Differential invariance of the multiplicity of real and complex analytic sets

José Edson Sampaio

Departamento de Matemática
Universidade Federal do Ceará

December 17, 2021

Outline

Initial considerations

Initial considerations

1. Essentially, the results presented here are in the preprint titled "Multiplicity, regularity and blow-spherical equivalence of real analytic sets", to appear in Math. Z.

Initial considerations

1. Essentially, the results presented here are in the preprint titled "Multiplicity, regularity and blow-spherical equivalence of real analytic sets", to appear in Math. Z.
2. You can find this preprint and others in my ResearchGate's webpage:
   `https://www.researchgate.net/profile/Jose-Edson-Sampaio`

Initial considerations

1. Essentially, the results presented here are in the preprint titled "Multiplicity, regularity and blow-spherical equivalence of real analytic sets", to appear in Math. Z.
2. You can find this preprint and others in my ResearchGate's webpage:

   `https://www.researchgate.net/profile/Jose-Edson-Sampaio`
3. And in my homepage:

   `https://sites.google.com/mat.ufc.br/edsonsampaio/`

Outline

1. Introduction and motivation

2. Main results

3. Preliminaries

4. Generalizations of Gau-Lipman's theorem
   - Complex case
   - Real case

Zariski's Multiplicity Conjecture

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions. If there is a homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, then $m(V(f), 0) = m(V(g), 0)$.

Invariance of the multiplicity in the differentiable case

Invariance of the multiplicity in the differentiable case

Ephraim-Trotman's theorem (1976)

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions. If there is a C^1 diffeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, then $m(V(f), 0) = m(V(g), 0)$.

Invariance of the multiplicity in the differentiable case

Ephraim-Trotman's theorem (1976)

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions. If there is a C^1 diffeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, then $m(V(f), 0) = m(V(g), 0)$.

Gau-Lipman's theorem (1981)

Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets. If there is a homeomorphism $\varphi\colon (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ such that φ and φ^{-1} have derivatives at 0, then $m(X, 0) = m(Y, 0)$.

Metric versions of Zariski's Multiplicity Conjecture

Metric versions of Zariski's Multiplicity Conjecture

Metric version of ZM Conjecture

Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $\dim X = \dim Y = d$. If their germs at zero are bi-Lipschitz homeomorphic, then their multiplicities $m(X,0)$ and $m(Y,0)$ are equal.

Metric versions of Zariski's Multiplicity Conjecture

Metric version of ZM Conjecture

Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $\dim X = \dim Y = d$. If their germs at zero are bi-Lipschitz homeomorphic, then their multiplicities $m(X,0)$ and $m(Y,0)$ are equal.

Global metric version of ZM Conjecture

Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex algebraic sets with $\dim X = \dim Y = d$. If X and Y are bi-Lipschitz homeomorphic at infinity, then $\deg(X) = \deg(Y)$.

Some answers to metric versions of ZM Conjecture

1

2

3

Some answers to metric versions of ZM Conjecture

- These two metric versions of Zariski's Multiplicity Conjecture are equivalent and hold true if $d = 1$ or 2 (Bobadilla, Fernandes and S. (2018)[1]);

[1] Bobadilla, J.F. de; Fernandes, A. and Sampaio, J. E. *Multiplicity and degree as bi-lipschitz invariants for complex sets*. Journal of Topology, vol. 11 (2018), 958-966.

[2]

[3]

Some answers to metric versions of ZM Conjecture

- These two metric versions of Zariski's Multiplicity Conjecture are equivalent and hold true if $d = 1$ or 2 (Bobadilla, Fernandes and S. (2018)[1]);
- They hold true when the involved Lipschitz constants are close enough to 1 (Comte (1998)[2]);

[1] Bobadilla, J.F. de; Fernandes, A. and Sampaio, J. E. *Multiplicity and degree as bi-lipschitz invariants for complex sets*. Journal of Topology, vol. 11 (2018), 958-966.

[2] Comte, G. *Multiplicity of complex analytic sets and bi-Lipschitz maps*. Real analytic and algebraic singularities (Nagoya/Sapporo/Hachioji, 1996) Pitman Res. Notes Math. Ser., vol. 381 (1998), 182-188.

[3]

Some answers to metric versions of ZM Conjecture

- These two metric versions of Zariski's Multiplicity Conjecture are equivalent and hold true if $d = 1$ or 2 (Bobadilla, Fernandes and S. (2018)[1]);
- They hold true when the involved Lipschitz constants are close enough to 1 (Comte (1998)[2]);
- They have negative answers if $d > 2$ (Birbrair, Fernandes, S., Verbitsky (2020)[3]).

[1] Bobadilla, J.F. de; Fernandes, A. and Sampaio, J. E. *Multiplicity and degree as bi-lipschitz invariants for complex sets*. Journal of Topology, vol. 11 (2018), 958-966.

[2] Comte, G. *Multiplicity of complex analytic sets and bi-Lipschitz maps*. Real analytic and algebraic singularities (Nagoya/Sapporo/Hachioji, 1996) Pitman Res. Notes Math. Ser., vol. 381 (1998), 182-188.

[3] Birbrair, L.; Fernandes, A.; Sampaio, J. E. and Verbitsky, M. *Multiplicity of singularities is not a bi-Lipschitz invariant*. Math. Ann., vol. 377 (2020), 115-121.

Fukui-Kurdyka-Paunescu Conjecture

Fukui-Kurdyka-Paunescu Conjecture

Fukui-Kurdyka-Paunescu conjecture

Let $X, Y \subset \mathbb{R}^n$ be two germs at the origin of irreducible real analytic subsets. If $h\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the germ of a subanalytic, arc-analytic and bi-Lipschitz homeomorphism such that $h(X) = Y$, then $m(X, 0) \equiv m(Y, 0) \bmod 2$.

Partial answers

4

5

6

7

8

Partial answers

- It is true for $n = 2$ (Risler (2001)[4]);

[4] Risler, J.-J. *Invariant Curves and Topological Invariants for Real Plane Analytic Vector Fields*. Journal of Differential Equations, vol. 172 (2001), 212–226.

5

6

7

8

Partial answers

- It is true for $n = 2$ (Risler (2001)[4]);
- It is true for $n = 2$ even with equality without mod 2 (Fukui, Kurdyka and Paunescu (2004)[5]);

[4]Risler, J.-J. *Invariant Curves and Topological Invariants for Real Plane Analytic Vector Fields*. Journal of Differential Equations, vol. 172 (2001), 212–226.

[5]Fukui, T.; Kurdyka, K. and Paunescu, L. *An inverse mapping theorem for arc-analytic homeomorphisms.* Geometric singularity theory, pp. 49–56, Banach Center Publ., 65, Polish Acad. Sci. Inst. Math., Warsaw, 2004.

6

7

8

Partial answers

- It is true for $n = 2$ (Risler (2001)[4]);
- It is true for $n = 2$ even with equality without mod 2 (Fukui, Kurdyka and Paunescu (2004)[5]);
- It is true for hypersurfaces, i.e., $\dim X = \dim Y = n - 1$ (Valette (2010)[6]);

[4] Risler, J.-J. *Invariant Curves and Topological Invariants for Real Plane Analytic Vector Fields*. Journal of Differential Equations, vol. 172 (2001), 212–226.

[5] Fukui, T.; Kurdyka, K. and Paunescu, L. *An inverse mapping theorem for arc-analytic homeomorphisms.* Geometric singularity theory, pp. 49–56, Banach Center Publ., 65, Polish Acad. Sci. Inst. Math., Warsaw, 2004.

[6] Valette, G. *Multiplicity mod 2 as a metric invariant*. Discrete Comput. Geom., vol. 43 (2010), 663-679.

[7]

[8]

Partial answers

- It is true for $n = 2$ (Risler (2001)[4]);
- It is true for $n = 2$ even with equality without mod 2 (Fukui, Kurdyka and Paunescu (2004)[5]);
- It is true for hypersurfaces, i.e., $\dim X = \dim Y = n - 1$ (Valette (2010)[6]);
- It is true for the case that h and h^{-1} are differentiable at 0 (S. (2021)[7]);

[4] Risler, J.-J. *Invariant Curves and Topological Invariants for Real Plane Analytic Vector Fields*. Journal of Differential Equations, vol. 172 (2001), 212–226.

[5] Fukui, T.; Kurdyka, K. and Paunescu, L. *An inverse mapping theorem for arc-analytic homeomorphisms.* Geometric singularity theory, pp. 49–56, Banach Center Publ., 65, Polish Acad. Sci. Inst. Math., Warsaw, 2004.

[6] Valette, G. *Multiplicity mod 2 as a metric invariant*. Discrete Comput. Geom., vol. 43 (2010), 663-679.

[7] Sampaio, J. E. *Multiplicity, regularity and blow-spherical equivalence of real analytic sets*. Accepted for a publication in Math. Z., arXiv:2105.09769 [math.AG].

[8]

Partial answers

- It is true for $n = 2$ (Risler (2001)[4]);
- It is true for $n = 2$ even with equality without mod 2 (Fukui, Kurdyka and Paunescu (2004)[5]);
- It is true for hypersurfaces, i.e., $\dim X = \dim Y = n - 1$ (Valette (2010)[6]);
- It is true for the case that h and h^{-1} are differentiable at 0 (S. (2021)[7]);
- It is true in general (Fernandes, Jelonek, S. (2021)[8]).

[4] Risler, J.-J. *Invariant Curves and Topological Invariants for Real Plane Analytic Vector Fields*. Journal of Differential Equations, vol. 172 (2001), 212–226.

[5] Fukui, T.; Kurdyka, K. and Paunescu, L. *An inverse mapping theorem for arc-analytic homeomorphisms.* Geometric singularity theory, pp. 49–56, Banach Center Publ., 65, Polish Acad. Sci. Inst. Math., Warsaw, 2004.

[6] Valette, G. *Multiplicity mod 2 as a metric invariant*. Discrete Comput. Geom., vol. 43 (2010), 663-679.

[7] Sampaio, J. E. *Multiplicity, regularity and blow-spherical equivalence of real analytic sets*. Accepted for a publication in Math. Z., arXiv:2105.09769 [math.AG].

[8] Fernandes, A.; Zbigniew, J. and Sampaio, J. E. *On the Fukui-Kurdyka-Paunescu Conjecture*, arXiv:2108.01179 [math.AG].

Outline

Generalization of Gau-Lipman's theorem

Generalization of Gau-Lipman's theorem

Theorem (S. '21)

Let $(X,0),(Y,0) \subset (\mathbb{C}^n,0)$ be germs of complex analytic sets and let $h\colon (X,0) \to (Y,0)$ be a homeomorphism which is blow-spherical differentiable at 0. Then $m(X,0) = m(Y,0)$.

Generalization of Gau-Lipman's theorem

Theorem (S. '21)

Let $(X,0),(Y,0)\subset(\mathbb{C}^n,0)$ be germs of complex analytic sets and let $h\colon(X,0)\to(Y,0)$ be a homeomorphism which is blow-spherical differentiable at 0. Then $m(X,0)=m(Y,0)$.

Corollary (Gau-Lipman '83)

Let $(X,0),(Y,0)\subset(\mathbb{C}^n,0)$ be germs of complex analytic sets and let $h\colon(\mathbb{C}^n,X,0)\to(\mathbb{C}^n,Y,0)$ be a homeomorphism such that h and h^{-1} are differentiable at 0. Then $m(X,0)=m(Y,0)$.

Real version of Gau-Lipman's theorem

Real version of Gau-Lipman's theorem

Theorem (S. '21)

Let $(X,0) \subset (\mathbb{R}^n,0), (Y,0) \subset (\mathbb{R}^m,0)$ be germs of real analytic sets and let $h\colon (X,0) \to (Y,0)$ be a homeomorphism which is blow-spherical differentiable at 0. Then $m(X,0) \equiv m(Y,0) \bmod 2$.

Real version of Gau-Lipman's theorem

Theorem (S. '21)

Let $(X,0) \subset (\mathbb{R}^n,0), (Y,0) \subset (\mathbb{R}^m,0)$ be germs of real analytic sets and let $h\colon (X,0) \to (Y,0)$ be a homeomorphism which is blow-spherical differentiable at 0. Then $m(X,0) \equiv m(Y,0) \mod 2$.

Corollary (S. '21)

Let $(X,0) \subset (\mathbb{R}^n,0), (Y,0) \subset (\mathbb{R}^m,0)$ be germs of real analytic sets and let $h\colon (X,0) \to (Y,0)$ be a homeomorphism such that h and h^{-1} are differentiable at 0. Then $m(X,0) \equiv m(Y,0) \mod 2$.

Outline

Tangent cone

Definition

Let $X \subset \mathbb{R}^n$ be a subset and let $p \in X$. We say that $v \in \mathbb{R}^n$ is **a tangent vector** of X at p if there is sequences $\{t_j\}_{j\in\mathbb{N}}$ of positive real numbers and $\{x_j\}_{j\in\mathbb{N}} \subset X$ such that $\lim\limits_{j\to\infty} t_j = 0$ and $\lim \frac{x_n - p}{t_n} = v$. We denote by $C(X,p)$ the set of all the tangent vectors of X at p and we call it **the tangent cone of** X **at** p.

Tangent cone

Definition

Let $X \subset \mathbb{R}^n$ be a subset and let $p \in X$. We say that $v \in \mathbb{R}^n$ is **a tangent vector** of X at p if there is sequences $\{t_j\}_{j\in\mathbb{N}}$ of positive real numbers and $\{x_j\}_{j\in\mathbb{N}} \subset X$ such that $\lim\limits_{j\to\infty} t_j = 0$ and $\lim \frac{x_n - p}{t_n} = v$. We denote by $C(X,p)$ the set of all the tangent vectors of X at p and we call it **the tangent cone of X at p**.

Definition

The **strict transform** of the subset X under the spherical blowing-up $\beta_n : \mathbb{S}^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n$ (given by $\beta_n(x,r) = rx$) is $X' := \overline{\beta_n^{-1}(X \setminus \{0\})}$ and the **boundary** $\partial X'$ of the **strict transform** is $\partial X' := X' \cap (\mathbb{S}^{n-1} \times \{0\})$.

Tangent cone

Definition

Let $X \subset \mathbb{R}^n$ be a subset and let $p \in X$. We say that $v \in \mathbb{R}^n$ is **a tangent vector** of X at p if there is sequences $\{t_j\}_{j\in\mathbb{N}}$ of positive real numbers and $\{x_j\}_{j\in\mathbb{N}} \subset X$ such that $\lim\limits_{j\to\infty} t_j = 0$ and $\lim \frac{x_n - p}{t_n} = v$. We denote by $C(X,p)$ the set of all the tangent vectors of X at p and we call it **the tangent cone of** X **at** p.

Definition

The **strict transform** of the subset X under the spherical blowing-up $\beta_n : \mathbb{S}^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n$ (given by $\beta_n(x,r) = rx$) is $X' := \overline{\beta_n^{-1}(X \setminus \{0\})}$ and the **boundary** $\partial X'$ of the **strict transform** is $\partial X' := X' \cap (\mathbb{S}^{n-1} \times \{0\})$.

Note that $\partial X' = C_X \times \{0\}$, where $C_X = C(X,0) \cap \mathbb{S}^{n-1}$.

Simple points

Definition

Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{X}$ is a non-isolated point. We say that $x \in \partial X'$ is **a simple point of** $\partial X'$, if there is an open $U \subset \mathbb{R}^{n+1}$ with $x \in U$ such that:

a) the connected components of $(X' \cap U) \setminus \partial X'$, say $X_1, ..., X_r$, are topological manifolds with $\dim X_i = \dim X$, $i = 1, ..., r$;

b) $(X_i \cup \partial X') \cap U$ are topological manifolds with boundary.

Let $Smp(\partial X')$ be the set of simple points of $\partial X'$.

Simple points

Definition

Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{X}$ is a non-isolated point. We say that $x \in \partial X'$ is **a simple point of** $\partial X'$, if there is an open $U \subset \mathbb{R}^{n+1}$ with $x \in U$ such that:

a) the connected components of $(X' \cap U) \setminus \partial X'$, say $X_1, ..., X_r$, are topological manifolds with $\dim X_i = \dim X$, $i = 1, ..., r$;

b) $(X_i \cup \partial X') \cap U$ are topological manifolds with boundary.

Let $Smp(\partial X')$ be the set of simple points of $\partial X'$.

$\mathrm{Smp}(\partial X')$ is an open dense subset of the $(d-1)$-dimensional part of $\partial X'$ whenever $\partial X'$ is a $(d-1)$-dimensional subset, where $d = \dim X$.

Relative multiplicities

Definition

Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in X$. We define $k_X : Smp(\partial X') \to \mathbb{N}$, with $k_X(x)$ the number of connected components of the germ $(\beta_n^{-1}(X \setminus \{0\}), x)$.

Relative multiplicities

Definition

Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in X$. We define $k_X : Smp(\partial X') \to \mathbb{N}$, with $k_X(x)$ the number of connected components of the germ $(\beta_n^{-1}(X \setminus \{0\}), x)$.

k_X is constant in each connected component C_j of $Smp(\partial X')$. Then, we define $k_X(C_j) := k_X(x)$ with $x \in C_j$.

Relative multiplicities

Definition

Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in X$. We define $k_X : Smp(\partial X') \to \mathbb{N}$, with $k_X(x)$ the number of connected components of the germ $(\beta_n^{-1}(X \setminus \{0\}), x)$.

k_X is constant in each connected component C_j of $Smp(\partial X')$. Then, we define $k_X(C_j) := k_X(x)$ with $x \in C_j$.

When X is a complex analytic set, there is a complex analytic set Σ with $\dim \Sigma < \dim X$ such that for each irreducible component X_j of the tangent cone $C(X, 0)$, $X_j \setminus \Sigma$ intersects only one connected component C_i of $Smp(\partial X')$. Then, we define $k_X(X_j) := k_X(C_i)$.

Odd part of the tangent link

Definition

Let $X \subset \mathbb{R}^n$ be a real analytic set. We denote by C'_X the closure of the union of all connected components C_j of $Smp(\partial X')$ such that $k_X(C_j)$ is an odd number. We call C'_X the **odd part of** $C_X \subset \mathbb{S}^{n-1}$.

An easy example

Blow-spherical homeomorphisms

Definition

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed sets. We say that a homeomorphism $h\colon (X,0) \to (Y,0)$ is a **blow-spherical homeomorphism** if the homeomorphism $\beta_m^{-1} \circ h \circ \beta_n\colon X' \setminus \partial X' \to Y' \setminus \partial Y'$ extends to a homeomorphism $h'\colon X' \to Y'$. In this case, we denote by $\nu_h\colon C_X \to C_Y$ the homeomorphism such that $h(x,0) = (\nu_h(x),0)$ for all $x \in C_X$.

Blow-spherical differentiability

Definition

Let $X, Y \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed sets. We say that a blow-spherical homeomorphism homeomorphism $h\colon (X,0) \to (Y,0)$ is **blow-spherical differentiable at** 0 if there exists an isomorphism linear $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

Blow-spherical differentiability

Definition

Let $X, Y \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed sets. We say that a blow-spherical homeomorphism homeomorphism $h\colon (X,0) \to (Y,0)$ is **blow-spherical differentiable at** 0 if there exists an isomorphism linear $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

Example

Let $(X,0),(Y,0) \subset (\mathbb{R}^n,0)$ be germs of closed sets and let $\varphi\colon (\mathbb{R}^n, X, 0) \to (\mathbb{R}^n, Y, 0)$ be a homeomorphism such that φ and φ^{-1} are differentiable at 0. Then $h = \varphi|_X\colon (X,0) \to (Y,0)$ is blow-spherical differentiable at 0 and $\nu_h = \frac{D\varphi_0}{\|D\varphi_0\|}$.

Multiplicity

Multiplicity

Proposition

Let X be an analytic set in $\mathbb{C}^n$ with $k = \dim X$ and $0 \in X$. Then, for an orthogonal projection $\pi : \mathbb{C}^n \to \mathbb{C}^k$ such that $\pi^{-1}(0) \cap C(X,0) = \{0\}$ and for U being a sufficiently small open neighborhood of $0 \in \mathbb{C}^n$, we have that $\#\pi^{-1}(t) \cap (X \cap U)$ is constant for $t \in \pi(U) \setminus \sigma$ for some σ, which is an analytic set in $\pi(U)$.

Multiplicity

Proposition

Let X be an analytic set in $\mathbb{C}^n$ with $k = \dim X$ and $0 \in X$. Then, for an orthogonal projection $\pi : \mathbb{C}^n \to \mathbb{C}^k$ such that $\pi^{-1}(0) \cap C(X,0) = \{0\}$ and for U being a sufficiently small open neighborhood of $0 \in \mathbb{C}^n$, we have that $\#\pi^{-1}(t) \cap (X \cap U)$ is constant for $t \in \pi(U) \setminus \sigma$ for some σ, which is an analytic set in $\pi(U)$.

Definition

In this case, we define the **multiplicity of** X **at** 0 to be $m(X,0) = \#\pi^{-1}(t) \cap (X \cap U)$ for $t \in \pi(U) \setminus \sigma$.

Real multiplicity

Real multiplicity

Definition

Let $X \subset \mathbb{R}^n$ be a real analytic set with $0 \in X$ and

$$X_{\mathbb{C}} = V(\mathcal{I}_{\mathbb{R}}(X,0)),$$

where $\mathcal{I}_{\mathbb{R}}(X,0)$ is the ideal in $\mathbb{C}\{z_1,\ldots,z_n\}$ generated by the complexifications of all germs of real analytic functions that vanish on the germ $(X,0)$. We define the multiplicity of X at the origin by $m(X,0) := m(X_{\mathbb{C}},0)$.

Outline

Generalization of Gau-Lipman's theorem

Generalization of Gau-Lipman's theorem

Theorem (S. '21)

Let $(X,0),(Y,0)\subset(\mathbb{C}^n,0)$ be germs of complex analytic sets and let $h\colon(X,0)\to(Y,0)$ be a homeomorphism which is blow-spherical differentiable at 0. Then $m(X,0)=m(Y,0)$.

Key results

Key results

Proposition 1

Let $h\colon (X,0) \to (Y,0)$ be a blow-spherical homeomorphism. Then, $h'(Smp(\partial X')) = Smp(\partial Y')$ and $k_X(v) = k_Y(h'(v))$ for all $v \in Smp(\partial X')$.

Key results

Proposition 1

Let $h\colon (X,0) \to (Y,0)$ be a blow-spherical homeomorphism. Then, $h'(Smp(\partial X')) = Smp(\partial Y')$ and $k_X(v) = k_Y(h'(v))$ for all $v \in Smp(\partial X')$.

Proposition 2

Let X be a complex analytic set of $\mathbb{C}^n$ with $0 \in X$ and let $X_1, ..., X_r$ be the irreducible components of $C(X,0)$. Then

$$m(X,0) = \sum_{j=1}^{r} k_X(X_j) \cdot m(X_j,0).$$

Sketch of the proof

9

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

9

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• ϕ is a homeomorphism between $C(X,0)$ and $C(Y,0)$, which implies that ϕ maps bijectively the irreducible components of $C(X,0)$ to the irreducible components of $C(Y,0)$.

9

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

• ϕ is a homeomorphism between $C(X,0)$ and $C(Y,0)$, which implies that ϕ maps bijectively the irreducible components of $C(X,0)$ to the irreducible components of $C(Y,0)$.

• Let $X_1, ..., X_r$ and $Y_1, ..., Y_r$ be the irreducible components of $C(X,0)$ and $C(Y,0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1, ..., r$.

9

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• ϕ is a homeomorphism between $C(X,0)$ and $C(Y,0)$, which implies that ϕ maps bijectively the irreducible components of $C(X,0)$ to the irreducible components of $C(Y,0)$.
• Let $X_1, ..., X_r$ and $Y_1, ..., Y_r$ be the irreducible components of $C(X,0)$ and $C(Y,0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1, ..., r$.
• Its complexification $\phi_{\mathbb{C}}\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a $\mathbb{C}$-linear isomorphism and $\phi_{\mathbb{C}}(X_{j\mathbb{C}}) = Y_{j\mathbb{C}}$ for all $j = 1, ..., r$.

9

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• ϕ is a homeomorphism between $C(X,0)$ and $C(Y,0)$, which implies that ϕ maps bijectively the irreducible components of $C(X,0)$ to the irreducible components of $C(Y,0)$.
• Let $X_1, ..., X_r$ and $Y_1, ..., Y_r$ be the irreducible components of $C(X,0)$ and $C(Y,0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1, ..., r$.
• Its complexification $\phi_{\mathbb{C}}\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a $\mathbb{C}$-linear isomorphism and $\phi_{\mathbb{C}}(X_{j\mathbb{C}}) = Y_{j\mathbb{C}}$ for all $j = 1, ..., r$.
• For each $j \in \{1, ..., r\}$, $X_{j\mathbb{C}}$ (resp. $Y_{j\mathbb{C}}$) is complex analytic isomorphic to $X_j \times c_n(X_j)$ (resp. $Y_j \times c_n(Y_j)$).[9]

[9] Ephraim, R. *The cartesian product structure and C^∞ equivalences of singularities*. Transactions of the American Mathematical Society, vol. 224 (1976), no. 2, 299–311.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• ϕ is a homeomorphism between $C(X,0)$ and $C(Y,0)$, which implies that ϕ maps bijectively the irreducible components of $C(X,0)$ to the irreducible components of $C(Y,0)$.
• Let $X_1, ..., X_r$ and $Y_1, ..., Y_r$ be the irreducible components of $C(X,0)$ and $C(Y,0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1, ..., r$.
• Its complexification $\phi_{\mathbb{C}}\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a $\mathbb{C}$-linear isomorphism and $\phi_{\mathbb{C}}(X_{j\mathbb{C}}) = Y_{j\mathbb{C}}$ for all $j = 1, ..., r$.
• For each $j \in \{1, ..., r\}$, $X_{j\mathbb{C}}$ (resp. $Y_{j\mathbb{C}}$) is complex analytic isomorphic to $X_j \times c_n(X_j)$ (resp. $Y_j \times c_n(Y_j)$).[9]
• $m(X_j,0)^2 = m(X_j \times c_n(X_j),0) = m(Y_j \times c_n(Y_j),0) = m(Y_j,0)^2$.

[9] Ephraim, R. *The cartesian product structure and C^∞ equivalences of singularities*. Transactions of the American Mathematical Society, vol. 224 (1976), no. 2, 299–311.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• ϕ is a homeomorphism between $C(X,0)$ and $C(Y,0)$, which implies that ϕ maps bijectively the irreducible components of $C(X,0)$ to the irreducible components of $C(Y,0)$.
• Let $X_1, ..., X_r$ and $Y_1, ..., Y_r$ be the irreducible components of $C(X,0)$ and $C(Y,0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1, ..., r$.
• Its complexification $\phi_{\mathbb{C}}\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a $\mathbb{C}$-linear isomorphism and $\phi_{\mathbb{C}}(X_{j\mathbb{C}}) = Y_{j\mathbb{C}}$ for all $j = 1, ..., r$.
• For each $j \in \{1, ..., r\}$, $X_{j\mathbb{C}}$ (resp. $Y_{j\mathbb{C}}$) is complex analytic isomorphic to $X_j \times c_n(X_j)$ (resp. $Y_j \times c_n(Y_j)$).[9]
• $m(X_j,0)^2 = m(X_j \times c_n(X_j),0) = m(Y_j \times c_n(Y_j),0) = m(Y_j,0)^2$.
• However, by Proposition 1, $k_X(X_j) = k_Y(Y_j)$ for all $j = 1, ..., r$. Thus,

[9] Ephraim, R. *The cartesian product structure and C^∞ equivalences of singularities*. Transactions of the American Mathematical Society, vol. 224 (1976), no. 2, 299–311.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

• ϕ is a homeomorphism between $C(X,0)$ and $C(Y,0)$, which implies that ϕ maps bijectively the irreducible components of $C(X,0)$ to the irreducible components of $C(Y,0)$.

• Let $X_1, ..., X_r$ and $Y_1, ..., Y_r$ be the irreducible components of $C(X,0)$ and $C(Y,0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1, ..., r$.

• Its complexification $\phi_{\mathbb{C}}\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a $\mathbb{C}$-linear isomorphism and $\phi_{\mathbb{C}}(X_{j\mathbb{C}}) = Y_{j\mathbb{C}}$ for all $j = 1, ..., r$.

• For each $j \in \{1, ..., r\}$, $X_{j\mathbb{C}}$ (resp. $Y_{j\mathbb{C}}$) is complex analytic isomorphic to $X_j \times c_n(X_j)$ (resp. $Y_j \times c_n(Y_j)$).[9]

• $m(X_j,0)^2 = m(X_j \times c_n(X_j),0) = m(Y_j \times c_n(Y_j),0) = m(Y_j,0)^2$.

• However, by Proposition 1, $k_X(X_j) = k_Y(Y_j)$ for all $j = 1, ..., r$. Thus,

$$\sum_{j=1}^{r} k_X(X_j) \cdot m(X_j,0) = \sum_{j=1}^{r} k_Y(Y_j) \cdot m(Y_j,0)$$

---

[9] Ephraim, R. *The cartesian product structure and C^∞ equivalences of singularities*. Transactions of the American Mathematical Society, vol. 224 (1976), no. 2, 299–311.

Gau-Lipman' theorem and an example

Corollary (Gau-Lipman '83)

Let $(X,0),(Y,0)\subset(\mathbb{C}^n,0)$ be germs of complex analytic sets and let $h\colon(\mathbb{C}^n,X,0)\to(\mathbb{C}^n,Y,0)$ be a homeomorphism such that h and h^{-1} are differentiable at 0. Then $m(X,0)=m(Y,0)$.

Gau-Lipman' theorem and an example

Corollary (Gau-Lipman '83)

Let $(X,0),(Y,0)\subset(\mathbb{C}^n,0)$ be germs of complex analytic sets and let $h\colon(\mathbb{C}^n,X,0)\to(\mathbb{C}^n,Y,0)$ be a homeomorphism such that h and h^{-1} are differentiable at 0. Then $m(X,0)=m(Y,0)$.

Example

Let $X=\{(x,y)\in\mathbb{C}^2;y^2=x^3\}$ and $Y=\{(x,y)\in\mathbb{C}^2;y^2=x^5\}$. There exists a blow-spherical homeomorphism $h\colon X\to Y$ such that $\nu_h=id_{C_X}$. Thus, h is a blow-spherical differentiable mapping. However, there is no homeomorphism $\varphi\colon(\mathbb{C}^2,X,0)\to(\mathbb{C}^2,Y,0)$.

Real version of Gau-Lipman's theorem

Real version of Gau-Lipman's theorem

Theorem (S. '21)

Let $(X,0) \subset (\mathbb{R}^n,0), (Y,0) \subset (\mathbb{R}^m,0)$ be germs of real analytic sets and let $h\colon (X,0) \to (Y,0)$ be a homeomorphism which is blow-spherical differentiable at 0. Then $m(X,0) \equiv m(Y,0) \bmod 2$.

Key results

Key results

Proposition 3

Let $h\colon (X,0) \to (Y,0)$ be a blow-spherical homeomorphism. Then, $h'(C'_X) = C'_Y$.

Key results

Proposition 3

Let $h\colon (X,0) \to (Y,0)$ be a blow-spherical homeomorphism. Then, $h'(C'_X) = C'_Y$.

Proposition 4

Let $X \subset \mathbb{R}^n$ be a $d-$dimensional real analytic set and $0 \in X$. Then, $\deg(C'_X)$ is defined and $\deg(C'_X) \equiv m(X) \mathrm{mod}\ 2$.

Sketch of the proof

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• $A = \phi(X)$ is a real analytic set such that $m(X, 0) = m(A, 0)$.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

• $A = \phi(X)$ is a real analytic set such that $m(X,0) = m(A,0)$.

• Thus, it is enough to show that $m(Y,0) \equiv m(A,0) \bmod 2$.

Sketch of the proof

- There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
- $A = \phi(X)$ is a real analytic set such that $m(X,0) = m(A,0)$.
- Thus, it is enough to show that $m(Y,0) \equiv m(A,0) \mod 2$.
- Consider the mapping $\psi\colon (Y,0) \to (A,0)$ given by $\psi = \phi \circ \varphi^{-1}$.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• $A = \phi(X)$ is a real analytic set such that $m(X,0) = m(A,0)$.
• Thus, it is enough to show that $m(Y,0) \equiv m(A,0) \bmod 2$.
• Consider the mapping $\psi\colon (Y,0) \to (A,0)$ given by $\psi = \phi \circ \varphi^{-1}$.
• ψ is a blow-spherical homeomorphism such that $\nu_\psi = id$. In particular, by Proposition 3, $C'_Y = \psi'(C'_Y) = C'_A$.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.

• $A = \phi(X)$ is a real analytic set such that $m(X,0) = m(A,0)$.

• Thus, it is enough to show that $m(Y,0) \equiv m(A,0) \bmod 2$.

• Consider the mapping $\psi\colon (Y,0) \to (A,0)$ given by $\psi = \phi \circ \varphi^{-1}$.

• ψ is a blow-spherical homeomorphism such that $\nu_\psi = id$. In particular, by Proposition 3, $C'_Y = \psi'(C'_Y) = C'_A$.

• By Proposition 4, $\deg(C'_Y) \equiv m(Y,0) \text{mod } 2$ and $\deg(C'_A) \equiv m(A,0) \text{mod } 2$.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• $A = \phi(X)$ is a real analytic set such that $m(X,0) = m(A,0)$.
• Thus, it is enough to show that $m(Y,0) \equiv m(A,0) \bmod 2$.
• Consider the mapping $\psi\colon (Y,0) \to (A,0)$ given by $\psi = \phi \circ \varphi^{-1}$.
• ψ is a blow-spherical homeomorphism such that $\nu_\psi = id$. In particular, by Proposition 3, $C'_Y = \psi'(C'_Y) = C'_A$.
• By Proposition 4, $\deg(C'_Y) \equiv m(Y,0)\text{mod } 2$ and $\deg(C'_A) \equiv m(A,0)\text{mod } 2$.
• Since $C'_Y = C'_A$, $\deg(C'_Y) \equiv \deg(C'_A)\text{mod } 2$.

Sketch of the proof

• There exists an $\mathbb{R}$-linear isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_h(x) = \frac{\phi(x)}{\|\phi(x)\|}$ for all $x \in C_X$.
• $A = \phi(X)$ is a real analytic set such that $m(X,0) = m(A,0)$.
• Thus, it is enough to show that $m(Y,0) \equiv m(A,0) \bmod 2$.
• Consider the mapping $\psi\colon (Y,0) \to (A,0)$ given by $\psi = \phi \circ \varphi^{-1}$.
• ψ is a blow-spherical homeomorphism such that $\nu_\psi = id$. In particular, by Proposition 3, $C'_Y = \psi'(C'_Y) = C'_A$.
• By Proposition 4, $\deg(C'_Y) \equiv m(Y,0) \text{mod } 2$ and $\deg(C'_A) \equiv m(A,0) \text{mod } 2$.
• Since $C'_Y = C'_A$, $\deg(C'_Y) \equiv \deg(C'_A) \text{mod } 2$.
• Therefore $m(Y,0) \equiv m(A,0) \bmod 2$.

Real case

Real case

Corollary (S. '21)

Let $(X,0) \subset (\mathbb{R}^n,0), (Y,0) \subset (\mathbb{R}^m,0)$ be germs of real analytic sets and let $h\colon (X,0) \to (Y,0)$ be a homeomorphism such that h and h^{-1} are differentiable at 0. Then $m(X,0) \equiv m(Y,0) \bmod 2$.

Final examples

The assumption that $D\varphi_0$ is an isomorphism cannot be removed.

Final examples

The assumption that $D\varphi_0$ is an isomorphism cannot be removed.

Example

Let $X = \{(x, y) \in \mathbb{R}^2;\ y^3 = x^2\}$ and $Y = \{(x, y) \in \mathbb{R}^2;\ y = 0\}$. Then $\varphi\colon (\mathbb{R}^2, X, 0) \to (\mathbb{R}^2, Y, 0)$ given by $\varphi(x, y) = (x, y^3 - x^2)$ is a homeomorphism, which has a derivative at the origin, but $D\varphi_0$ is not an isomorphism. In this case, $m(X, 0) = 2$ and $m(Y, 0) = 1$.

Final examples

The assumption that $D\varphi_0$ is an isomorphism cannot be removed.

Example

Let $X = \{(x,y) \in \mathbb{R}^2;\, y^3 = x^2\}$ and $Y = \{(x,y) \in \mathbb{R}^2;\, y = 0\}$. Then $\varphi\colon (\mathbb{R}^2, X, 0) \to (\mathbb{R}^2, Y, 0)$ given by $\varphi(x,y) = (x, y^3 - x^2)$ is a homeomorphism, which has a derivative at the origin, but $D\varphi_0$ is not an isomorphism. In this case, $m(X,0) = 2$ and $m(Y,0) = 1$.

We cannot obtain equality (without modulo 2).

Final examples

The assumption that $D\varphi_0$ is an isomorphism cannot be removed.

Example

Let $X = \{(x, y) \in \mathbb{R}^2;\ y^3 = x^2\}$ and $Y = \{(x, y) \in \mathbb{R}^2;\ y = 0\}$. Then $\varphi\colon (\mathbb{R}^2, X, 0) \to (\mathbb{R}^2, Y, 0)$ given by $\varphi(x, y) = (x, y^3 - x^2)$ is a homeomorphism, which has a derivative at the origin, but $D\varphi_0$ is not an isomorphism. In this case, $m(X, 0) = 2$ and $m(Y, 0) = 1$.

We cannot obtain equality (without modulo 2).

Example

Let $V = \{(x, y, z) \in \mathbb{R}^3;\ z^3 = x^5 y + x y^5\}$. Then the mapping $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $\varphi(x, y, z) = (x, y, z - (x^5 y + x y^5)^{\frac{1}{3}})$ is a homeomorphism which has a derivative at the origin and its inverse has also a derivative at the origin. Moreover, $\varphi(V) = \mathbb{R}^2 \times \{0\}$, but $m(V, 0) = 3$ and $m(\mathbb{R}^2 \times \{0\}, 0) = 1$.

Thanks!