A McKay correspondence for the Poincaré series of some finite subgroups of $\mathrm{SL}_3(\mathbb{C})$

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$G \subset \mathrm{SL}_2(\mathbb{C})$ finite subgroup: $\mathcal{C}_\ell$, $\mathcal{D}_n$, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$ (binary polyhedral groups).

$(\mathbb{C}^2/G, 0)$ rational surface singularity (*Kleinian singularity*)
Resolution: $A_{\ell-1}$, D_{n+2}, E_6, E_7, E_8

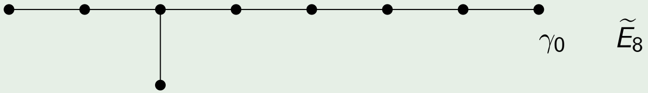
McKay correspondence: (1980)
$\gamma_0 (=$ trivial representation$), \gamma_1, \ldots, \gamma_l$ irreducible representations of G,
γ natural 2-dimensional representation

$$\gamma_j \otimes \gamma = \bigoplus_i b_{ij}\gamma_i, \quad B = (b_{ij}) \quad (l+1) \times (l+1)\text{-matrix}$$

$2I - B$ affine Cartan matrix of corresponding root system
$\rightsquigarrow$ extended Coxeter-Dynkin diagram

Example

Binary icosahedral group $\mathcal{I}$
$\leftrightarrow$ extended Coxeter-Dynkin diagram of E_8:



$G \subset \mathrm{SL}_3(\mathbb{C})$ finite subgroup (Blichfeldt, Miller, Dickson 1916; Yau, Yu 1993):
4 infinite series (A), (B) (abelian), (C) ($\Delta(3 \cdot n^2)$) (D) ($\Delta(6 \cdot n^2)$);
8 exceptional groups (E), (F), (G), (H), (I), (J), (K), (L)

We are interested in those cases where the natural 3-dimensional representation is irreducible $\Rightarrow G$ non abelian

Theorem (Yau–Yu)

G non abelian $\Longleftrightarrow (\mathbb{C}^3/G, 0)$ non-isolated Gorenstein singularity.

$\mathbb{C}^3/G$ hypersurface in $\mathbb{C}^4$ or complete intersection in $\mathbb{C}^5$.
Equations computed by Yau–Yu.

Example

$G = G_{168} = \mathrm{PSL}(2, \mathbb{F}_7)$ Kleinian group of order 168.
$\mathbb{C}^3/G = \{R(w,x,y,z) = 0\}$ hypersurface in $\mathbb{C}^4$,

$$\begin{aligned} R(w,x,y,z) &= z^2 - y^3 - 1728x^7 + 88w^2xy^2 - 1008wx^4y \\ &\quad -1088w^4x^2y + 256w^7y + 60032w^3x^5 \\ &\quad -22016w^6x^3 + 2048w^9x \end{aligned}$$

$(\deg w = 4,\ \deg x = 6,\ \deg y = 14,\ \deg z = 21)$

We shall consider the hyperplane sections $w = 0$ of the singularities $\mathbb{C}^3/G$!

Example (continued)

Setting $w = 0$ in $R(w, x, y, z)$ gives

$$R(0, x, y, z) = z^2 - y^3 - 1728x^7$$

$\rightsquigarrow$ Arnold's exceptional unimodal singularity E_{12}.
$\leftrightarrow$ Fuchsian singularity with signature $(2, 3, 7)$.

Fuchsian singularity (of genus 0):

Consider m-gon $\square$ with angles $\frac{\pi}{\alpha_1}, \ldots, \frac{\pi}{\alpha_m}$ in the Poincaré disc $\mathbb{D}$.
$\Sigma \subset \mathrm{PSU}(1,1)$ group generated by reflections in edges of $\square$,
$\Sigma_+ \subset \Sigma$ subgroup of orientation preserving automorphisms.
$P := \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 - |z_2|^2 > 0\}$
$\pi : P \to \mathbb{D}$, $(z_1, z_2) \mapsto \frac{z_2}{z_1}$

Definition

$\Gamma \subset \mathrm{SU}(1,1)$ "binary group" corresponding to $\Sigma_+ \subset \mathrm{PSU}(1,1)$, *Fuchsian group* (of the first kind) *of signature* $(\alpha_1, \ldots, \alpha_m)$.

$((P/\Gamma) \cup \{0\}, 0)$ *Fuchsian singularity* (of genus 0) *of signature* $(\alpha_1, \ldots, \alpha_m)$.

Kleinian or Fuchsian singularity is a normal surface singularity (X, x) with a good $\mathbb{C}^*$-action.

Coordinate algebra: $A_X = \oplus_{k=0}^{\infty} A_{X,k}$ graded $\mathbb{C}$-algebra with $A_{X,0} = \mathbb{C}$.

$X = \mathrm{Spec}(A_X)$ normal 2-dimensional affine algebraic variety over $\mathbb{C}$, x defined by $\mathfrak{m} = \oplus_{k=1}^{\infty} A_{X,k}$.

Natural compactification: $\overline{X} := \mathrm{Proj}(A_X[t])$, t degree 1.

$\overline{X}$ has singularities at $\overline{X}_\infty := \overline{X} \setminus X = \mathrm{Proj}(A_X)$,

Kleinian singularity: Resolution of x
Fuchsian singularity: Configuration at ∞ (after resolution)

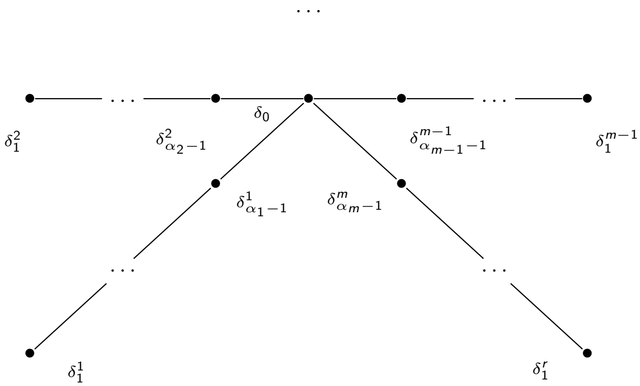
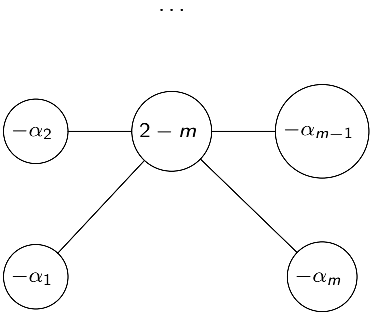


Figure: The graph $T^-_{\alpha_1,\ldots,\alpha_m}$

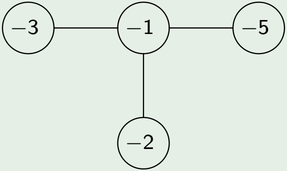
Kleinian singularity: Configuration at ∞ (after resolution)
Fuchsian singularity: Resolution of x



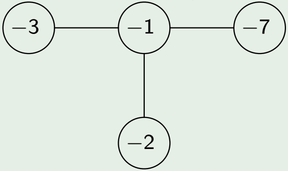
Example

$\mathbb{C}^2/\mathcal{I}$: Resolution: $T^-_{2,3,5}$,

Configuration at ∞:



E_{12}: Resolution:



Configuration at ∞: $T^-_{2,3,7}$

G	$\lvert G \rvert$	Sing	$\alpha_1, \ldots, \alpha_m$.
(C): $T = \Delta(3 \cdot 2^2)$	12	E_6	2,3,3
$\Delta(3 \cdot 4^2)$	48	E_{14}	3,3,4
(D): $O = \Delta(6 \cdot 2^2)$	24	E_7	2,3,4
$\Delta(6 \cdot 3^2)$	54	$\delta 1$	2,2,2,2,2,2
$\Delta(6 \cdot 4^2)$	96	Z_{11}	2,3,8
$\Delta(6 \cdot 6^2)$	216	$Z_{1,0}$	2,2,2,4
(E)	108	$K'_{1,0}$	2,2,4,4
(F)	216	U_{12}	4,4,4
(G)	648	$U_{1,0}$	2,3,3,3
(H)=I	60	E_8	2,3,5
(I)	168	E_{12}	2,3,7
(J)	180	$Q_{2,0}$	2,2,2,5
(K)	504	Q_{11}	2,4,7
(L)	1080	E_{13}	2,4,5

$G \subset \mathrm{SL}_3(\mathbb{C})$ finite subgroup,
$\gamma_0 (= \text{trivial}), \gamma_1, \ldots, \gamma_l$ irreducible representations of G,
γ natural 3-dimensional representation,
γ^* contragredient representation

$$\gamma_j \otimes \gamma = \bigoplus_i b_{ij}\gamma_i, B = (b_{ij}), \quad \gamma_j \otimes \gamma^* = \bigoplus_i b^*_{ij}\gamma_i, B^* = (b^*_{ij}).$$

Example

$G = G_{168}$:

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Definition

Poincaré series of graded algebra A.

$$p_A(t) = \sum_{k=0}^{\infty} \dim(A_k) t^k.$$

(X, x) Kleinian or Fuchsian singularity with coordinate algebra A,

$$p_X(t) := p_A(t).$$

$G \subset \mathrm{SL}_n(\mathbb{C})$ finite subgroup, $S := S(\mathbb{C}^n) = \bigoplus_{k=0}^{\infty} S^k(\mathbb{C}^n)$,
S^G algebra of invariant polynomials with respect to G.

$$p_G(t) := p_{S^G}(t).$$

S^G is generated by homogeneous polynomials $f_1, \dots, f_N$ of degree $d_1, \dots, d_N$ respectively.

$$c_G := \gcd(d_1, \dots, d_n).$$

$G \subset \mathrm{SL}_2(\mathbb{C})$: $X = \mathbb{C}^2/G$, $p_G(t) = p_X(t^{c_G})$

$G \subset \mathrm{SL}_3(\mathbb{C})$: Consider singularity (X, x) given by hyperplane section $w = 0$ of $\mathbb{C}^3/G$,

$$p_G(t) = \frac{p_X(t^{c_G})}{(1 - t^{\deg w})}$$

McKay correspondence for Poincaré series of $G \subset \mathrm{SL}_2(\mathbb{C})$:

Theorem (–, 2002)

For a Kleinian singularity (X, x) not of type A_{2n}:

$$p_X(t) = \frac{\textit{char. pol. of Coxeter element}}{\textit{char. pol. of affine Coxeter element}}$$

Example

$X = \mathbb{C}^2/\mathcal{I}$:

$$p_X(t) = \frac{\text{char. pol. of Coxeter element of } E_8}{\text{char. pol. of Coxeter element of } \widetilde{E}_8}$$

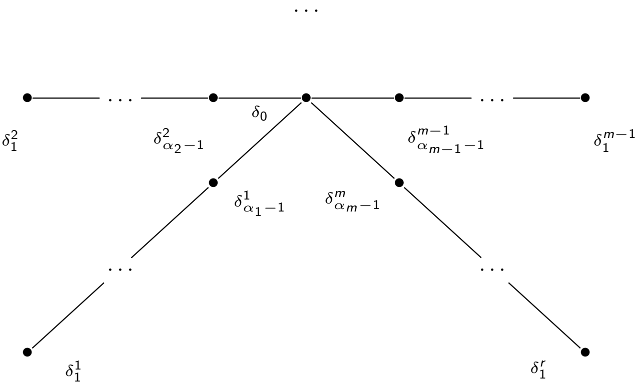


Figure: The graph $T^-_{\alpha_1,\ldots,\alpha_m}$

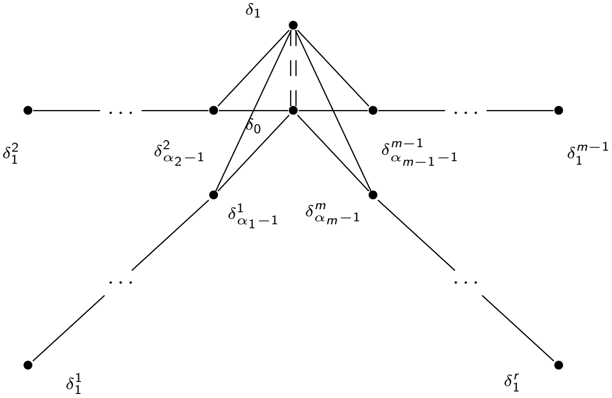


Figure: The graph $T_{\alpha_1,\ldots,\alpha_m}$

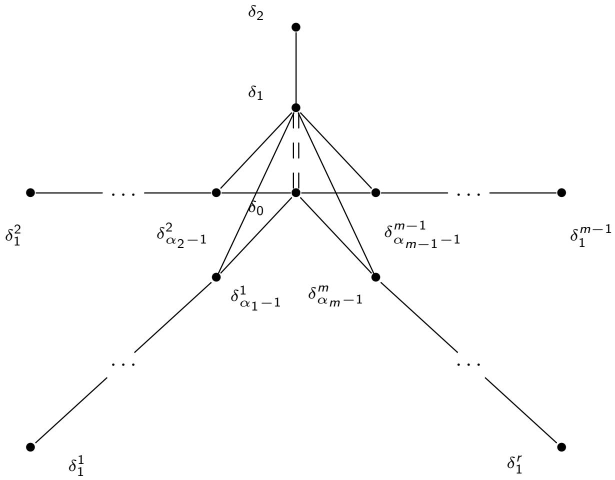


Figure: The graph $T^+_{\alpha_1,\dots,\alpha_m}$

$(V, \langle -, - \rangle)$ lattice spanned by one of the graphs $T^-_{\alpha_1,\ldots,\alpha_m}$, $T_{\alpha_1,\ldots,\alpha_m}$, $T^+_{\alpha_1,\ldots,\alpha_m}$, $\langle \delta, \delta \rangle = -2$ (root).

$$s_\delta(x) = x - \frac{2\langle x, \delta \rangle}{\langle \delta, \delta \rangle}\delta = x + \langle x, \delta \rangle \delta \quad \text{for } x \in V.$$

Definition

D Coxeter-Dynkin diagram corresponding to ordered basis of roots $\mathcal{B} = (\delta_1, \ldots, \delta_\mu)$. *Coxeter element* corresponding to D:

$$\tau_D = s_{\delta_1} s_{\delta_2} \cdots s_{\delta_\mu}$$

$\Delta_D(t) := \det(1 - \tau_D^{-1} t)$ characteristic polynomial.

$$A_{\ell-1} = T^-_{1,1,\ell},\ D_{n+2} = T^-_{2,2,n},\ E_6 = T^-_{2,3,3},\ E_7 = T^-_{2,3,4},\ E_8 = T^-_{2,3,5}$$
$$\widetilde{A}_{\ell-1} = T_{1,1,\ell},\ \widetilde{D}_{n+2} = T_{2,2,n},\ \widetilde{E}_6 = T_{2,3,3},\ \widetilde{E}_7 = T_{2,3,4},\ \widetilde{E}_8 = T_{2,3,5}$$

Theorem (–,Ploog 2010)

(i) *For a Kleinian singularity not of type A_{2n} we have*

$$p_X(t) = \frac{\Delta_{T^-_{\alpha_1,\ldots,\alpha_m}}(t)}{\Delta_{T_{\alpha_1,\ldots,\alpha_m}}(t)}.$$

(ii) *For a Fuchsian singularity we have*

$$p_X(t) = \frac{\Delta_{T^+_{\alpha_1,\ldots,\alpha_m}}(t)}{\Delta_{T_{\alpha_1,\ldots,\alpha_m}}(t)}.$$

G	$\|G\|$	c_G	Sing	$\alpha_1, \ldots, \alpha_m$.
(C): $T = \Delta(3 \cdot 2^2)$	12	1	E_6	2,3,3
$\Delta(3 \cdot 4^2)$	48	1	E_{14}	3,3,4
(D): $O = \Delta(6 \cdot 2^2)$	24	1	E_7	2,3,4
$\Delta(6 \cdot 3^2)$	54	3	$\delta 1$	2,2,2,2,2,2
$\Delta(6 \cdot 4^2)$	96	1	Z_{11}	2,3,8
$\Delta(6 \cdot 6^2)$	216	3	$Z_{1,0}$	2,2,2,4
(E)	108	3	$K'_{1,0}$	2,2,4,4
(F)	216	3	U_{12}	4,4,4
(G)	648	6	$U_{1,0}$	2,3,3,3
(H)=I	60	1	E_8	2,3,5
(I)	168	1	E_{12}	2,3,7
(J)	180	3	$Q_{2,0}$	2,2,2,5
(K)	504	3	Q_{11}	2,4,7
(L)	1080	3	E_{13}	2,4,5

Theorem

(i) *For the Kleinian cases $G = T, O, I$*

$$p_G(t) = \frac{q_G(t)\Delta_{T^-_{\alpha_1,\alpha_2,\alpha_3}}(t)}{(1-t)q_G(t)\Delta_{T_{2,\alpha_1,\alpha_2,\alpha_3}}(t)},$$

$q_G(t) = (1-t)^a(1-t^e)^b$ *polynomial.*

(ii) *For the Fuchsian cases except $\Delta(6 \cdot 3^2)$ and (G)*

$$p_G(t) = \frac{q_G(t^{c_G})\Delta_{T^+_{\alpha_1,\ldots,\alpha_m}}(t^{c_G})}{(1-t^{c_G})q_G(t^{c_G})\Delta_{T_{\alpha_0,\alpha_1,\ldots,\alpha_m}}(t^{c_G})}, \quad \alpha_0 = \frac{\deg w}{c_G},$$

$q_G(t) = (1-t)^a(1-t^e)^b$ *polynomial.*

Theorem

(iii) $G = \Delta(6 \cdot 3^2)$, $m = 6$

$$p_G(t) = \frac{(1-t^{c_G})q_G(t^{c_G})\Delta_{T^+_{\alpha_1,\ldots,\alpha_6}}(t^{c_G})}{q_G(t^{c_G})\Delta_{T_{\alpha_2,\ldots,\alpha_6}}(t^{c_G})}$$

$q_G(t)$ rational function.

(iv) $G = (G)$, $c_G = 6$, $\deg w = 9$, $m = 4$

$$p_G(t) = \frac{q_G(t^3)\Delta_{T^+_{\alpha_1,\ldots,\alpha_4}}(t^{c_G})}{(1-t^9)q_G(t^3)\Delta_{T_{\alpha_1,\ldots,\alpha_4}}(t^{c_G})},$$

$q_G(t)$ rational function.

Idea of proof:
ρ_k representation of G on $S^k(\mathbb{C}^n)$ induced by natural action on $\mathbb{C}^n$.
$\rho_k = \sum_{i=0}^{l} v_{ki}\gamma_i$, $v_k = (v_{k0}, \dots, v_{kl})^t \in \mathbb{Z}^{l+1}$.

Idea of Kostant:

$$P_G(t) := \sum_{k=0}^{\infty} v_k t^k \quad \text{formal power series with coefficients in } \mathbb{Z}^{l+1}$$

$P_G(t)_i := \sum_{k=0}^{\infty} v_{ki} t^k$, $P_G(t)_0 = p_G(t)$

$V = \{x = \sum_{k=0}^{\infty} x_k t^k \, : \, x_k \in \mathbb{Z}^{l+1}\}$
free module of rank $l+1$ over the ring R of formal power series with integer coefficients.

Clebsch-Gordon formulas $\Rightarrow x = P_G(t)$ solution of linear equation in V:

$$\begin{aligned} n = 2: &\qquad ((1+t^2)I - tB)x = v_0 \\ n = 3: &\qquad ((1-t^3)I - tB + t^2B^*)x = v_0 \end{aligned}$$

Write this as

$$M(t)x = v_0$$

$M_0(t)$ matrix obtained from $M(t)$ by replacing first column by $v_0 = (1, 0, \ldots, 0)^t$.

Cramer's rule $\Rightarrow$

$$P_G(t)_0 = \frac{\det M_0(t)}{\det M(t)}$$

$n = 2$: Exercise in Bourbaki (Groupes et algèbres de Lie, Chap. 4,5,6)
$\Rightarrow$ (A_{2k} excluded)

$$\det M(t) = \det(t^2 I - \tau_a), \quad \det M_0(t) = \det(t^2 I - \tau)$$

τ_a affine Coxeter element, τ Coxeter element

G finite group, γ_0(=trivial), $\gamma_1, \ldots, \gamma_l$ irreducible representations, γ d-dimensional faithful (complex) representation,

$$\gamma_j \otimes \gamma = \bigoplus_i b_{ij}\gamma_i, \quad B = (b_{ij})$$

Proposition (Steinberg)

The column $(\chi_1(g), \ldots, \chi_l(g))^t$ ($g \in G$ fixed) of the character table of G is an eigenvector of the matrix B with eigenvalue $\chi_\gamma(g)$.

$G \subset \mathrm{SL}_3(\mathbb{C})$:

$$\begin{aligned} \det M(t) &= \det((1-t^3)I - tB + t^2B^*) \\ &= \prod_{g \in \mathrm{Conj}\, G} (1 - \chi_\gamma(g)t + \chi_{\gamma^*}(g)t^2 - t^3) \end{aligned}$$

Example

$G = G_{168}$ revisited: $c_G = 1$, $\deg w = 4$,
E_{12}: Coxeter-Dynkindiagram (with respect to a distinguished basis of vanishing cycles): $T^+_{2,3,7}$

$$p_G(t) = \frac{(1-t^{42})}{(1-t^4)(1-t^6)(1-t^{14})(1-t^{21})}$$

$$\begin{aligned} \det M(t) &= (1-t)^2(1-t^2)(1-t^3)(1-t^4)(1-t^7) \\ &= (1-t)^4 \Delta_{T_{2,3,4,7}}(t) \\ \det M_0(t) &= (1-t)^3 \frac{(1-t^2)(1-t^3)(1-t^7)(1-t^{42})}{(1-t)(1-t^6)(1-t^{14})(1-t^{21})} \\ &= (1-t)^3 \Delta_{T^+_{2,3,7}}(t) \end{aligned}$$

Thank you!

Reference: Journal of Singularities **18** (2018), 397–408