Newton Polyhedra and Whitney Equisingularity For Determinantal Singularities

Thaís Maria Dalbelo - UFSCar
Joint work with Luiz Hartmann and Maicom Varella

thaisdalbelo@ufscar.br

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* Consider $M_{n,k}$ the set of complex matrices of size $n \times k$;

* $M^s_{n,k}$ the set of matrices with rank less than s, $0 < s \leq n \leq k$;

* $M^s_{n,k}$ is an irreducible variety of $M_{n,k}$ with codimension $(n-s+1)(k-s+1)$ and it is called *generic determinantal variety*;

* Let $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a holomorphic map germ defined by $A(x) = (a_{ij})(x)$ with $a_{ij} \in \mathcal{O}_m$.

Definition

Let $([A], 0) \subset (\mathbb{C}^m, 0)$ be the germ of the variety defined by $[A] = A^{-1}(M^s_{n,k})$. We say that $[A]$ is a *determinantal variety* of type $(n, k; s)$ in $(\mathbb{C}^m, 0)$ if its dimension is equal to

$$m - (n-s+1)(k-s+1).$$

The analytical structure of $[A]$ is the one defined by A and $M^s_{n,k}$, i.e., given by the minors of size s of $A(x)$. Consequently,

$$[A] = \{x \in \mathbb{C}^m : \det M_I(x) = 0,\ I = 1, \dots, C^n_s \cdot C^k_s\}.$$

When $s = 1$, $[A]$ is a complete intersection variety.

Definition

Let $([A], 0) \subset (\mathbb{C}^m, 0)$ be a determinantal variety of type $(n, k; s)$ satisfying the condition

$$s = 1 \text{ or } m < (n - s - 2)(k - s - 2).$$

The variety $[A]$ is said to be an *isolated determinantal singularity* (IDS) if $[A]$ is smooth at x and $\mathrm{rank}\, A(x) = s - 1$ for all $x \neq 0$ in a neighbourhood of the origin.

Vanishing Euler characteristic

* Pereira and Ruas;
* Nuño-Ballesteros, Oréfice-Okamoto and Tomazella.
* $A_B : \mathbb{C}^m \to M_{n,k}$, $A_B(x) = A(x) + B$, where $B \in M_{n,k}$.

Definition

The *vanishing Euler characteristic* of an IDS $[A]$, denoted by $\nu([A], 0)$, is defined as

$$\nu([A], 0) = (-1)^d (\chi([A_B]) - 1),$$

where $B \in M_{n,k}$ is a matrix such that $[A_B]$ is a determinantal smoothing of $[A]$ and $d = \dim\, [A]$.

* $m_d([A], 0) = \nu([A], 0) + \nu([A^p], 0)$, where $p : \mathbb{C}^m \to \mathbb{C}$ is a generic linear function, $[A^p] = [A] \cap p^{-1}(0)$ and $d = dim[A]$.

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Newton Polyhedra

We denote by x^a the monomial $x_1^{a_1} \cdots x_m^{a_m} \in \mathbb{C}^m$, where $a = (a_1, \cdots, a_m) \in \mathbb{Z}^m$.

Definition

The **Newton polyhedron** Δ_f of a germ of a function $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$, given by

$$f(x) = \sum_{a \in \mathbb{Z}^m} c_a x^a$$

is the convex hull of the union $\bigcup_{a : c_a \neq 0} (a + \mathbb{R}_+^m)$. The coefficient c_a is said to be a **leading coefficient** of the power series f, if a is contained in a bounded face of the Newton Polyhedron Δ_f.

We denote by f^l the lowest order non-zero l-quasihomogeneous component of f, where $l = (l_1, \dots, l_m)$ is a collection of positive weights, assigned to the variables $x_1, \dots, x_m$.

Definition

Let $A = (a_{ij}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a matrix with polynomial entries. The leading coefficients of A are said to be *in general position*, if, for every collection l of positive weights and every subset $\mathcal{I} \subset \{1, \dots, n\}$, the set of all points $x \in (\mathbb{C} \setminus 0)^m$, such that the matrix

$$(a^l_{ij}(x))_{i \in \mathcal{I},\, j \in \{1,\dots,k\}}$$

is degenerate, has the maximal possible codimension $k - |\mathcal{I}| + 1$.

If the leading coefficients of A are in general position, then the set of all $x \in (\mathbb{C} \setminus 0)^m$ such that, the rank of the matrix

$$(a^l_{ij}(x))_{i\in\{1,\dots,n\},\, j\in\{1,\dots,k\}}$$

is less than n, is a determinantal set, since it has codimension $k - n + 1$.

Definition

The leading coefficients of A are said to be *in strong general position*, if, for every collection l of positive weights, the polynomial matrix (a^l_{ij}) defines a nonsingular determinantal set in $(\mathbb{C} \setminus 0)^m$. In this case, the leading coefficients of a germ a 1-form w are said to be *in general position with respect to* A, if, for every collection l of positive weights, the restriction of w^l to the determinantal set, defined by the matrix $(a^l_{ij}(x))$ in $(\mathbb{C} \setminus 0)^m$, has no zeros.

Theorem (Esterov, 2007)

Let Δ_j be the Newton polyhedron of each entry of the polynomial matrix $A = (a_{ij}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ and $[A] = A^{-1}(M^n_{n,k})$. If $\mathbb{R}^m_+ \setminus \Delta_j$ is bounded for all $j = 1, \ldots, k$ and the leading coefficients of A are in general position, then A defines the germ of a determinantal singularity $[A]$, whose multiplicity is

$$m([A], 0) = \sum_{0 < j_0 < \cdots < j_{k-n} \leq k} m! \cdot MV(\tilde{\Delta}_{j_0}, \cdots, \tilde{\Delta}_{j_{k-n}}, \underbrace{L, \cdots, L}_{m-k+n-1}),$$

where L denotes the standard m-dimensional simplex and $\tilde{\Delta} = \mathbb{R}^m_+ \setminus \Delta$.

Example

Let $A : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$ be the germ given by the matrix

$$A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}.$$

None of the entries of A have Newton polyhedron Δ_j, with $\tilde{\Delta}_j$ bounded.

Example

Let $A : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$ be the germ given by the matrix

$$A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}.$$

None of the entries of A have Newton polyhedron Δ_j, with $\tilde{\Delta}_j$ bounded. Through a series of elementary operations on A, we obtain the germ

$$\tilde{A} = \begin{bmatrix} x+y+z+w & x+2y+z+2w & 2x+2y+z+w \\ x+2y+2z+w & x+3y+2z+3w & 2x+4y+2z+w \end{bmatrix}$$

The Newton polyhedron Δ_j of all entries of $\tilde{A}$ have $\tilde{\Delta}_j$ bounded, furthermore, all Δ_j are the same.

* $R_i \leftrightarrow R_i + R_j$ $(C_i \leftrightarrow C_i + C_j)$;
* $R_i \leftrightarrow R_j$ $(C_i \leftrightarrow C_j)$;
* $R_i \leftrightarrow h \cdot R_i$ $(C_i \leftrightarrow h \cdot C_i)$.

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Proposition

Let $A, \tilde{A} : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be germs of matrices such that $\tilde{A}$ is obtained from A by a finite number of elementary operations between rows and columns. Then $[\tilde{A}] = [A]$.

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Proposition

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Definition

The germs of matrices $A, \tilde{A}$ are said to be *elementary equivalent* if they can be obtained from each other using the elementary operations.

Proposition: (____, Hartmann, Varella)

Let $A, \tilde{A} : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be germs of elementary equivalent matrices. If $[A]$ is an IDS, then $[A]$ and $[\tilde{A}]$ have the same smoothing and, consequently,

$$\nu([A], 0) = \nu([\tilde{A}], 0).$$

* $A_B : \mathbb{C}^m \to M_{n,k}$, $A_B(x) = A(x) + B$, where $B \in M_{n,k}$.

Definition

Let $A = (a_{ij}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a matrix with holomorphic entries. We say that $\mathcal{B} = \{f_1, \cdots, f_r\}$ is a *basis* for the germ A if $\mathcal{B}$ is linearly independent and each a_{ij} can be written as linear combination of elements of $\mathcal{B}$, where $f_s : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ for $s = 1, \cdots, r$.

Definition

Let $A = (a_{ij}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a matrix with holomorphic entries. We say that $\mathcal{B} = \{f_1, \cdots, f_r\}$ is a *basis* for the germ A if $\mathcal{B}$ is linearly independent and each a_{ij} can be written as linear combination of elements of $\mathcal{B}$, where $f_s : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ for $s = 1, \cdots, r$.

Remark

If $\mathcal{B} = \{f_1, \cdots, f_r\}$ is a basis for the germ A, then there is a germ $\tilde{A} = (\tilde{a}_{ij})$ elementary equivalent to A such that $\tilde{a}_{ij} = \sum_{s=1}^{r} b_{ij}^s f_s$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, k$, where $b_{ij}^s \neq 0$ for all $s = 1, \cdots, r$. In this case, we say that the germ $\tilde{A}$ is *elementary equivalent to A with respect to $\mathcal{B}$*.

Remark

The Newton polyhedra of all the $\tilde{a}_{ij}(x)$'s on the previous Remark are equal to the Newton polyhedron of the function $(f_1 + \cdots + f_r)(x)$.

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Definition

Let $\mathcal{B} = \{f_1, \cdots, f_r\}$ be a basis for a germe $A = (a_{ij}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ of a matrix with holomorphic entries. The Newton polyhedron of the function $(f_1 + \cdots + f_r)(x)$ is called *Newton polyhedron of the matrix A*.

Theorem (____, Hartmann, Varella)

Let Δ be the Newton polyhedron of the polynomial matrix $A = (a_{ij}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ and $[A] = A^{-1}(M^n_{n,k})$. If $\tilde{\Delta}$ is bounded and the leading coefficients of A are in general position, then A defines the germ of a determinantal singularity $[A]$, whose multiplicity is

$$m([A], 0) = C^k_{k-n+1} \cdot m! \cdot MV(\underbrace{\tilde{\Delta}, \cdots, \tilde{\Delta}}_{k-n+1}, \underbrace{L, \cdots, L}_{m-k+n-1}).$$

where L denotes the standard m-dimensional simplex and $\tilde{\Delta} = \mathbb{R}^m_+ \setminus \Delta$.

Example

Let $A : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$ be the germ given by the matrix

$$A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} x+y+z+w & x+2y+z+2w & 2x+2y+z+w \\ x+2y+2z+w & x+3y+2z+3w & 2x+4y+2z+w \end{bmatrix}$$

$$\mathcal{B} = \{x, y, z, w\}$$

$$m([A], 0) = C_2^3 \cdot 4! \cdot MV(\tilde{\Delta}, \tilde{\Delta}, L, L) = C_2^3 \cdot 4! \cdot MV(L, L, L, L)$$

Contents

Euler Obstruction

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Theorem (Brasselet, Lê and Seade)

Let $(X_0, 0)$ be an equidimensional complex analytic singularity germ with a Whitney stratification $\mathcal{V} = \{V_i\}$ and let $l : U \to \mathbb{C}$ be a generic complex linear form in an open set U containing 0 in $\mathbb{C}^m$. Then

$$Eu_{X_0}(0) = \sum_i \chi(V_i \cap B_\epsilon \cap l^{-1}(t_0)) \cdot Eu_{X_0} V_i,$$

where B_ϵ is a small closed ball around 0 in $\mathbb{C}^m$, $t_0 \in \mathbb{C} \setminus \{0\}$ is sufficiently near 0 and $Eu_{X_0} V_i$ is the Euler obstruction at any point of the stratum V_i.

Theorem (____, Hartmann, Varella)

Let Δ be the Newton polyhedron of a holomorphic matrix $A = (a_{ij}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$, $n \leq k$ and $([A], 0) = (A^{-1}(M^n_{n,k}), 0)$. Suppose that $\tilde{\Delta}$ is bounded and $m \leq 2(k - n + 2)$.

i) *If the leading coefficients of A are in strong general position, then the determinantal singularity $([A], 0)$ is smooth outside the origin.*

ii) *If the Newton polyhedron $\Delta_f \subset \mathbb{R}^m_+$ of a germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ intersects all coordinate axes, and the leading coefficients of df are in general position with respect to A, then the Euler characteristic of a Milnor fiber of $f|_{[A]}$ is equal to*

$$\chi([A] \setminus \{0\} \cap B_\epsilon \cap f^{-1}(t_0)) = \sum_{q=k-n+1}^{k} \sum_{\substack{I \subset \{1, \cdots, m\} \\ |I| \geq q+1}} \sum_{a=1}^{|I|-q}$$

$$(-1)^{|I|+k+n} C_{|I|+q-a-2}^{n+q-k-1} C_q^k C_{q-1}^{|I|-a-1} |I|! MV(\underbrace{\tilde{\Delta}_f^I, \cdots, \tilde{\Delta}_f^I}_{a}, \tilde{\Delta}^I, \cdots, \tilde{\Delta}^I).$$

Corollary (____, Hartmann, Varella)

Let Δ be the Newton polyhedron of the matrix $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$, $n \leq k$ and $([A], 0) = (A^{-1}(M^n_{n,k}), 0)$. Suppose that $\tilde{\Delta}$ is bounded and $m \leq 2(k - n + 2)$. If the generic linear form $l : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ has a bounded Newton polyhedron and the leading coefficients of l are in general position with respect to A, then

$$Eu_{[A]}(0) = \sum_{q=k-n+1}^{k} \sum_{\substack{I \subset \{1, \cdots, m\} \\ |I| \geq q+1}} \sum_{a=1}^{|I|-q}$$

$$(-1)^{|I|+k+n} C^{n+q-k-1}_{|I|+q-a-2} C^k_q C^{|I|-a-1}_{q-1} |I|! MV(\underbrace{L^I, \cdots, L^I}_{a}, \tilde{\Delta}^I, \cdots, \tilde{\Delta}^I),$$

where L is the standard m-dimensional simplex.

Example

When a germ A satisfies the conditions for the previously Corollary and the matrix A has only linear entries, we have the following formula for the local Euler obstruction of $[A]$:

$$Eu_{[A]}(0) = \sum_{q=k-n+1}^{k} \sum_{\substack{I\subset\{1,\ldots,m\} \\ |I|\geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+k+n} C_{|I|+q-a-2}^{n+q-k-1} \cdot C_q^k \cdot C_{q-1}^{|I|-a-1}.$$

This is just a sum of binomial coefficients and we can computate it using a computer programm.

Example

Let $A : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$ be the germ given by the matrix

$$A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}.$$

Then,

$$Eu_{[A]}(0) = \sum_{q=2}^{3} \sum_{\substack{I \subset \{1,2,3,4\} \\ |I| \geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+5} C_{|I|+q-a-2}^{q-2} \cdot C_q^3 \cdot C_{q-1}^{|I|-a-1} = -1.$$

Contents

Whitney equisingularity

* $\mathcal{A} : (\mathbb{C}^m \times \mathbb{C}, 0) \to (M_{n,k}, 0)$; $\mathcal{A}(x, t) = A_t(x)$ and $\mathcal{A}(x, 0) = A(x)$;

* If $[A_t]$ is determinantal for all t, $\mathcal{A}$ is said to be a *determinantal deformation* of $[A]$;

* $[A_t]$ is origin preserving if $0 \in S([A_t])$, for all t in D, where $S([A_t])$ denotes the singular set of $[A_t]$ and $D \subset \mathbb{C}$ is a disc around the origin;

* $\{([A_t], 0)\}_{t \in D}$ is a good family if there exists $\varepsilon > 0$ with $S([A_t]) = \{0\}$ on B_ε, for all t in D;

* $\{([A_t], 0)\}_{t \in D}$ is Whitney equisingular if it is a good family and $\{[\mathcal{A}] \setminus T, T\}$ satisfies the Whitney conditions, where $T = \{0\} \times D$.

Theorem (Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, 2018)

A good family of d-dimensional IDS $\{([A_t],0)\}_{t\in D}$ is Whitney equisingular if and only if all the polar multiplicities $m_i([A_t],0)$, $i=0,\cdots,d$ are constant on $t\in D$.

Theorem (Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, 2018)

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Theorem (____, Hartmann, Varella)

Let $[A]$ be a d-dimensional determinantal singularity, defined by the germ of a matrix $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ with polynomial entries, Newton polyhedron Δ and $m \leq 2(k - n + 2)$. Let $\{([A_t], 0)\}_{t\in D}$, be a determinantal family, with $A_0 = A$. If the Newton polyhedron of A_t is Δ, Δ is elementary bounded and the leading coefficients of A_t are in strong general position, for all $t \in D$, then the family $\{([A_t], 0)\}_{t\in D}$ is Whitney equisingular.

Proof: Firstly, since Δ is elementary bounded, $m \leq 2(k-n+2)$ and the leading coefficients of A_t are in strong general position, then $[A_t]$ is smooth outside the origin, consequently $\{([A_t], 0)\}_{t \in D}$ is a good family. Moreover, $\chi([A_t]) = 1$, once $[A_t]$ is contractible. Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, proved that $\nu([A_t], 0)$ is constant on t, i.e. $\nu([A_t], 0) = \nu([A_0], 0)$, for all $t \in D$. Moreover,

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$$m_d([A_t], 0) = \nu([A_t], 0) + \nu([A_t] \cap H_t, 0),$$

where $H_t \subset \mathbb{C}^m$ is a generic hyperplane for all $t \in D$.

Proof: Firstly, since Δ is elementary bounded, $m \leq 2(k-n+2)$ and the leading coefficients of A_t are in strong general position, then $[A_t]$ is smooth outside the origin, consequently $\{([A_t], 0)\}_{t \in D}$ is a good family. Moreover, $\chi([A_t]) = 1$, once $[A_t]$ is contractible. Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, proved that $\nu([A_t], 0)$ is constant on t, i.e. $\nu([A_t], 0) = \nu([A_0], 0)$, for all $t \in D$. Moreover,

$$m_d([A_t], 0) = \nu([A_t], 0) + \nu([A_t] \cap H_t, 0),$$

where $H_t \subset \mathbb{C}^m$ is a generic hyperplane for all $t \in D$.

* H_t is a generic hyperplane and $[A_t]$ is smooth outside the origin.

* $[A_t] \cap H_t$ is a $(d-1)$-dimensional determinantal singularity which is smooth outside the origin;

* $\nu([A_t] \cap H_t, 0) = \nu([A_0] \cap H_0, 0)$;

$$\begin{aligned} m_d([A_t],0) &= \nu([A_t],0) + \nu([A_t] \cap H_t, 0) \\ &= \nu([A_0],0) + \nu([A_0] \cap H_0, 0) \\ &= m_d([A_0],0). \end{aligned}$$

Gaffney, Grulha Jr. and Ruas, proved that:

$$m_{d-l}([A_t] \cap H_t^1 \cap \cdots \cap H_t^l, 0) = m_{d-l}([A_t],0),$$

where $H_t^1, \ldots, H_t^l$ are generic hyperplanes for all $t \in D$.

$$\begin{aligned} m_d([A_t],0) &= \nu([A_t],0)+\nu([A_t]\cap H_t,0) \\ &= \nu([A_0],0)+\nu([A_0]\cap H_0,0) \\ &= m_d([A_0],0). \end{aligned}$$

Gaffney, Grulha Jr. and Ruas, proved that:

$$m_{d-l}([A_t]\cap H_t^1\cap\cdots\cap H_t^l,0)=m_{d-l}([A_t],0),$$

where $H_t^1,\ldots,H_t^l$ are generic hyperplanes for all $t\in D$. Since,

$$dim_{\mathbb{C}^{m-l}}[A_t]\cap H_t^1\cap\cdots\cap H_t^l=d-l,$$

$$\begin{aligned} m_{d-l}([A_t],0) &= m_{d-l}([A_t]\cap H_t^1\cap\cdots\cap H_t^l,0) \\ &= m_{d-l}([A_0]\cap H_0^1\cap\cdots\cap H_0^l,0) \\ &= m_{d-l}([A_0],0), \end{aligned}$$

for all $l=1,\ldots,d$.

Example

Let $\{([A_t], 0)\}_{t \in D}$ be the family of 2-dimensional determinantal singularities defined by the germ $A_t : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$, with

$$A_t = \begin{bmatrix} x + tx^2 & y & z \\ y & z & w \end{bmatrix}.$$

For each t an elementary equivalent matrix to A_t is

$$\begin{bmatrix} x + tx^2 + y + z + w & x + tx^2 + 2y + z + 2w & 2x + 2tx^2 + 2y + z \\ x + tx^2 + 2y + 2z + w & x + tx^2 + 3t + 2z + 3w & 2x + 2tx^2 + 4y + 2 \end{bmatrix}$$

which we denote by $\tilde{A}_t$.

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Thank you for the attention!