On jumps of the Lê numbers in the family of line singularities

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Introduction

- $f\colon (U,0) \to (\mathbb{C},0)$ - complex analytic function with singularity at 0 (not necessarily isolated), U - open neighbourhood of 0 in $\mathbb{C}^n$, $z = (z_1, \ldots, z_n)$ - fixed system of linear coordinates, Σf - critical locus of f.

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- $\dim_0 \Sigma f = d \geq 0$,
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- $\dim_0 \Sigma f = 0 \Rightarrow \lambda^0_{f,z}(0) = \mu_f(0)$

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Gap sheaves

- $(X, \mathscr{O}_X)$ - complex analytic space, $W \subseteq X$ - analytic subset, $\mathscr{I}$ - coherent sheaf of ideals in $\mathscr{O}_X$, $V(\mathscr{I})$ - analytic space defined by the vanishing of $\mathscr{I}$.

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- Take these components of $V(\mathscr{I})$, which are contained in W. Let $x \in V(\mathscr{I})$. Take a minimal primary decomposition of the stalk $\mathscr{I}_x$ of $\mathscr{I}$ in $\mathscr{O}_{X,x}$. Define the ideal $\mathscr{I}_x \neg W$ in $\mathscr{O}_{X,x}$ consisting of the intersection of those (possibly embedded) primary components Q of $\mathscr{I}_x$ such that $V(Q) \not\subseteq W$.

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- This definition does not depend on the choice of minimal primary decomposition of $\mathscr{I}_x$.
- Perform the operation described above at all points of $V(\mathscr{I})$. We obtain a coherent sheaf of ideals called a *gap sheaf* and denoted by $\mathscr{I} \neg W$. We denote by $V(\mathscr{I}) \neg W$ complex space $V(\mathscr{I} \neg W)$.

Lê cycle and Lê numbers

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- Analytic cycle

$$[\Lambda^k_{f,z}] := \left[\Gamma^{k+1}_{f,z} \cap V\left(\frac{\partial f}{\partial z_{k+1}}\right)\right] - \left[\Gamma^k_{f,z}\right]$$

is called kth *Lê cycle* of f with respect to z.

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- We define kth *Lê number* of f at $0 \in \mathbb{C}^n$ with respect to z

$$\lambda^k_{f,z}(0) := \left([\Lambda^k_{f,z}]\cdot[V(z_1,\dots,z_k)]\right)_0$$

provided that this intersection is 0-dimensional or empty at 0;

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- $$\begin{aligned}[\Lambda^1_{f,z}] &= \left[\Gamma^2_{f,z} \cap V\left(\frac{\partial f}{\partial z_2}\right)\right] - \left[\Gamma^1_{f,z}\right] \\ &= [V(z_3^3) \cap V(z_2(2z_1^2 + 4z_2^2))] - [V(2z_1^2 + 4z_2^2, z_3^3)] \\ &= [V(z_2, z_3^3)]\end{aligned}$$

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- $$\begin{aligned}[\Lambda^0_{f,z}] &= \left[\Gamma^1_{f,z} \cap V\left(\frac{\partial f}{\partial z_1}\right)\right] \\ &= [V(2z_1^2 + 4z_2^2, z_3^3) \cap V(z_1 z_2^2)] \\ &= [V(z_1, z_2^2, z_3^3)] + [V(z_1^2, z_2^2, z_3^3)].\end{aligned}$$

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- $$\lambda^1_{f,z} = ([\Lambda^1_{f,z}] \cdot [V(z_1)])_0 = [V(z_1, z_2, z_3^3)]_0 = 3;$$
$$\lambda^0_{f,z} = ([\Lambda^0_{f,z}] \cdot \mathbb{C}^3)_0 = 6 + 12 = 18.$$

Uniform Iomdine-Lê-Massey formula

- $(f_t)\colon (D \times U, D \times \{0\}) \to (\mathbb{C}, 0)$ - deformation of f, $0 \in D \subset \mathbb{C}$

Theorem (Uniform Iomdine-Lê-Massey formula)

For sufficiently large integer j and any sufficiently small complex number t, we have:

- $\Sigma(f_t + z_1^j) = \Sigma f_t \cap V(z_1)$ *in a neighbourhood of the origin;*

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- $\dim_0 \Sigma(f_t + z_1^j) = d - 1$;
- *Lê numbers* $\lambda^k_{f_t + z_1^j, \tilde{z}}(0)$ *exist for every* $0 \leq k \leq d-1$ *and*

$$\lambda^0_{f_t+z_1^j,\tilde{z}}(0) = \lambda^0_{f_t,z}(0) + (j-1)\lambda^1_{f_t,z}(0); \tag{0.1}$$

$$\lambda^k_{f_t+z_1^j,\tilde{z}}(0) = (j-1)\lambda^{k+1}_{f_t,z}(0) \quad \text{for} \quad 1 \leq k \leq d-1; \tag{0.2}$$

where $\tilde{z} = (z_2, \ldots, z_n, z_1)$.

Lê numbers of deformation

Corollary

$\lambda^k_{f_t,z}(0)$ - *defined for every* $k \leq d$ *and independent of* t *for sufficiently small* t

Lê numbers of deformation

Theorem

The tuple of Lê numbers $\left(\lambda^d_{f_t,z}(0),\ldots,\lambda^0_{f_t,z}(0)\right)$ *is lexicographically upper-semicontinuous in the variable* t*, i.e. for all* t *sufficiently small, either* $\lambda^d_{f,z}(0) > \lambda^d_{f_t,z}(0)$
or $\lambda^d_{f,z}(0) = \lambda^d_{f_t,z}(0)$ *and* $\lambda^{d-1}_{f,z}(0) > \lambda^{d-1}_{f_t,z}(0)$

or

$$\vdots$$

or $\lambda^d_{f,z}(0) = \lambda^d_{f_t,z}(0),\ldots,\lambda^1_{f,z}(0) = \lambda^1_{f_t,z}(0), \lambda^0_{f,z}(0) \geq \lambda^0_{f_t,z}(0).$

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- $\lambda_{f,z}(0) - \lambda_{f_t,z}(0)$ - jump of Lê numbers of deformation of f_t

Example

$f(x,y,z) = y^2 + z^3,\ \lambda_{f,z}(0) = (2,0)$
$f_t^k = f + tx^k z^2,\ \lambda_{f_t^k,z}(0) = (1, 3k-1)$
$\lambda_{f,z}(0) - \lambda_{f_t,z}(0) = (1, 1-3k)$.

Line singularities

$f\colon (U,0) \to (\mathbb{C},0)$ - line singularity i.e.
$\Sigma f = \{z \in \mathbb{C}^n \colon z_2 = \dots = z_n = 0\}$
$f|_{V(z_1)}$ - has an isolated singularity at 0

Theorem (Massey)

A tuple of Lê numbers

$$\lambda_{f,z}(0) = (\lambda^d_{f,z}(0), \lambda^{d-1}_{f,z}(0), \dots, \lambda^0_{f,z}(0))$$

is a topological invariant in the class of line singularities.

(f_t) - deformation of f such that f_t line singularity for small t

Theorem (Main Theorem)

f - nondegenerate line singularity, $\lambda^0_{f,z}(0) > 0$. Then there exists a deformation (f_t) such that

$$\lambda^0_{f_t,z}(0) = 0$$

and

$$\lambda^1_{f_t,z}(0) = \lambda^1_{f,z}(0).$$

Example

$f(x,y,z) = y^2 + z^3 + x^2z^2$, $\lambda_{f,z}(0) = (1,5)$.
$f_t = f + tz^2$ - deformation f, $\lambda_{f_t,z}(0) = (1,0)$.
$\lambda_{f,z}(0) - \lambda_{f_t,z}(0) = (0,5)$.

This deformation "straightens" the line singularity along its critical locus.

Minimal jumps of Lê numbers

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- $\min_{(f_t)\in D}\{\lambda_{f,z}(0) - \lambda_{f_t,z}(0)\}$ - the minimal jump of Lê numbers of deformation f_t

Geometric interpretation

- $[\Sigma f] = \lambda^1_{f,z}(0)[z_1 - axis] + \lambda^0_{f,z}(0)[0]$ - critical cycle of f

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- $[\Sigma f_t] = \lambda^1_{f_t,z}(0)[z_1 - axis] + \lambda^0_{f_t,z}(0)[0]$ - critical cycle of f_t
- Minimal jump of $(\lambda^1_{f,z}(0), \lambda^0_{f,z}(0))$ we may interpret as "nearness" of the above cycles

Jump of $\lambda^0_{f,z}(0)$

Proposition

f - line sigularity. Then

$$\lambda^0_{f,z}(0) \neq 1.$$

Example

$f(x,y,z) = y^2 + z^3 + xz^2$, $\lambda_{f,z}(0) = (1,2)$
$f_t = f + tz^2$ - deformacja f, $\lambda_{f_t,z}(0) = (1,0)$
$\lambda_{f,z}(0) - \lambda_{f_t,z}(0) = (0,2)$ - minimal jump

We consider the jump of $\lambda^1_{f,z}(0)$
$D^1 = \{(f_t) : \lambda^1_{f,z}(0) - \lambda^1_{f_t,z}(0) \neq 0\}$

Proposition

There exists a singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *such, that*

$$\min_{(f_t)\in D^1} \{\lambda^1_{f,z}(0) - \lambda^1_{f_t,z}(0)\} > 1.$$

Example

Sz. Brzostowski, T. Krasiński

$$f(x, y, z) = y^4 + z^4 + y^2 z^2$$

$$\lambda^1_{f,z}(0) = \mu_0(f|_{x=x_0}) = 9.$$

$$(f_t) - \text{deformation of f}$$

$$\lambda^1_{f_t,z}(0) = \mu_0(f_t|_{x=x_0}) \leq 7.$$

Thank you very much for your attention!