Multiplicity of singular points as a bi-Lipschitz invariant

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THEOREM 1

Bobadilla, _, Sampaio, 2018

idea of the proof. →

Multiplicity of 2-dimensional complex analytic singularities is a bi-Lipschitz invariant

Theorem 2

Birbrair, _, Sampaio, Verbitsky, 2020

details of the proof. →

Multiplicity of 3-dimensional (or higher dimensional) complex analytic singularities is not a bi-Lipschitz invariant.

Multiplicity

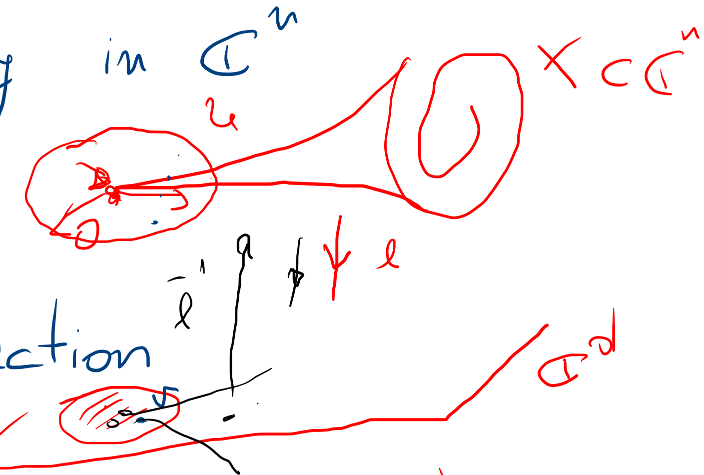
Pure dimension

$(X,0)$ complex analytic singularity in $\mathbb{C}^n$

d = dimension $(X,0)$

$\mathrm{mult}(X,0)=1$

$\ell : \mathbb{C}^n \to \mathbb{C}^d$ linear generic projection



$\therefore\ \ell|_X : (X,0) \to (\mathbb{C}^d, 0)$ finite mapping-germ

$\mathrm{Ker}(\ell) \cap C_0X = \{0\}$

$m = \# \ell^{-1}(r)$ $\quad r$ generic

$\mathrm{mult}(X,0) :=$ topological degree of $\ell|_X$

multiplicity of $(X,0)$

Examples

• $X: \{y^2 = x^3\}$ in $\mathbb{C}^2$

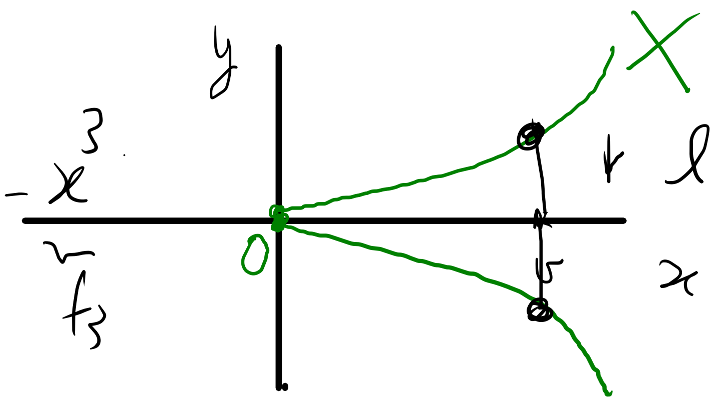
$\text{mult}(X,0) = ? = 2$

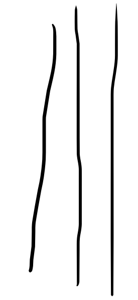
$\# \ell^{-1}(v) \cap X = 2$

$f(x,y) = y^2 - x^3$

$\tilde{f}_3$

$f_2 + f_3$





• $X: \{\underbrace{f(z)=0}_{\text{Reduced equation}}\}$ in $\mathbb{C}^n$.

$\text{mult}(X,0) = ?$
$= m$

$\overset{\neq 0}{f_m}(z) + f_{m+1}(z) + \dots + f_k(z) + \dots = 0$

Bi-Lipschitz equivalence

$(X,0)$ in $\mathbb{C}^n$ is bi-Lipschitz equivalent to $(Y,0)$ in $\mathbb{C}^m$ w.r.t outer

if $\exists$ $\lambda \geq 1$ and a homeomorphism $\phi : (X,0) \to (Y,0)$

$$\frac{1}{\lambda}|p-q| \leq |\phi(p)-\phi(q)| \leq \lambda |p-q| \quad \forall\, p,q \in X$$

close to 0.

Main question:

¿ $(X,0) \underset{\text{bi-Lipschitz equivalent}}{\sim} (Y,0) \Longrightarrow \operatorname{mult}(X,0) = \operatorname{mult}(Y,0)$?

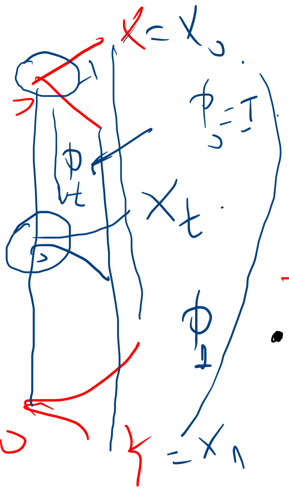
Some references

Zariski ('71) : Is the multiplicity of codimension 1 in $\mathbb{C}^n$ complex analytic singularities an invariant of the embedded topology?

(C. Eyral 2007)

- $n=2$ (Zariski 1932)
- mult $=1$ (A'Campo and Lê 1973)
- $n=3$, mult $=2$ (Navarro Aznar 1980)

- C^1-equivalence (Ephraim 1976, Trotman 1977)
- Diff equivalence (Gau-Lipman 1983)
  any dimension, any codimension!
- Greuel, Saia, O'Shea, Eyral, ...)



- Trotman – Risler 1997

$X = f^{-1}(0)$ $\quad Y = g^{-1}(0)$ $\quad \Longrightarrow \quad \mathrm{mult}(X,0) = \mathrm{mult}(Y,0)$

$\phi : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$

bi-Lipschitz ; $\frac{1}{c}|g| \leq |f \circ \phi| \leq c|g|$

- Comte 1998

$(X,0) \xrightarrow{\phi} (Y,0)$ $\quad \Longrightarrow \quad \mathrm{mult}(X,0) = \mathrm{mult}(Y,0)$

λ bi-Lipschitz $\quad d = \dim(X,0) = \dim(Y,0)$

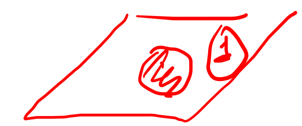
$$1 \leq \lambda \leq \left(1 + \frac{1}{\max\{\mathrm{mult}(X,0), \mathrm{mult}(Y,0)\}}\right)^{\frac{1}{4d}}$$

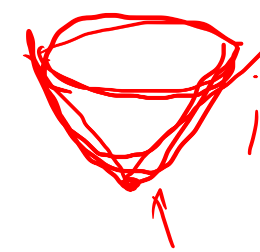
- Neumann, Pichon 2012

multiplicity of normal 2-dimensional complex analytic singularities is a bi-Lipschitz invariant.

- Birbrair, F., Lê, Sampaio 2015

Multiplicity 1 is a bi-Lipschitz invariant.





$|x|$

Example (Proof of theorem 2)

two singularities which are bi-Lipschitz equivalent and having different multiplicities.

$\mathbb{C}P^1 \xhookrightarrow{\text{degree } a} \mathbb{C}P^n$

$[x : y] \longmapsto [x^a : x^{a-1}y : \dots : xy^{a-1} : y^a]$.

$\mathbb{C}P^1 \xhookrightarrow{\text{degree } b} \mathbb{C}P^m$

$\mathbb{C}P^n \times \mathbb{C}P^m \xhookrightarrow{\text{degree } 2} \mathbb{C}P^N$.

$([s_0 : \dots : s_{n-1}], [t_0 : \dots : t_{m-1}]) \to$

$[\dots : s_j t_i : \dots]$

$(X_{a'b'}, 0)$

$\mathbb{CP}^1 \times \mathbb{CP}^1 \xhookrightarrow{\text{degree } 2ab} \mathbb{CP}^N$

$\uparrow$ Im (embedded) $= \tilde{X}_{ab}$.

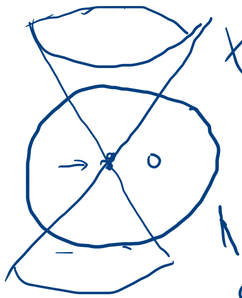
$(X_{ab}, 0)$

$\nparallel$

mult $= 2ab$.

$\boxed{(a,b)\ ;\ (a',b') \quad ab \neq a'b'.}$

$\mathbb{C}^{N+1}$



X_{ab} ≈ homogeneous affine cone associated to $\tilde{X}_{ab}$ $\cong \mathbb{CP}^1 \times \mathbb{CP}^1$

$S^N \cap X_{ab} = \mathrm{link}(X_{ab}, 0)$

$\boxed{\text{compact, 5-dim manifold.}}$ Real

(a,b), (a',b') ;

$S^N \cap X_{ab}$ diffeomorphic to $S^N \cap X_{a'b'}$.

$\boxed{\text{Rmk: } \rightarrow \underline{a} \text{ \& } \underline{b} \text{ coprime.} \Rightarrow \text{link}(X_{ab,0}) \text{ is simply connect.}}$

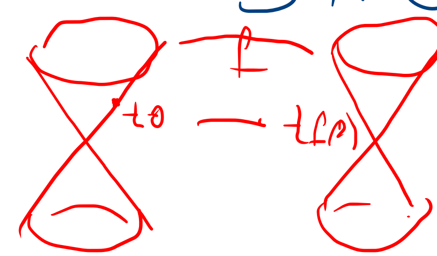
B-S theory $\Rightarrow$ **Proposition**: If M is a compact, simply connected 5-manifold defined as a total space of an S^1-bundle on $\mathbb{C}P^1 \times \mathbb{C}P^1$, then M is diffeomorphic to $S^2 \times S^3$.

$2 \cdot \overset{12}{2 \cdot 3}$.

$(2,3) \rightarrow \text{link}(X_{2,3,0}) \simeq S^2 \times S^3$

$(4,5) \rightarrow \text{lin}(X_{4,5,0}) \simeq S^2 \times S^3$.

$2 \cdot 4 \cdot 5 = 40$.



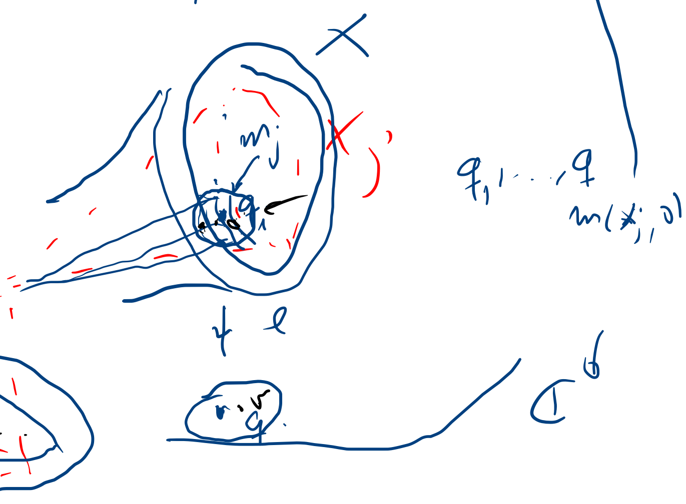
Proof of Theorem 1.

$$\boxed{\operatorname{mult}(X,0) = \sum_{i=1}^{n} m_j \operatorname{mult}(X_j,0)}$$

$X_1, \dots, X_n$ are the irreducible component of the $\underbrace{\text{tangent cone}}_{\text{Reduced}}$ of X at 0.

$$\# \ell^{-1}(v) \cap X = \sum_{j=1}^{n} m_j \operatorname{mult}(X_j, 0)$$

$||$



PROPOSITION Bobadilla, —, Sampaio 2018

Relative multiplicities $\underbrace{m(X, X_j)}_{m_j}$ are bi-Lipschitz invariant.

HOMOGENEOUS CASE

Proposition Bobadilla, –, Sampaio 2018.

2-dim
alg.
Homogeneous
irreducible

Let $(S,0)$ be a 2-dimensional algebraic irreducible homogeneous singularity in $\mathbb{C}^n$
In this case

$$\boxed{\text{Torsion part } H^2(S\setminus 0, \mathbb{Z}) = \mathbb{Z}/\mathrm{mult}(X,0)\,\mathbb{Z}}$$

integral

Thank you