Extension of the Gusein-Zade Theorem on the jumps equal to 1 of the Milnor number in deformations of curve plane singularities

Aleksandra Zakrzewska

Faculty of Mathematics and Computer Science University of Lodz

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Deformation of the singularity

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A **deformation of the singularity** f_0 is the germ of a holomorphic function $f = f(s, z) : (\mathbb{C} \times \mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ such that

1. $f(0, z) = f_0(z)$,
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The deformation $f(s, z)$ of the singularity f_0 will also be treated as a family (f_s) of functions germs, taking $f_s(z) := f(s, z)$.

Milnor number

Definition

Since f_0 is an isolated singularity, f_s for sufficiently small s also has isolated singularities near 0. By the above for sufficiently small s one can define μ_s

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_2/(\nabla f_s),$$

called the **Milnor number of** f_s, where $\mathcal{O}_2$ is the ring of the holomorphic function germs at 0, and (∇f_s) is the ideal in $\mathcal{O}_2$ generated by ∇f_s.

Jump of the Milnor number

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities, there exists an open neighbourhood S of the point 0 such that

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Definition

The constant difference $\mu_0 - \mu_s$ (for $s \in S \setminus \{0\}$) will be called **the jump of the deformation** (f_s) and denoted by $\lambda((f_s))$. The smallest non-zero value among all the jumps of deformations of the singularity f_0 will be called **the jump of the Milnor number of the singularity** f_0 and denoted by $\lambda(f_0)$.

Example

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- For example:
  - $f_s^1(x, y) = x^6 + y^6 + s(x + \varepsilon y)^5$, where $\varepsilon^6 = -1$,
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- Their Milnor numbers: $\mu((f_s^1)) = 21$, $\mu((f_s^2)) = 22$.
- In work [2] Brzostowski, Krasinski and Walewska proved that f_s^2 realizes the jump of the Milnor number, so $\lambda(f_0) = 3$.

The linear jump of Milnor number

We will consider the jump of the Milnor for all **linear deformations** of f_0 i.e. deformations of the form $f_s = f_0 + sg$, where g is a holomorphic function in the neighbourhood of 0 such that $g(0) = 0$. The smallest non-zero value among all the jumps of linear deformations of the singularity f_0 will be denoted $\lambda^{lin}(f_0)$.

Singularities with the jump of Milnor number 1

In 1993 Gusein-Zade ([4]) proved that:

Theorem

If for a singularity f_0 there exists a maximal exceptional divisor in the resolution process of the singularity f_0 which intersects no more than three other components of the total preimage of the curve $f_0 = 0$, then $\lambda(f_0) = 1$.

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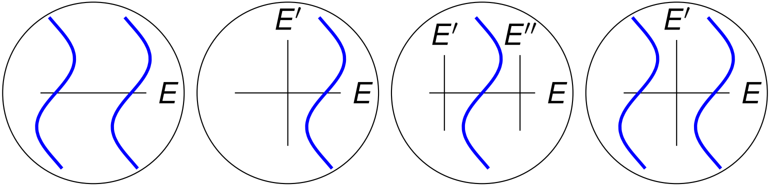
- In this work I will present additional conditions for a singularity f_0 that $\lambda(f_0) = 1$.
- These conditions (with those of Gusein-Zade) are the sufficient and necessary conditions for that $\lambda^{lin}(f_0) = 1$.

Gusein-Zade Theorem

If there exists a maximal exceptional divisor (E) which intersects no more than three other components of the total preimage it means that we have one of below situation:



Enriques diagram

To get the main result we use the **Enriques diagrams** and a result of M. Alberich-Carraminñana and J.Roé.

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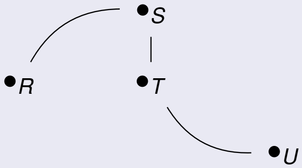
To any singularity we can assign its Enriques diagram D which represents the whole resolution process of this singularity. It is a finite graph with two types of edges and distinguished root R.

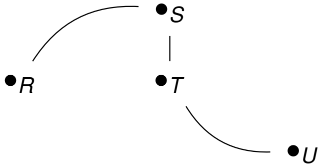


More details in E. Casas-Alvero *Singularities of Plane Curve* ([3]).

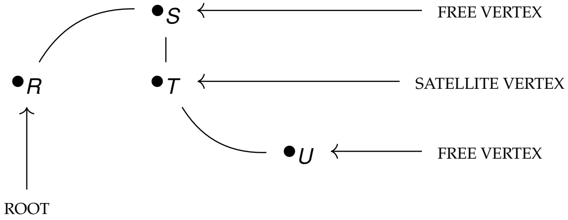
Proximity relation

The proximity relation between vertices in above example is defined as follows:

$R \to S, T \to R, T \to S, U \to T$



The vertex can be one of three types.



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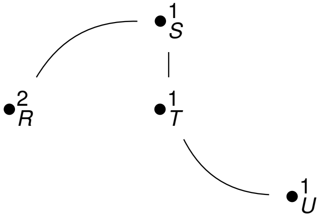
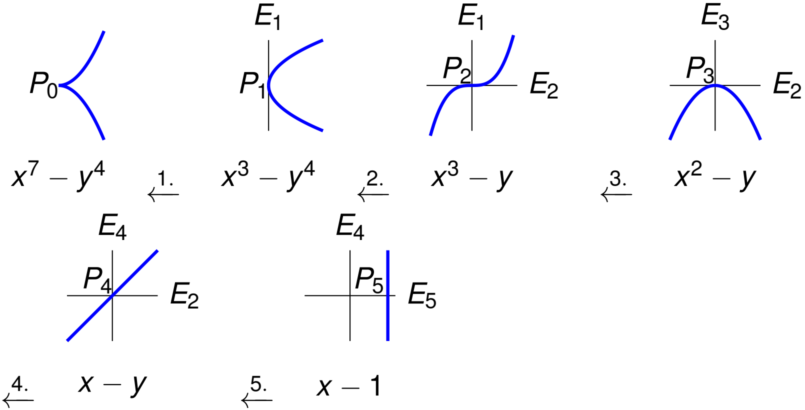


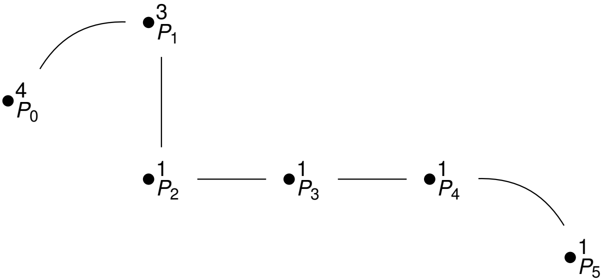
Example

Let $f_0(x, y) = x^7 - y^4$. Then the process of resolutions:



Example

The Enriques diagram of this singularity:



Minimal diagrams

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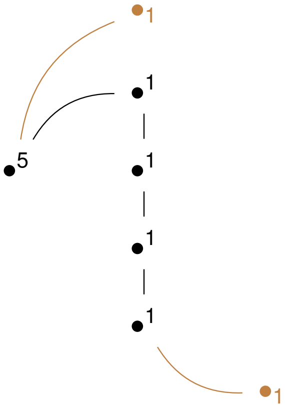
Enriques diagram of $f_0 = y(x^4 + y^5)$ and its minimal diagram (black).



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- gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. This a numeric condition that can be checked for any two diagrams;
- proved that $(\tilde{D}, \tilde{\nu})$ is linear adjacent to (D, ν) if and only if for every singularity f_0 with the Enriques diagram $(\tilde{D}, \tilde{\nu})$ there exists a linear deformation (f_s) such that the Enriques diagram of a generic element f_s is (D, ν).

Corollary

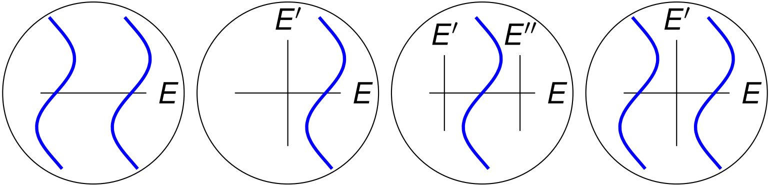
Corollary (from Theorem of M. Alberich-Carramiñana and J.Roé)

1. *To check if one singularity is linear deformation of another it is sufficient to compare their Enriques diagrams (check if they are in some numeric relation.)*
2. *Linear adjacency depends only on topological types of singularities.*
3. $\lambda^{lin}(f_0)$ *is a topological invariant.*

Singularities with the jump of Milnor number 1

Gusein-Zade ([4]) proved that if for a singularity f_0 there exists a maximal exceptional divisor which intersects no more than three other components of the total preimage of the curve $f_0 = 0$, then $\lambda(f_0) = 1$.

Gusein-Zade Theorem



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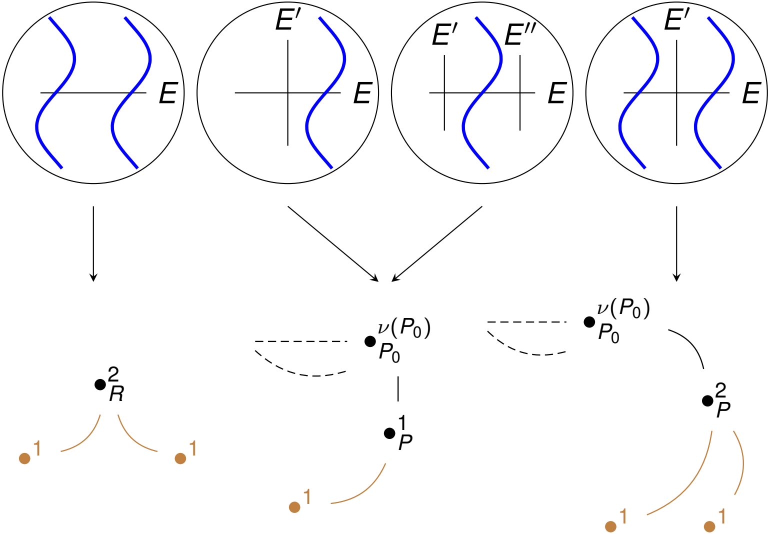

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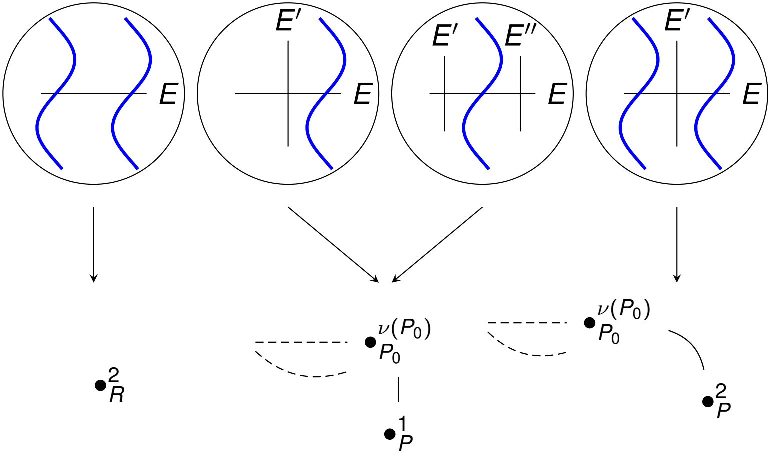

Singularities with the jump of Milnor number 1

Theorem (Gusein-Zade)

Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be a singularity and (D, ν) its minimal diagram. If one of below conditions is true:

1. *the diagram D contains only root with weight 2,*
2. *there exists a leaf $P \in D$ such that P is satellite with weight 1,*
3. *there exists a leaf $P \in D$ such that P is free with weight 2,*

then $\lambda(f_0) = \lambda^{lin}(f_0) = 1$.

Additional conditions for the jump of Milnor number 1

Theorem

Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be a singularity and (D, ν) its minimal diagram. If one of below conditions is true:

4. *there exists $P_1, \ldots, P_k \in D$ ($k \geq 2$) such that, P_1 is the root, $P_i \to P_{i-1}$ for $i = 2, \ldots, k$, P_2 it is the only vertex proximate to P_1, P_k it is the only vertex proximate to P_{k-1} and $\nu(P_k) = \nu(P_1) - 2$, $\nu(P_i) = \nu(P_1) - 1$ for $i = 2, \ldots, k-1$ when $k > 2$.*

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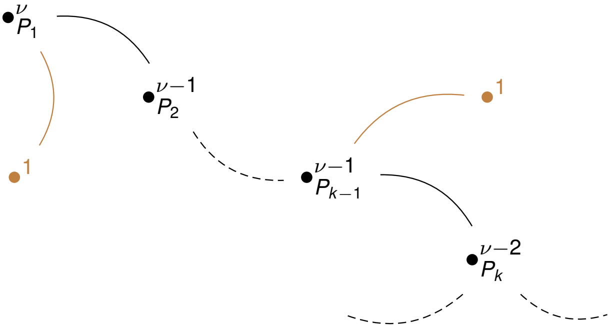

Theorem

5. *there exists* $P_0, P_1, \ldots, P_k \in D$ *(*$k \geq 2$*) such that,* P_1 *is free,* $P_1, \ldots, P_k \to P_0$*,* $P_i \to P_{i-1}$ *for* $i = 2, \ldots, k$*,* P_2 *it is the only vertex proximate to* P_1*,* P_k *is the only vertex proximate to* P_{k-1} *and* $\nu(P_k) = \nu(P_1) - 2$*,* $\nu(P_i) = \nu(P_1) - 1$ *for* $i = 2, \ldots, k-1$ *when* $k > 2$*.*

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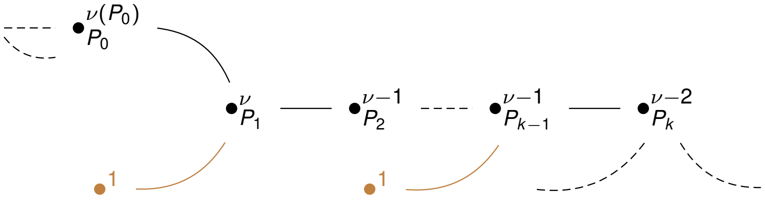

Main theorem

Theorem

Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be a singularity and (D, ν) its minimal diagram. We have $\lambda^{lin}(f_0) = 1$ if and only if one of the $(1) - (5)$ condition is satisfied.

Proof $\Leftarrow$

For the diagram (D, ν) of f_0 we will construct the diagram (E, λ) such that :

- (D, ν) is linear adjacent to (E, λ)
- $\mu((E, \lambda)) = \mu((D, \nu)) - 1$.

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Then there exists a linear deformation that its generic element has the Enriques diagram (E, λ).

Proof $\Leftarrow$ - condition (1)

The diagram *D* contains only root with weight 2.

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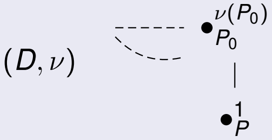
Proof $\Leftarrow$ - condition (2)

There exists a leaf $P \in D$ such that P is satellite with weight 1.



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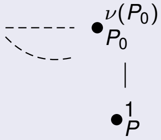

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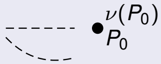

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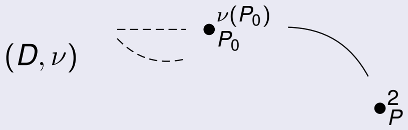

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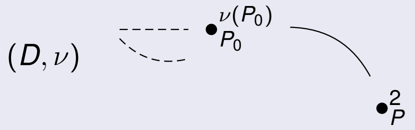

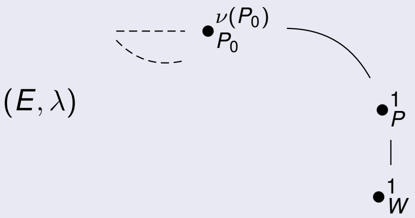

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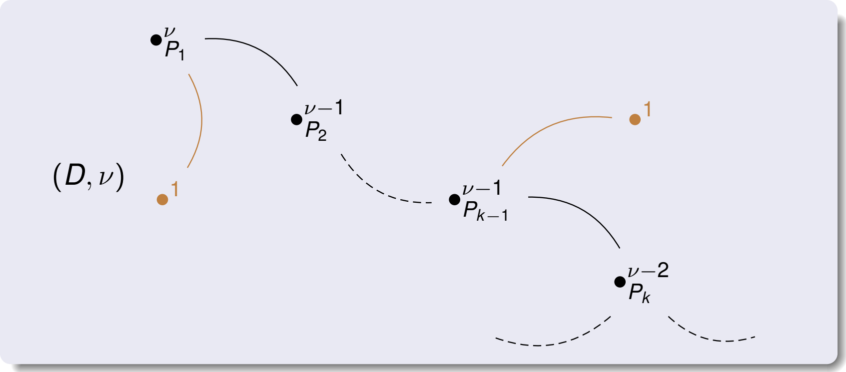
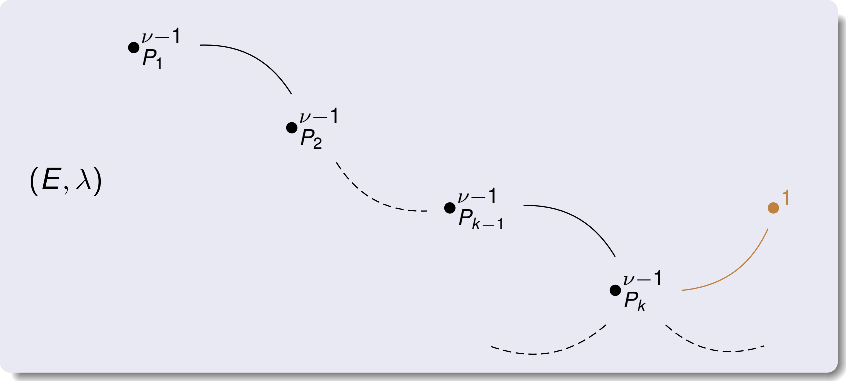
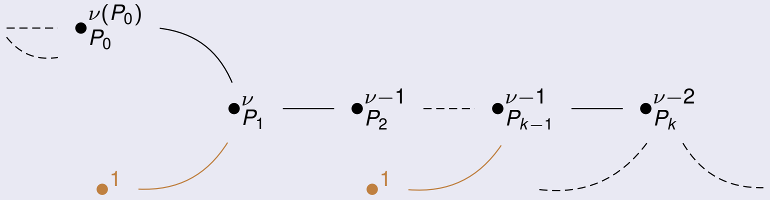


Proof $\Leftarrow$ - condition (4)

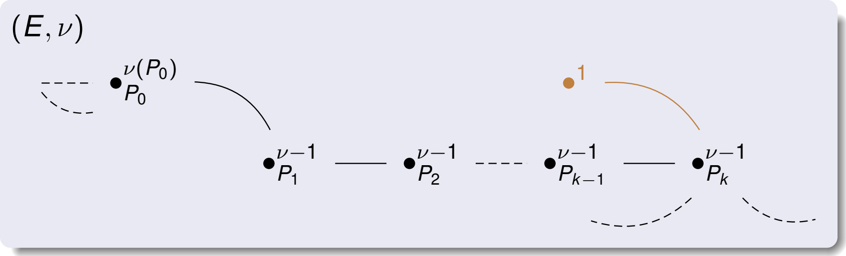


There exists $P_0, P_1, \ldots, P_k \in D$ ($k \geq 2$) such that, P_1 is free, $P_1, \ldots, P_k \to P_0$, $P_i \to P_{i-1}$ for $i = 2, \ldots, k$, P_2 it is the only vertex proximate to P_1, P_k is the only vertex proximate to P_{k-1} and $\nu(P_k) = \nu(P_1) - 2$, $\nu(P_i) = \nu(P_1) - 1$ for $i = 2, \ldots, k-1$ when $k > 2$.

(D, ν)



Proof $\Leftarrow$ - condition (5)



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- Moreover $\mu(E, \lambda) = \mu(D, \nu) - 1$.
- We will show that if (D, ν) does not satisfy the conditions (1) $-$ (3) then it has to satisfy either condition (4) or (5).

Proof $\Rightarrow$

- Let (D', ν') be a modification of (D, ν) such that for two appropriately chosen $P_1, S \in D$:
  - $D = D'$
  - $\nu'(P) = \nu(P)$ for $P \in D \setminus \{P_1, S\}$
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- (D', ν') is linear adjacent to (E, λ).
- Since the Milnor number is upper semi-continuous we get: $\mu((D, \nu)) - 1 = \mu((E, \lambda)) \leq \mu((D', \nu')) \leq$ $\mu((D, \nu)) - 2(\nu(P) - \nu(S)) + 2 + r_D - r_{D'}$.

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- It means that $1 \geq 2(\nu(P) - \nu(S)) - 2 - r_D + r_{D'}$.
- This condition is only true when (D, ν) satisfies the condition (4) or (5).

The linear deformation - condition (4)

Example

Let $f_0(x, y) = y(x + y^{k-1})(x^2 + y^{2k})$, for $k \geq 2$.

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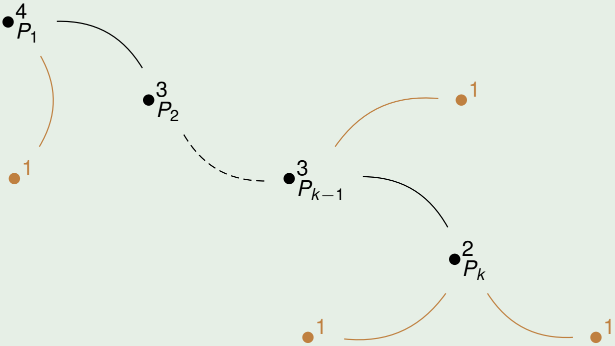

Example

Then the deformation is
$f_s(x,y) = y(x+y^{k-1})(x^2+y^{2k}) + sx(x^2+y^{2k}) =$
$(y(x+y^{k-1})+sx)(x^2+y^{2k})$.

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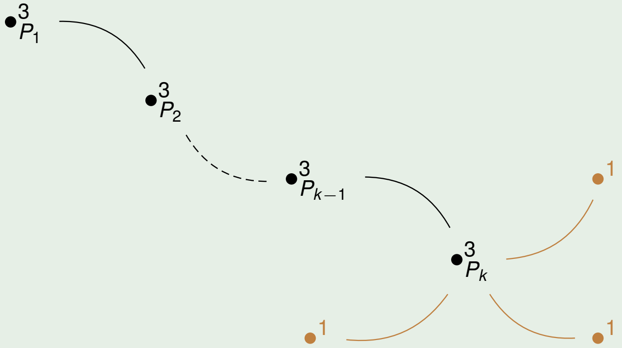

Example

The conditions (1) $-$ (5) are not necessary conditions for $\lambda(f_0) = 1$.

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The conditions (1) – (5) are not necessary conditions for $\lambda(f_0) = 1$.

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Let $f_0(x, y) = x^4 + y^6$. Then:

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Let $f_0(x, y) = x^4 + y^6$. Then:

- $\mu(f_0) = 15$.

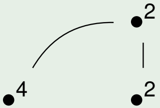
Example

The conditions (1) – (5) are not necessary conditions for $\lambda(f_0) = 1$.

Example

Let $f_0(x, y) = x^4 + y^6$. Then:

- $\mu(f_0) = 15$.
- Its minimal Enriques diagram doesn't satisfy any condition (1) – (5).



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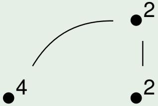

- Let $f_s(x, y) = x^4 + (y^2 + sx)^3$ (nonlinear deformation) – $\mu(f_s) = 14$.

Example

The conditions (1) – (5) are not necessary conditions for $\lambda(f_0) = 1$.

Example

Let $f_0(x, y) = x^4 + y^6$. Then:

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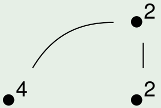

- Let $f_s(x, y) = x^4 + (y^2 + sx)^3$ (nonlinear deformation) – $\mu(f_s) = 14$.
- It means that $\lambda(f_0) = 1$.

Thank you for your attention!

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