_1})!(\mathbf{n_2})! \cdots (\mathbf{n_r})!$.