Mixed Bruce-Roberts numbers

Maria Aparecida Ruas (ICMC-USP)
Joint work with Carles Bivià-Ausina (U.P. València) and
Konstantinos Kourliouros (ICMC-USP)

Gdansk-Krakow-Lodz-Warszawa seminar in Singularity Theory

February 26, 2021

I will report on results of two works

I will report on results of two works

- *Mixed Bruce-Roberts numbers,* Carles Bivià-Ausina and M.A.S. Ruas, 2020.

I will report on results of two works

- *Mixed Bruce-Roberts numbers,* Carles Bivià-Ausina and M.A.S. Ruas, 2020.
- *Bruce-Roberts numbers and weighted homogeneous functions,* Carles Bivià-Ausina, Konstantinos Kourliouros and M.A.S. Ruas, work in progress.

I will report on results of two works

- *Mixed Bruce-Roberts numbers,* Carles Bivià-Ausina and M.A.S. Ruas, 2020.
- *Bruce-Roberts numbers and weighted homogeneous functions,* Carles Bivià-Ausina, Konstantinos Kourliouros and M.A.S. Ruas, work in progress.

Aim: Invariants of germs functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point defined on analytic varieties $(X, 0) \subset (\mathbb{C}^n, 0)$.

I will report on results of two works

- *Mixed Bruce-Roberts numbers,* Carles Bivià-Ausina and M.A.S. Ruas, 2020.
- *Bruce-Roberts numbers and weighted homogeneous functions,* Carles Bivià-Ausina, Konstantinos Kourliouros and M.A.S. Ruas, work in progress.

Aim: Invariants of germs functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point defined on analytic varieties $(X, 0) \subset (\mathbb{C}^n, 0)$.

- $\mu_X(f)$

I will report on results of two works

- *Mixed Bruce-Roberts numbers,* Carles Bivià-Ausina and M.A.S. Ruas, 2020.
- *Bruce-Roberts numbers and weighted homogeneous functions,* Carles Bivià-Ausina, Konstantinos Kourliouros and M.A.S. Ruas, work in progress.

Aim: Invariants of germs functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point defined on analytic varieties $(X, 0) \subset (\mathbb{C}^n, 0)$.

- $\mu_X(f)$ and $\tau_X(f)$

I will report on results of two works

- *Mixed Bruce-Roberts numbers,* Carles Bivià-Ausina and M.A.S. Ruas, 2020.
- *Bruce-Roberts numbers and weighted homogeneous functions,* Carles Bivià-Ausina, Konstantinos Kourliouros and M.A.S. Ruas, work in progress.

Aim: Invariants of germs functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point defined on analytic varieties $(X, 0) \subset (\mathbb{C}^n, 0)$.

- $\mu_X(f)$ and $\tau_X(f)$
- Sectional Bruce-Roberts Milnor numbers $\mu_X^{(i)}(f)$

- Bill Bruce and Mark Roberts, 1988.

- Bill Bruce and Mark Roberts, 1988.
- Nivaldo Grulha, 2012

- Bill Bruce and Mark Roberts, 1988.
- Nivaldo Grulha, 2012
- Imran Ahmed, M.A.S. Ruas and João Nivaldo Tomazella, 2013

- Bill Bruce and Mark Roberts, 1988.
- Nivaldo Grulha, 2012
- Imran Ahmed, M.A.S. Ruas and João Nivaldo Tomazella, 2013
- Juan Nuño-Ballesteros, Bruna Oréfice Okamoto, João Nivaldo Tomazella, 2013

- Bill Bruce and Mark Roberts, 1988.
- Nivaldo Grulha, 2012
- Imran Ahmed, M.A.S. Ruas and João Nivaldo Tomazella, 2013
- Juan Nuño-Ballesteros, Bruna Oréfice Okamoto, João Nivaldo Tomazella, 2013
- Konstantinos Kourliouros, 2020

- Bill Bruce and Mark Roberts, 1988.
- Nivaldo Grulha, 2012
- Imran Ahmed, M.A.S. Ruas and João Nivaldo Tomazella, 2013
- Juan Nuño-Ballesteros, Bruna Oréfice Okamoto, João Nivaldo Tomazella, 2013
- Konstantinos Kourliouros, 2020
- Juan Nuño-Ballesteros, Bruna Oréfice-Okamoto, Bárbara Lima-Pereira and João Nivaldo Tomazella, 2020.

The Milnor number

• $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

(2) f has an isolated singularity .

(3) f has an $\mathcal{R}$-versal unfolding.

(4) f is finitely $\mathcal{R}$-determined.

The Milnor number

- $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

- Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$,

(2) f has an isolated singularity .

(3) f has an $\mathcal{R}$-versal unfolding.

(4) f is finitely $\mathcal{R}$-determined.

The Milnor number

• $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

• Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f.

(2) f has an isolated singularity .

(3) f has an $\mathcal{R}$-versal unfolding.

(4) f is finitely $\mathcal{R}$-determined.

The Milnor number

• $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

• Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f.

• $\mathcal{R} = \{\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, holomorphic diffeomorphisms$\}$

(2) f has an isolated singularity .

(3) f has an $\mathcal{R}$-versal unfolding.

(4) f is finitely $\mathcal{R}$-determined.

The Milnor number

• $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

• Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f.

• $\mathcal{R} = \{\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, holomorphic diffeomorphisms$\}$

Theorem

Let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu(f) < \infty$.

The Milnor number

- $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

- Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f.

- $\mathcal{R} = \{\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, holomorphic diffeomorphisms$\}$

Theorem

Let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu(f) < \infty$.

(2) f has an isolated singularity .

The Milnor number

- $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

- Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f.

- $\mathcal{R} = \{\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, holomorphic diffeomorphisms$\}$

Theorem

Let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu(f) < \infty$.

(2) f has an isolated singularity .

(3) f has an $\mathcal{R}$-versal unfolding.

The Milnor number

• $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

• Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f.

• $\mathcal{R} = \{\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, holomorphic diffeomorphisms$\}$

Theorem

Let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu(f) < \infty$.

(2) f has an isolated singularity .

(3) f has an $\mathcal{R}$-versal unfolding.

(4) f is finitely $\mathcal{R}$-determined.

The Milnor number

• $\mathcal{O}_n := \{f : (\mathbb{C}^n, 0) \to \mathbb{C}\}$, local ring of germs of holomorphic functions at 0.

• Milnor number of f: $\mu(f) = \dim \frac{\mathcal{O}_n}{J(f)}$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f.

• $\mathcal{R} = \{\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, holomorphic diffeomorphisms$\}$

Theorem

Let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu(f) < \infty$.

(2) f has an isolated singularity .

(3) f has an $\mathcal{R}$-versal unfolding.

(4) f is finitely $\mathcal{R}$-determined.

The Bruce-Roberts Milnor number

Bruce and Roberts, 1988

- $X \subset (\mathbb{C}^n, 0)$ analytic subvariety

$$\mu_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)}. \quad (1)$$

$\mu_X(f)$ is the *Bruce-Roberts Milnor number of f with respect to X*.

The Bruce-Roberts Milnor number

Bruce and Roberts, 1988

- $X \subset (\mathbb{C}^n, 0)$ analytic subvariety $\quad I(X) \subset \mathcal{O}_n$ ideal defining X.

$$\mu_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)}. \tag{1}$$

$\mu_X(f)$ is the *Bruce-Roberts Milnor number of f with respect to X*.

The Bruce-Roberts Milnor number

Bruce and Roberts, 1988

- $X \subset (\mathbb{C}^n, 0)$ analytic subvariety $I(X) \subset \mathcal{O}_n$ ideal defining X.
- Θ_X: $\mathcal{O}_n$-module of germs at 0 of vector fields in $\mathbb{C}^n$ tangent to X:

$$\Theta_X = \left\{\delta \in \Theta_n : \delta(I(X)) \subseteq I(X)\right\}.$$

$$\mu_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)}. \quad (1)$$

$\mu_X(f)$ is the *Bruce-Roberts Milnor number of f with respect to X*.

The Bruce-Roberts Milnor number

Bruce and Roberts, 1988

- $X \subset (\mathbb{C}^n, 0)$ analytic subvariety $\quad I(X) \subset \mathcal{O}_n$ ideal defining X.
- Θ_X: $\mathcal{O}_n$-module of germs at 0 of vector fields in $\mathbb{C}^n$ tangent to X:

$$\Theta_X = \left\{\delta \in \Theta_n : \delta(I(X)) \subseteq I(X)\right\}.$$

This module is also denoted by $\mathrm{Derlog}(X)$, or Der(-log$X$) (Saito).

$$\mu_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)}. \tag{1}$$

$\mu_X(f)$ is the *Bruce-Roberts Milnor number of f with respect to X*.

The Bruce-Roberts Milnor number

Bruce and Roberts, 1988

- $X \subset (\mathbb{C}^n, 0)$ analytic subvariety $I(X) \subset \mathcal{O}_n$ ideal defining X.
- Θ_X: $\mathcal{O}_n$-module of germs at 0 of vector fields in $\mathbb{C}^n$ tangent to X:

$$\Theta_X = \left\{\delta \in \Theta_n : \delta(I(X)) \subseteq I(X)\right\}.$$

This module is also denoted by $\mathrm{Derlog}(X)$, or Der(-log$X$) (Saito).

Definition

Let $X \subset (\mathbb{C}^n, 0)$ and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

The Bruce-Roberts Milnor number

Bruce and Roberts, 1988

- $X \subset (\mathbb{C}^n, 0)$ analytic subvariety $I(X) \subset \mathcal{O}_n$ ideal defining X.
- Θ_X: $\mathcal{O}_n$-module of germs at 0 of vector fields in $\mathbb{C}^n$ tangent to X:

$$\Theta_X = \left\{ \delta \in \Theta_n : \delta(I(X)) \subseteq I(X) \right\}.$$

This module is also denoted by $\mathrm{Derlog}(X)$, or Der(-log$X$) (Saito).

Definition

Let $X \subset (\mathbb{C}^n, 0)$ and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

$$\mu_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)}. \tag{1}$$

$\mu_X(f)$ is the *Bruce-Roberts Milnor number of f with respect to X*.

Let $X \subset U$ be an analytic variety, U open neighbourhood of 0.

$\forall x \in U$, $\Theta_X(x)$: linear subspace of $\mathbb{C}^n$ generated by $\{\delta(x) : \delta \in \Theta_{X,x}\}$.

Let $X \subset U$ be an analytic variety, U open neighbourhood of 0.

$\forall x \in U,\ \Theta_X(x)$: linear subspace of $\mathbb{C}^n$ generated by $\{\delta(x) : \delta \in \Theta_{X,x}\}$.

Theorem (Saito)

There exists a unique stratification $\{X_\alpha\}_{\alpha\in\Lambda}$ of U with the following properties.

(1) Each X_α is a smooth connected submanifold of U and U is the disjoint union $\cup_\alpha X_\alpha$.

(2) For all $\alpha \in \Lambda$: $T_x X_\alpha = \Theta_X(x)$, for all $x \in X_\alpha$.

(3) Let $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$, such that $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, then $X_\alpha \subseteq \partial X_\beta$.

Let $X \subset U$ be an analytic variety, U open neighbourhood of 0.
$\forall x \in U$, $\Theta_X(x)$: linear subspace of $\mathbb{C}^n$ generated by $\{\delta(x) : \delta \in \Theta_{X,x}\}$.

Theorem (Saito)

There exists a unique stratification $\{X_\alpha\}_{\alpha\in\Lambda}$ of U with the following properties.

(1) Each X_α is a smooth connected submanifold of U and U is the disjoint union $\cup_\alpha X_\alpha$.

(2) For all $\alpha \in \Lambda$: $T_x X_\alpha = \Theta_X(x)$, for all $x \in X_\alpha$.

(3) Let $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$, such that $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, then $X_\alpha \subseteq \partial X_\beta$.

Definition

$\{X_\alpha\}_{\alpha\in A}$ is the *logarithmic stratification of U relative to X* and each X_α is called a logarithmic stratum. The collection $\{X_\alpha \cap X\}_{\alpha\in A}$ is a stratification of X called the *logarithmic stratification of X*.

Let $\mathcal{R}(X)$ denote the group of diffeomorphims $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $\phi(X) = X$.

(2) $V(df(\Theta_X)) \subseteq \{0\}$.

(3) f has an $\mathcal{R}(X)$-versal unfolding.

(4) f is finitely $\mathcal{R}(X)$-determined.

(5) The restriction of f to each logarithmic stratum of X is a submersion except, possibly, at 0.

Let $\mathcal{R}(X)$ denote the group of diffeomorphims $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $\phi(X) = X$.

Theorem (Bruce-Roberts)

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu_X(f)$ is finite.

Let $\mathcal{R}(X)$ denote the group of diffeomorphims $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $\phi(X) = X$.

Theorem (Bruce-Roberts)

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu_X(f)$ is finite.

(2) $V(df(\Theta_X)) \subseteq \{0\}$.

Let $\mathcal{R}(X)$ denote the group of diffeomorphims $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $\phi(X) = X$.

Theorem (Bruce-Roberts)

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu_X(f)$ is finite.

(2) $V(df(\Theta_X)) \subseteq \{0\}$.

(3) f has an $\mathcal{R}(X)$-versal unfolding.

Let $\mathcal{R}(X)$ denote the group of diffeomorphims $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $\phi(X) = X$.

Theorem (Bruce-Roberts)

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu_X(f)$ is finite.

(2) $V(df(\Theta_X)) \subseteq \{0\}$.

(3) f has an $\mathcal{R}(X)$-versal unfolding.

(4) f is finitely $\mathcal{R}(X)$-determined.

Let $\mathcal{R}(X)$ denote the group of diffeomorphims $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $\phi(X) = X$.

Theorem (Bruce-Roberts)

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. Then the following conditions are equivalent:

(1) $\mu_X(f)$ is finite.

(2) $V(df(\Theta_X)) \subseteq \{0\}$.

(3) f has an $\mathcal{R}(X)$-versal unfolding.

(4) f is finitely $\mathcal{R}(X)$-determined.

(5) The restriction of f to each logarithmic stratum of X is a submersion except, possibly, at 0.

Example: Let $X = \{(x, y, z) \in \mathbb{C}^3 : xyz = 0\}$, then

$$\Theta_X = \langle (x, 0, 0), (0, y, 0), (0, 0, z) \rangle.$$

Example: Let $X = \{(x, y, z) \in \mathbb{C}^3 : xyz = 0\}$, then

$$\Theta_X = \langle (x, 0, 0), (0, y, 0), (0, 0, z) \rangle.$$

(1) Let $f \in \mathcal{O}_3$ be given by $f(x, y, z) = xy + xz + yz$, for all $(x, y, z) \in \mathbb{C}^3$. Therefore $Jf(\Theta_X) = \langle xy + xz, xy + yz, xz + yz \rangle$, which has not finite colength, that is, $\mu_X(f)$ is not finite, whereas f has an isolated singularity at the origin.

Example: Let $X = \{(x, y, z) \in \mathbb{C}^3 : xyz = 0\}$, then

$$\Theta_X = \langle (x, 0, 0), (0, y, 0), (0, 0, z) \rangle.$$

(1) Let $f \in \mathcal{O}_3$ be given by $f(x, y, z) = xy + xz + yz$, for all $(x, y, z) \in \mathbb{C}^3$. Therefore $Jf(\Theta_X) = \langle xy + xz, xy + yz, xz + yz \rangle$, which has not finite colength, that is, $\mu_X(f)$ is not finite, whereas f has an isolated singularity at the origin.

(2) If f is a generic linear form, $\mu_X(f) = 1$

Morse functions on X

Let χ be a Whitney stratification of X, and X_0 the stratum containing 0.

Morse functions on X

Let χ be a Whitney stratification of X, and X_0 the stratum containing 0.

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is Morse for χ:

Morse functions on X

Let χ be a Whitney stratification of X, and X_0 the stratum containing 0.

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is Morse for χ:

  (i) If $X_0 \neq \{0\}$, then 0 is a nondegenerate critical point of $f|_{X_0}$,

Morse functions on X

Let χ be a Whitney stratification of X, and X_0 the stratum containing 0.

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is Morse for χ:
  (i) If $X_0 \neq \{0\}$, then 0 is a nondegenerate critical point of $f|_{X_0}$,
  (ii) If $x_n \in X_\alpha,\ x_n \to 0,\ T_{x_n}X_\alpha \to T$, and $0 \notin X_\alpha$ then $df(0)$ must have maximal rank at T.

Morse functions on X

Let χ be a Whitney stratification of X, and X_0 the stratum containing 0.

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is Morse for χ:
  (i) If $X_0 \neq \{0\}$, then 0 is a nondegenerate critical point of $f|_{X_0}$,
  (ii) If $x_n \in X_\alpha,\ x_n \to 0,\ \ T_{x_n}X_\alpha \to T$, and $0 \notin X_\alpha$ then $df(0)$ must have maximal rank at T.

- A Morsification of f is a family $f_t : U \to \mathbb{C}$ such that $f_0 = f$ and $f_t,\ t \neq 0$ has only Morse singularities on X inside U.

Morse functions on X

Let χ be a Whitney stratification of X, and X_0 the stratum containing 0.

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is Morse for χ:
  (i) If $X_0 \neq \{0\}$, then 0 is a nondegenerate critical point of $f|_{X_0}$,
  (ii) If $x_n \in X_\alpha,\ x_n \to 0,\ T_{x_n}X_\alpha \to T$, and $0 \notin X_\alpha$ then $df(0)$ must have maximal rank at T.

- A Morsification of f is a family $f_t : U \to \mathbb{C}$ such that $f_0 = f$ and $f_t,\ t \neq 0$ has only Morse singularities on X inside U.

Does the Bruce-Roberts Milnor number of f gives information about the topology of the Morsification f_t ?

The logarithmic caracteristic variety

X holonomic

The logarithmic caracteristic variety

X holonomic

Definition

The Logarithmic Caracteristic Variety of X, $LC(X)$, is defined as follows. Suppose that $\delta_1, \ldots, \delta_s$ generate the module Θ_X in some neighbourhood U of $0 \in \mathbb{C}^n$. Let $T_U^*\mathbb{C}^n$ denote the restriction to U of the cotangent bundle of $\mathbb{C}^n$. We define $LC_U(X)$ to be

$$\{(x, \xi) \in T_U^*(\mathbb{C}^n) : \xi(\delta_i) = 0, 1 \leq i \leq s\}$$

$LC(X)$ is the germ of $LC_U(X)$ at $0 \in \mathbb{C}^n$.

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.

$\mu_X(f) \geq \Sigma_\alpha n_\alpha m_\alpha$, with equality if and only if $LC(X)$ is Cohen-Macaulay.

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.
- m_α : multiplicity of Y_α in $LC(X)$.

$\mu_X(f) \geq \Sigma_\alpha n_\alpha m_\alpha$, with equality if and only if $LC(X)$ is Cohen-Macaulay.

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.
- m_α : multiplicity of Y_α in $LC(X)$.
- The derivative of $f : \mathbb{C}^n \to \mathbb{C}$ defines a section of $T^*\mathbb{C}^n$,

$\mu_X(f) \geq \Sigma_\alpha n_\alpha m_\alpha$, with equality if and only if $LC(X)$ is Cohen-Macaulay.

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.
- m_α : multiplicity of Y_α in $LC(X)$.
- The derivative of $f : \mathbb{C}^n \to \mathbb{C}$ defines a section of $T^*\mathbb{C}^n$,

$$Df : \mathbb{C}^n \to T^*\mathbb{C}^n, \;\; Df(x) = (x, df(x))$$

$\mu_X(f) \geq \Sigma_\alpha n_\alpha m_\alpha$, with equality if and only if $LC(X)$ is Cohen-Macaulay.

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.
- m_α : multiplicity of Y_α in $LC(X)$.
- The derivative of $f : \mathbb{C}^n \to \mathbb{C}$ defines a section of $T^*\mathbb{C}^n$,

$$Df : \mathbb{C}^n \to T^*\mathbb{C}^n, \;\; Df(x) = (x, df(x))$$

Theorem (Bruce-Roberts)

If f has a critical point at 0 and n_α is the number of critical points of a Morsification of f on X_α, then

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.
- m_α : multiplicity of Y_α in $LC(X)$.
- The derivative of $f : \mathbb{C}^n \to \mathbb{C}$ defines a section of $T^*\mathbb{C}^n$,

$$Df : \mathbb{C}^n \to T^*\mathbb{C}^n, \quad Df(x) = (x, df(x))$$

Theorem (Bruce-Roberts)

If f has a critical point at 0 and n_α is the number of critical points of a Morsification of f on X_α, then $\mu_X(f) \geq \Sigma_\alpha n_\alpha m_\alpha,$

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.
- m_α : multiplicity of Y_α in $LC(X)$.
- The derivative of $f : \mathbb{C}^n \to \mathbb{C}$ defines a section of $T^*\mathbb{C}^n$,

$$Df : \mathbb{C}^n \to T^*\mathbb{C}^n, \;\; Df(x) = (x, df(x))$$

Theorem (Bruce-Roberts)

If f has a critical point at 0 and n_α is the number of critical points of a Morsification of f on X_α, then $\mu_X(f) \geq \Sigma_\alpha n_\alpha m_\alpha$, with equality if and only if $LC(X)$ is Cohen-Macaulay.

- Y_α : irreducible components of the logarithmic characteristic subvariety $LC(X) \subset T^*\mathbb{C}^n$.
- m_α : multiplicity of Y_α in $LC(X)$.
- The derivative of $f : \mathbb{C}^n \to \mathbb{C}$ defines a section of $T^*\mathbb{C}^n$,

$$Df : \mathbb{C}^n \to T^*\mathbb{C}^n, \quad Df(x) = (x, df(x))$$

Theorem (Bruce-Roberts)

If f has a critical point at 0 and n_α is the number of critical points of a Morsification of f on X_α, then $\mu_X(f) \geq \Sigma_\alpha n_\alpha m_\alpha$, with equality if and only if $LC(X)$ is Cohen-Macaulay.

• If cod$X > 1$, $LC(X)$ is not Cohen-Macaulay.

Theorem (Aleksandrov-Kersken, Wahl)

Let $w \in \mathbb{Z}^n_{\geq 1}$, and $\theta_w = w_1 x_1 \frac{\partial}{\partial x_1} + \cdots + w_n x_n \frac{\partial}{\partial x_n}$ (Euler v.f.).

Theorem (Aleksandrov-Kersken, Wahl)

Let $w \in \mathbb{Z}^n_{\geq 1}$, and $\theta_w = w_1 x_1 \frac{\partial}{\partial x_1} + \cdots + w_n x_n \frac{\partial}{\partial x_n}$ (Euler v.f.).
Let $X = h^{-1}(0),\ h = (h_1, \ldots, h_p)$ be a weighted homogeneous ICIS with respect to w, $n - p \geq 1$.

Theorem (Aleksandrov-Kersken, Wahl)

Let $w \in \mathbb{Z}^n_{\geq 1}$, and $\theta_w = w_1 x_1 \frac{\partial}{\partial x_1} + \cdots + w_n x_n \frac{\partial}{\partial x_n}$ (Euler v.f.).
Let $X = h^{-1}(0)$, $h = (h_1, \ldots, h_p)$ be a weighted homogeneous ICIS with respect to w, $n - p \geq 1$. Then Θ_X is generated by $\{\theta_w, h_i \frac{\partial}{\partial x_j} : i = 1, \ldots, p, j = 1, \ldots, n\}$, and the derivations coming from

$$I_{p+1} \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}. \tag{2}$$

In particular, given any function $f \in \mathcal{O}_n$, we have

$$Jf(\Theta_X) = \langle \theta_w(f) \rangle + \langle h_1, \ldots, h_p \rangle J(f) + \mathbf{J}(f, h_1, \ldots, h_p). \tag{3}$$

Theorem (Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, 2013)

- If $X = h^{-1}(0)$ is a weighted homogeneous hypersurface with isolated singularity, and $\mu_X(f) < \infty$, then

$$\mu_X(f) = \mu(f) + \mu(X \cap f^{-1}(0))$$

Theorem (Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, 2013)

- If $X = h^{-1}(0)$ is a weighted homogeneous hypersurface with isolated singularity, and $\mu_X(f) < \infty$, then

$$\mu_X(f) = \mu(f) + \mu(X \cap f^{-1}(0))$$

- $LC(X)$ is Cohen-Macaulay.

Theorem (Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, 2013)

- If $X = h^{-1}(0)$ is a weighted homogeneous hypersurface with isolated singularity, and $\mu_X(f) < \infty$, then

$$\mu_X(f) = \mu(f) + \mu(X \cap f^{-1}(0))$$

- $LC(X)$ is Cohen-Macaulay.

Recall Lê-Greuel formula

Theorem (Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, 2013)

- If $X = h^{-1}(0)$ is a weighted homogeneous hypersurface with isolated singularity, and $\mu_X(f) < \infty$, then

$$\mu_X(f) = \mu(f) + \mu(X \cap f^{-1}(0))$$

- $LC(X)$ is Cohen-Macaulay.

Recall Lê-Greuel formula

$$\mu(f) + \mu(X \cap f^{-1}(0)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f, J(f,h)\rangle}$$

Theorem (Nuño-Ballesteros, Oréfice-Okamoto, Tomazella, 2013)

- If $X = h^{-1}(0)$ is a weighted homogeneous hypersurface with isolated singularity, and $\mu_X(f) < \infty$, then

$$\mu_X(f) = \mu(f) + \mu(X \cap f^{-1}(0))$$

- $LC(X)$ is Cohen-Macaulay.

Recall Lê-Greuel formula

$$\mu(f) + \mu(X \cap f^{-1}(0)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f, J(f,h) \rangle}$$

The above result is not verified for non-homogeneous hypersurfaces and neither for homogeneous ICIS.

The formula $\mu_X(f) = \mu(f) + \mu(X \cap f^{-1}(0))$, X weighted homogeneous has been generalized recently independently

Theorem (K. Kourliouros; B. de Lima Pereira, J. Nuño-Ballesteros, B. Oréfice-Okamoto, J. Tomazella)

Let $h \in \mathcal{O}_n$ with isolated singularity at the origin and let $X = h^{-1}(0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then $(f, h) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ is an ICIS whose Milnor number satisfies the relation

$$\mu_X(f) = \mu(f) + \mu(f, h) + \mu(h) - \tau(h). \tag{4}$$

The formula $\mu_X(f) = \mu(f) + \mu(X \cap f^{-1}(0))$, X weighted homogeneous has been generalized recently independently

Theorem (K. Kourliouros; B. de Lima Pereira, J. Nuño-Ballesteros, B. Oréfice-Okamoto, J. Tomazella)

Let $h \in \mathcal{O}_n$ with isolated singularity at the origin and let $X = h^{-1}(0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then $(f, h) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ is an ICIS whose Milnor number satisfies the relation

$$\mu_X(f) = \mu(f) + \mu(f, h) + \mu(h) - \tau(h). \tag{4}$$

Lima Pereira, Nuño-Ballesteros, Oréfice-Okamoto, Tomazella: If X is a hypersurface with isolated singularities, LC(X) is Cohen-Macaulay

The Bruce-Roberts Tjurina number

Definition

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. We define

$$\tau_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f \rangle + df(\Theta_X)}. \tag{5}$$

When the colength on the right is finite, we refer to $\tau_X(f)$ as the *Bruce-Roberts Tjurina number of f with respect to X*.

The Bruce-Roberts Tjurina number

Definition

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. We define

$$\tau_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f \rangle + df(\Theta_X)}. \tag{5}$$

When the colength on the right is finite, we refer to $\tau_X(f)$ as the *Bruce-Roberts Tjurina number of f with respect to X*.

$\tau_X(f) \leq \mu_X(f)$

The Bruce-Roberts Tjurina number

Definition

Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. We define

$$\tau_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f \rangle + df(\Theta_X)}. \tag{5}$$

When the colength on the right is finite, we refer to $\tau_X(f)$ as the *Bruce-Roberts Tjurina number of* f *with respect to* X.

$\tau_X(f) \leq \mu_X(f)$

Goals : Compare the two invariants and find conditions for equality to be verified

Let R be a ring and let I be an ideal of R. For any $f \in R$, we define

$$r_f(I) = \min \left\{ r \in \mathbb{Z}_{\geq 1} : f^r \in I \right\}.$$

If no such r exist, then we set $r_f(I) = \infty$.

$$\varphi_{f,I} : R/I \to R/I, \ \ \varphi_{f,I}(g) = fg$$

Let R be a ring and let I be an ideal of R. For any $f \in R$, we define

$$r_f(I) = \min\left\{r \in \mathbb{Z}_{\geq 1} : f^r \in I\right\}.$$

If no such r exist, then we set $r_f(I) = \infty$.

$$\varphi_{f,I} : R/I \to R/I, \ \ \varphi_{f,I}(g) = fg$$

Theorem (Bivià-Ausina and Ruas)

Let $(R, \mathbf{m})$ be a Noetherian local ring. Let I be an ideal of R of finite colength and let $f \in R$ such that $r_f(I) < \infty$. Then

$$\frac{\ell\left(\frac{R}{I}\right)}{\ell\left(\frac{R}{\langle f\rangle + I}\right)} \leq r_f(I) \tag{6}$$

and equality holds if and only if $\ker(\varphi_{f,I}) = \frac{\langle f^{r-1}\rangle + I}{I}$, where $r = r_f(I)$.

Estimating $\mu_X(f)/\tau_X(f)$

Corollary (Bivià-Ausina, Ruas)

$X \subset (\mathbb{C}^n, 0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then

$$\frac{\mu_X(f)}{\tau_X(f)} \leq r_f(df(\Theta_X)) \tag{7}$$

and equality holds if and only if $\ker(\varphi_{f,df(\Theta_X)}) = \frac{\langle f^{r-1}\rangle + df(\Theta_X)}{df(\Theta_X)}$, where $r = r_f(df(\Theta_X))$.

$\Theta_X = \langle(-2x^4y^3, 5y^6 + 2x^3y^4 + 5x^9), (2x, 3y)\rangle$.Given $f(x,y) = x + y$, $\mu_X(f) = 6$ and $\tau_X(f) = 1$, $\quad r_f(J_X(f)) = 6$.

Estimating $\mu_X(f)/\tau_X(f)$

Corollary (Bivià-Ausina, Ruas)

$X \subset (\mathbb{C}^n, 0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then

$$\frac{\mu_X(f)}{\tau_X(f)} \leq r_f(df(\Theta_X)) \tag{7}$$

and equality holds if and only if $\ker(\varphi_{f,df(\Theta_X)}) = \frac{\langle f^{r-1}\rangle + df(\Theta_X)}{df(\Theta_X)}$, where $r = r_f(df(\Theta_X))$.

Example

Let $h(x,y) = xy^6 + x^4y^4 + x^{10}, \quad X = h^{-1}(0),$

Estimating $\mu_X(f)/\tau_X(f)$

Corollary (Bivià-Ausina, Ruas)

$X \subset (\mathbb{C}^n, 0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then

$$\frac{\mu_X(f)}{\tau_X(f)} \leq r_f(df(\Theta_X)) \tag{7}$$

and equality holds if and only if $\ker(\varphi_{f,df(\Theta_X)}) = \frac{\langle f^{r-1}\rangle + df(\Theta_X)}{df(\Theta_X)}$, where $r = r_f(df(\Theta_X))$.

Example

Let $h(x,y) = xy^6 + x^4y^4 + x^{10}$, $\quad X = h^{-1}(0)$,
$\Theta_X = \langle(-2x^4y^3, 5y^6 + 2x^3y^4 + 5x^9), (2x, 3y)\rangle$.

Estimating $\mu_X(f)/\tau_X(f)$

Corollary (Bivià-Ausina, Ruas)

$X \subset (\mathbb{C}^n, 0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then

$$\frac{\mu_X(f)}{\tau_X(f)} \leq r_f(df(\Theta_X)) \tag{7}$$

and equality holds if and only if $\ker(\varphi_{f,df(\Theta_X)}) = \frac{\langle f^{r-1}\rangle + df(\Theta_X)}{df(\Theta_X)}$, where $r = r_f(df(\Theta_X))$.

Example

Let $h(x,y) = xy^6 + x^4y^4 + x^{10}, \quad X = h^{-1}(0)$,
$\Theta_X = \langle(-2x^4y^3, 5y^6 + 2x^3y^4 + 5x^9), (2x, 3y)\rangle$.Given $f(x,y) = x + y$,
$\mu_X(f) = 6$ and $\tau_X(f) = 1, \quad r_f(J_X(f)) = 6$.

Theorem (Bivià-Ausina and Ruas)

$h = (h_1, \ldots, h_p)$ weighted homogeneous ICIS with respect to w, $p \leq n - 1$. Let $f \in \mathcal{O}_n,\ \ \mu_X(f) < \infty$. Then (f, h) is also an ICIS and

$$\mu(h) \leq (r-1)\mu(f, h), \tag{8}$$

$r = r_{\pi(\theta_w(f))}\left(\pi(J(f, h_1, \ldots, h_p))\right)$, and $\pi : \mathcal{O}_n \to \frac{\mathcal{O}_n}{\langle h_1, \ldots, h_p \rangle}$. Moreover, if $R = \mathcal{O}_n / \left(\langle h_1, \ldots, h_p \rangle\right)$, then equality holds if and only if the kernel of the automorphism of R defined by multiplication by $\theta_w(f)$ is the ideal generated by the image of $\theta_w(f)^{r-1}$ in R.

$\mu_X(f) = \tau_X(f)$

Bivià-Ausina, Kourliouros, Ruas, (in progress).

$\mu_X(f) = \tau_X(f)$

Bivià-Ausina, Kourliouros, Ruas, (in progress).

The following conditions are equivalent:

$\mu_X(f) = \tau_X(f)$

Bivià-Ausina, Kourliouros, Ruas, (in progress).

The following conditions are equivalent:

- The equality $\tau_X(f) = \mu_X(f)$

$\mu_X(f) = \tau_X(f)$

Bivià-Ausina, Kourliouros, Ruas, (in progress).

The following conditions are equivalent:

- The equality $\tau_X(f) = \mu_X(f)$
- $f \in df(\Theta_X)$,

$\mu_X(f) = \tau_X(f)$

Bivià-Ausina, Kourliouros, Ruas, (in progress).

The following conditions are equivalent:

- The equality $\tau_X(f) = \mu_X(f)$
- $f \in df(\Theta_X)$,
- There exists a vector field $\delta \in \Theta_X$ such that $\delta(f) = f$.

$\mu_X(f) = \tau_X(f)$

Bivià-Ausina, Kourliouros, Ruas, (in progress).

The following conditions are equivalent:

- The equality $\tau_X(f) = \mu_X(f)$
- $f \in df(\Theta_X)$,
- There exists a vector field $\delta \in \Theta_X$ such that $\delta(f) = f$.

In particular $f \in Jf$, so that f is weighted homogeneous, but δ might not be an Euler vector field for f.

$\mu_X(f) = \tau_X(f)$

Bivià-Ausina, Kourliouros, Ruas, (in progress).

The following conditions are equivalent:

- The equality $\tau_X(f) = \mu_X(f)$
- $f \in df(\Theta_X)$,
- There exists a vector field $\delta \in \Theta_X$ such that $\delta(f) = f$.

In particular $f \in Jf$, so that f is weighted homogeneous, but δ might not be an Euler vector field for f.

Definition

The pair (f, X) is relatively weighted homogeneous if there exist a coordinate system such that both f and X are weighted homogeneous with the same rational weights $w \in \mathbb{Q}_+^n$.

Relative Saito's theorem

Conjectura (Bivià-Ausina, Kourliouros and Ruas)

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then $\mu_X(f) = \tau_X(f)$ if and only if (f, X) is relatively quasihomogeneous.

Relative Saito's theorem

Conjectura (Bivià-Ausina, Kourliouros and Ruas)

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then $\mu_X(f) = \tau_X(f)$ if and only if (f, X) is relatively quasihomogeneous.

- X may have nonisolated singularities

Relative Saito's theorem

Conjectura (Bivià-Ausina, Kourliouros and Ruas)

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then $\mu_X(f) = \tau_X(f)$ if and only if (f, X) is relatively quasihomogeneous.

- *X* may have nonisolated singularities
  - Clearly if (f, X) is relatively quasihomogeneous, then $\mu_X(f) = \tau_X(f)$.

Relative Saito's theorem

Conjectura (Bivià-Ausina, Kourliouros and Ruas)

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then $\mu_X(f) = \tau_X(f)$ if and only if (f, X) is relatively quasihomogeneous.

- X may have nonisolated singularities
  - Clearly if (f, X) is relatively quasihomogeneous, then $\mu_X(f) = \tau_X(f)$.
  - Notice that the equality $\mu_X(f) = \tau_X(f)$ implies $\mu(f) = \tau(f)$, so that f is weighted homogeneous in some coordinate system. But we don't know if X is weighted homogeneous with respect to the same weights.

Theorem (Bivià-Ausina, Kourliouros and Ruas)

The conjecture is true in the following cases:

(Saito (1971), Scheja and Wiebe (1973, 1977, 1980), Domitrz, Janeczko and Zhitomirskii (2004), Kruff (2020).

Poincaré-Dulac form

$$\delta = \delta_S + \delta_N$$

Theorem (Bivià-Ausina, Kourliouros and Ruas)

The conjecture is true in the following cases:

- (a) $f \in \mathcal{M}^3$

(Saito (1971), Scheja and Wiebe (1973, 1977, 1980), Domitrz, Janeczko and Zhitomirskii (2004), Kruff (2020).

Poincaré-Dulac form

$$\delta = \delta_S + \delta_N$$

Theorem (Bivià-Ausina, Kourliouros and Ruas)

The conjecture is true in the following cases:

- (a) $f \in \mathcal{M}^3$
- (b) X is an IHS (isolated hypersurface singularity).

(Saito (1971), Scheja and Wiebe (1973, 1977, 1980), Domitrz, Janeczko and Zhitomirskii (2004), Kruff (2020).
Poincaré-Dulac form

$$\delta = \delta_S + \delta_N$$

Theorem (Bivià-Ausina, Kourliouros and Ruas)

The conjecture is true in the following cases:

- (a) $f \in \mathcal{M}^3$
- (b) X is an IHS (isolated hypersurface singularity).

Theorem

X is weighted homogeneous if and only if there is a logarithmic vector field $\delta \in \Theta_X$, $\delta(0) = 0$ such that δ has positive rational eigenvalues $w \in \mathbb{Q}^n_+$.

Theorem (Bivià-Ausina, Kourliouros and Ruas)

The conjecture is true in the following cases:

- (a) $f \in \mathcal{M}^3$
- (b) X is an IHS (isolated hypersurface singularity).

Theorem

X is weighted homogeneous if and only if there is a logarithmic vector field $\delta \in \Theta_X$, $\delta(0) = 0$ such that δ has positive rational eigenvalues $w \in \mathbb{Q}^n_+$. (Saito (1971), Scheja and Wiebe (1973, 1977, 1980), Domitrz, Janeczko and Zhitomirskii (2004), Kruff (2020).

Theorem (Bivià-Ausina, Kourliouros and Ruas)

The conjecture is true in the following cases:

- (a) $f \in \mathcal{M}^3$
- (b) X is an IHS (isolated hypersurface singularity).

Theorem

X is weighted homogeneous if and only if there is a logarithmic vector field $\delta \in \Theta_X$, $\delta(0) = 0$ such that δ has positive rational eigenvalues $w \in \mathbb{Q}_+^n$. (Saito (1971), Scheja and Wiebe (1973, 1977, 1980), Domitrz, Janeczko and Zhitomirskii (2004), Kruff (2020).

Poincaré-Dulac form

$$\delta = \delta_S + \delta_N$$

Idea of the proof of the theorem

Lemma 1

Let (f, X) be a pair with $\mu_X(f) < \infty$. Let $Y = f^{-1}(0)$, and $H_Y := \ker df(.) \subset \Theta_Y$, where $df(.) : \Theta_n \to \mathcal{O}_n$. Then

$$\mu_X(f) = \mu(f) + \mu(f_X),$$

where $\mu(f_X) = \dim_{\mathbb{C}} \frac{\Theta_n}{\Theta_X + H_Y}$.

Idea of the proof of the theorem

Lemma 1

Let (f, X) be a pair with $\mu_X(f) < \infty$. Let $Y = f^{-1}(0)$, and $H_Y := \ker df(.) \subset \Theta_Y$, where $df(.) : \Theta_n \to \mathcal{O}_n$. Then

$$\mu_X(f) = \mu(f) + \mu(f_X),$$

where $\mu(f_X) = \dim_{\mathbb{C}} \frac{\Theta_n}{\Theta_X + H_Y}$.

Lemma 2

Let (f, X) be a pair with $\mu_X(f) < \infty$. Let $Y = f^{-1}(0)$,

$$\tau_X(f) = \tau(f) + \tau(f_X),$$

where $\tau(f_X) = \dim_{\mathbb{C}} \frac{\Theta_n}{\Theta_X + \Theta_Y}$.

Write

$$q_X(f) = \mu_X(f) - \tau_X(f), \quad q(f) = \mu(f) - \tau(f), \quad q(f_X) = \mu(f_X) - \tau(f_X)$$

where $q(f) = \dim_{\mathbb{C}} \frac{df(\Theta_n) + <f>}{df(\Theta_n)}$ and $q(f_X) = \dim_{\mathbb{C}} \frac{\Theta_Y}{\Theta_X + H_Y}$

Write

$$q_X(f) = \mu_X(f) - \tau_X(f), \quad q(f) = \mu(f) - \tau(f), \quad q(f_X) = \mu(f_X) - \tau(f_X)$$

Corollary

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then

$$q_X(f) = q(f) + q(f_X),$$

Write

$$q_X(f) = \mu_X(f) - \tau_X(f), \quad q(f) = \mu(f) - \tau(f), \quad q(f_X) = \mu(f_X) - \tau(f_X)$$

Corollary

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then

$$q_X(f) = q(f) + q(f_X),$$

where $q(f) = \dim_{\mathbb{C}} \frac{df(\Theta_n) + <f>}{df(\Theta_n)}$

Write

$$q_X(f) = \mu_X(f) - \tau_X(f), \quad q(f) = \mu(f) - \tau(f), \quad q(f_X) = \mu(f_X) - \tau(f_X)$$

Corollary

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then

$$q_X(f) = q(f) + q(f_X),$$

where $q(f) = \dim_{\mathbb{C}} \frac{df(\Theta_n) + <f>}{df(\Theta_n)}$ and $q(f_X) = \dim_{\mathbb{C}} \frac{\Theta_Y}{\Theta_X + H_Y}$

Write

$$q_X(f) = \mu_X(f) - \tau_X(f), \quad q(f) = \mu(f) - \tau(f), \quad q(f_X) = \mu(f_X) - \tau(f_X)$$

Corollary

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then

$$q_X(f) = q(f) + q(f_X),$$

where $q(f) = \dim_{\mathbb{C}} \frac{df(\Theta_n) + <f>}{df(\Theta_n)}$ and $q(f_X) = \dim_{\mathbb{C}} \frac{\Theta_Y}{\Theta_X + H_Y}$

If $q_X(f) = 0$, then $q(f) = 0$ and $q(f_X) = 0$:

Write

$$q_X(f) = \mu_X(f) - \tau_X(f), \quad q(f) = \mu(f) - \tau(f), \quad q(f_X) = \mu(f_X) - \tau(f_X)$$

Corollary

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then

$$q_X(f) = q(f) + q(f_X),$$

where $q(f) = \dim_{\mathbb{C}} \frac{df(\Theta_n) + <f>}{df(\Theta_n)}$ and $q(f_X) = \dim_{\mathbb{C}} \frac{\Theta_Y}{\Theta_X + H_Y}$

If $q_X(f) = 0$, then $q(f) = 0$ and $q(f_X) = 0$:

- $\exists$ an Euler vector field $\theta_w^f \in \Theta_Y$ such that $\theta_w^f(f) = f$

Write

$$q_X(f) = \mu_X(f) - \tau_X(f), \quad q(f) = \mu(f) - \tau(f), \quad q(f_X) = \mu(f_X) - \tau(f_X)$$

Corollary

Let (f, X) be a pair with $\mu_X(f) < \infty$. Then

$$q_X(f) = q(f) + q(f_X),$$

where $q(f) = \dim_{\mathbb{C}} \frac{df(\Theta_n) + <f>}{df(\Theta_n)}$ and $q(f_X) = \dim_{\mathbb{C}} \frac{\Theta_Y}{\Theta_X + H_Y}$

If $q_X(f) = 0$, then $q(f) = 0$ and $q(f_X) = 0$:

- $\exists$ an Euler vector field $\theta_w^f \in \Theta_Y$ such that $\theta_w^f(f) = f$
- $\exists \delta^X \in \Theta_X \cap \Theta_Y$, $\exists \eta^f \in H_Y$ such that $\delta^X = \theta_w^f + \eta^f$

- (a) If $f \in \mathcal{M}^3$, then all vector fields in H_Y are nilpotent, with zero eigenvalues,

- (a) If $f \in \mathcal{M}^3$, then all vector fields in H_Y are nilpotent, with zero eigenvalues,so that taking the semisimple part of δ^X we get

- (a) If $f \in \mathcal{M}^3$, then all vector fields in H_Y are nilpotent, with zero eigenvalues,so that taking the semisimple part of δ^X we get

$$\delta_S^X = \theta_w^f.$$

- (a) If $f \in \mathcal{M}^3$, then all vector fields in H_Y are nilpotent, with zero eigenvalues,so that taking the semisimple part of δ^X we get

$$\delta_S^X = \theta_w^f.$$

- (b) If X is IHS, then it suffices to consider $m_0(f) \leq 2$ and $m_0(h) \leq 2$

- (a) If $f \in \mathcal{M}^3$, then all vector fields in H_Y are nilpotent, with zero eigenvalues,so that taking the semisimple part of δ^X we get

$$\delta_S^X = \theta_w^f.$$

- (b) If X is IHS, then it suffices to consider $m_0(f) \leq 2$ and $m_0(h) \leq 2$ (as in this case the roles of f and h can be interchanged)

- (a) If $f \in \mathcal{M}^3$, then all vector fields in H_Y are nilpotent, with zero eigenvalues,so that taking the semisimple part of δ^X we get

$$\delta_S^X = \theta_w^f.$$

- (b) If X is IHS, then it suffices to consider $m_0(f) \leq 2$ and $m_0(h) \leq 2$ (as in this case the roles of f and h can be interchanged)

  (this can be done with the help of Saito's normal forms)

Remark: The missing case

X has nonisolated singularities, $\mu_X(f) < \infty$, $m_0(f) = 1, 2$.

Remark: The missing case

X has nonisolated singularities, $\mu_X(f) < \infty$, $m_0(f) = 1, 2$.

In particular, when $m_0(f) = 1$, i.e. , regular sections of X :

Remark: The missing case

X has nonisolated singularities, $\mu_X(f) < \infty$, $m_0(f) = 1, 2$.

In particular, when $m_0(f) = 1$, i.e. , regular sections of X :

- (Hyperplane) Sections $H = f^{-1}(0)$ of discriminants $X = \Delta(G)$ of a stable mapping G. (almost free divisors) (Damon).

Remark: The missing case

X has nonisolated singularities, $\mu_X(f) < \infty$, $m_0(f) = 1, 2$.

In particular, when $m_0(f) = 1$, i.e. , regular sections of X :

- (Hyperplane) Sections $H = f^{-1}(0)$ of discriminants $X = \Delta(G)$ of a stable mapping G. (almost free divisors) (Damon). Example: normal crossing divisors.

Remark: The missing case

X has nonisolated singularities, $\mu_X(f) < \infty$, $m_0(f) = 1, 2$.

In particular, when $m_0(f) = 1$, i.e. , regular sections of X :

- (Hyperplane) Sections $H = f^{-1}(0)$ of discriminants $X = \Delta(G)$ of a stable mapping G. (almost free divisors) (Damon). Example: normal crossing divisors.
- Determinantal varieties represented by square matrices and their (hyperplane) sections. (Mond and Goryunov)

Remark: The missing case

X has nonisolated singularities, $\mu_X(f) < \infty$, $m_0(f) = 1, 2$.

In particular, when $m_0(f) = 1$, i.e. , regular sections of X :

- (Hyperplane) Sections $H = f^{-1}(0)$ of discriminants $X = \Delta(G)$ of a stable mapping G. (almost free divisors) (Damon). Example: normal crossing divisors.
- Determinantal varieties represented by square matrices and their (hyperplane) sections. (Mond and Goryunov)
- Sections of line singularities. (Massey, Siersma, Eyral)

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$

- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f), \quad \alpha^* = (1, 2, 3, 4)$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \ldots, n$, H^i linear space of dimension i, define
  $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad$ set $\mu^{(0)}(f) = 1$

- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f), \quad \alpha^* = (1, 2, 3, 4)$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \ldots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad$ set $\mu^{(0)}(f) = 1$
- Teissier: $\mu^*(f) = (\mu^{(n)}(f), \ldots, \mu^{(1)}(f), \mu^{(0)}(f))$.

- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f), \quad \alpha^* = (1, 2, 3, 4)$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \ldots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad$ set $\mu^{(0)}(f) = 1$
- Teissier: $\quad \mu^*(f) = (\mu^{(n)}(f), \ldots, \mu^{(1)}(f), \mu^{(0)}(f))$.

(1) $\mu^{(1)}(f) = \text{ord}(f) - 1$.

- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f), \quad \alpha^* = (1, 2, 3, 4)$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \dots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad$ set $\mu^{(0)}(f) = 1$
- Teissier: $\mu^*(f) = (\mu^{(n)}(f), \dots, \mu^{(1)}(f), \mu^{(0)}(f))$.

(1) $\mu^{(1)}(f) = \text{ord}(f) - 1$.

(2) $\mu^{(n)}(f) \geq \mu^{(n-1)}(f) \geq \cdots \geq \mu^{(1)}(f)$.

- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f), \quad \alpha^* = (1, 2, 3, 4)$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \ldots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad$ set $\mu^{(0)}(f) = 1$
- Teissier: $\quad \mu^*(f) = (\mu^{(n)}(f), \ldots, \mu^{(1)}(f), \mu^{(0)}(f))$.

(1) $\mu^{(1)}(f) = \text{ord}(f) - 1$.

(2) $\mu^{(n)}(f) \geq \mu^{(n-1)}(f) \geq \cdots \geq \mu^{(1)}(f)$.

- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f), \quad \alpha^* = (1, 2, 3, 4)$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \dots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad \text{set } \mu^{(0)}(f) = 1$
- Teissier: $\mu^*(f) = (\mu^{(n)}(f), \dots, \mu^{(1)}(f), \mu^{(0)}(f)).$

(1) $\mu^{(1)}(f) = \text{ord}(f) - 1.$

(2) $\mu^{(n)}(f) \geq \mu^{(n-1)}(f) \geq \cdots \geq \mu^{(1)}(f).$

Example

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \ldots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad$ set $\mu^{(0)}(f) = 1$
- Teissier: $\quad \mu^*(f) = (\mu^{(n)}(f), \ldots, \mu^{(1)}(f), \mu^{(0)}(f)).$

(1) $\mu^{(1)}(f) = \mathrm{ord}(f) - 1.$

(2) $\mu^{(n)}(f) \geq \mu^{(n-1)}(f) \geq \cdots \geq \mu^{(1)}(f).$

Example

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$
- $\mu^*(f) = (1, 1, 1, 1)$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \ldots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad \text{set } \mu^{(0)}(f) = 1$
- Teissier: $\mu^*(f) = (\mu^{(n)}(f), \ldots, \mu^{(1)}(f), \mu^{(0)}(f)).$

(1) $\mu^{(1)}(f) = \text{ord}(f) - 1.$

(2) $\mu^{(n)}(f) \geq \mu^{(n-1)}(f) \geq \cdots \geq \mu^{(1)}(f).$

Example

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$
- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f),$

Teissier's μ^*-sequence

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad \mu(f) < \infty$
- For $i = 1, \ldots, n$, H^i linear space of dimension i, define $\mu^{(i)}(f) := \mu(f|_{H^{(i)}}), \quad$ set $\mu^{(0)}(f) = 1$
- Teissier: $\quad \mu^*(f) = (\mu^{(n)}(f), \ldots, \mu^{(1)}(f), \mu^{(0)}(f))$.

(1) $\mu^{(1)}(f) = \text{ord}(f) - 1$.

(2) $\mu^{(n)}(f) \geq \mu^{(n-1)}(f) \geq \cdots \geq \mu^{(1)}(f)$.

Example

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$
- $\mu^*(f) = (1, 1, 1, 1)$
- $\alpha^i(f) := \mu_{H^i}(f), \quad \alpha^* = (1, 2, 3, 4)$

Bruce-Roberts numbers and linear sections

- $f \in \mathcal{O}_n,\ \mu(f) < \infty$ and $X \subseteq (\mathbb{C}^n, 0)$.

Bruce-Roberts numbers and linear sections

- $f \in \mathcal{O}_n, \; \mu(f) < \infty$ and $X \subseteq (\mathbb{C}^n, 0)$.
- For $i \in \{1, \ldots, n\}$, let $L_{i,n} := \{p : \mathbb{C}^i \to \mathbb{C}^n, \text{ linear embedding}\}$

Bruce-Roberts numbers and linear sections

- $f \in \mathcal{O}_n,\ \mu(f) < \infty$ and $X \subseteq (\mathbb{C}^n, 0)$.
- For $i \in \{1, \ldots, n\}$, let $L_{i,n} := \{p : \mathbb{C}^i \to \mathbb{C}^n,$ linear embedding$\}$
- $L_{i,n}(f, X) := \{p \in L_{i,n}$ such that $\mu_{p^{-1}(X)}(f \circ p)$ is finite $\}$

Bruce-Roberts numbers and linear sections

- $f \in \mathcal{O}_n,\ \mu(f) < \infty$ and $X \subseteq (\mathbb{C}^n, 0)$.
- For $i \in \{1, \ldots, n\}$, let $L_{i,n} := \{p : \mathbb{C}^i \to \mathbb{C}^n,$ linear embedding$\}$
- $L_{i,n}(f, X) := \{p \in L_{i,n}$ such that $\mu_{p^{-1}(X)}(f \circ p)$ is finite $\}$

Definition

For any $i \in \{1, \ldots, n\}$ such that $L_{i,n}(f, X) \neq \emptyset$, we define the number

$$\mu_X^{(i)}(f) = \min_{p \in L_{i,n}(f,X)} \mu_{p^{-1}(X)}(f \circ p).$$

Bruce-Roberts numbers and linear sections

- $f \in \mathcal{O}_n,\ \mu(f) < \infty$ and $X \subseteq (\mathbb{C}^n, 0)$.
- For $i \in \{1, \ldots, n\}$, let $L_{i,n} := \{p : \mathbb{C}^i \to \mathbb{C}^n,$ linear embedding$\}$
- $L_{i,n}(f, X) := \{p \in L_{i,n}$ such that $\mu_{p^{-1}(X)}(f \circ p)$ is finite $\}$

Definition

For any $i \in \{1, \ldots, n\}$ such that $L_{i,n}(f, X) \neq \emptyset$, we define the number

$$\mu_X^{(i)}(f) = \min_{p \in L_{i,n}(f,X)} \mu_{p^{-1}(X)}(f \circ p).$$

If $L_{i,n}(f, X) = \emptyset$, then we set $\mu_X^{(i)}(f) = \infty$.

Bruce-Roberts numbers and linear sections

- $f \in \mathcal{O}_n,\ \mu(f) < \infty$ and $X \subseteq (\mathbb{C}^n, 0)$.
- For $i \in \{1, \ldots, n\}$, let $L_{i,n} := \{p : \mathbb{C}^i \to \mathbb{C}^n,$ linear embedding$\}$
- $L_{i,n}(f, X) := \{p \in L_{i,n}$ such that $\mu_{p^{-1}(X)}(f \circ p)$ is finite $\}$

Definition

For any $i \in \{1, \ldots, n\}$ such that $L_{i,n}(f, X) \neq \emptyset$, we define the number

$$\mu_X^{(i)}(f) = \min_{p \in L_{i,n}(f,X)} \mu_{p^{-1}(X)}(f \circ p).$$

If $L_{i,n}(f, X) = \emptyset$, then we set $\mu_X^{(i)}(f) = \infty$.

- *Mixed Bruce-Roberts numbers of f with respect to X.*

Bruce-Roberts numbers and linear sections

- $f \in \mathcal{O}_n,\ \mu(f) < \infty$ and $X \subseteq (\mathbb{C}^n, 0)$.
- For $i \in \{1, \ldots, n\}$, let $L_{i,n} := \{p : \mathbb{C}^i \to \mathbb{C}^n,$ linear embedding$\}$
- $L_{i,n}(f, X) := \{p \in L_{i,n}$ such that $\mu_{p^{-1}(X)}(f \circ p)$ is finite $\}$

Definition

For any $i \in \{1, \ldots, n\}$ such that $L_{i,n}(f, X) \neq \emptyset$, we define the number

$$\mu_X^{(i)}(f) = \min_{p \in L_{i,n}(f,X)} \mu_{p^{-1}(X)}(f \circ p).$$

If $L_{i,n}(f, X) = \emptyset$, then we set $\mu_X^{(i)}(f) = \infty$.

- *Mixed Bruce-Roberts numbers of f with respect to X.*

$$\mu_X^*(f) := (\mu_X^{(n)}(f), \ldots, \mu_X^{(1)}(f))$$

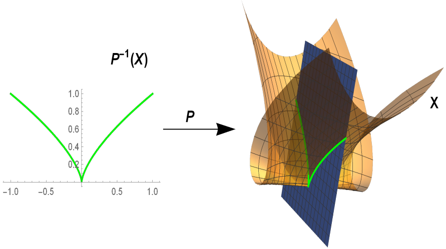

Figure: $X : 2x^2y^2 + y^3 - z^2 + x^4y = 0;\ \ P(u,v) = (0,u,v)$

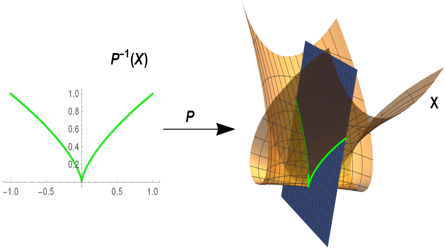

Figure: $X : 2x^2y^2 + y^3 - z^2 + x^4y = 0;\ P(u, v) = (0, u, v)$

Question: Is the sequence $\mu^*_X(f)$ a complete invariant of the Whitney equisingularity of families $f_t,\ \mu_X(f_t) < \infty$ defined on X?

Derlog and lowerable vector fields

Definition

- $\text{Low}_X(p) = \{\theta \in \Theta_i : dp \circ \theta = \eta \circ p, \text{ for some } \eta \in \Theta_X\}$ *lowerable vector fields with respect to p and X*.

Derlog and lowerable vector fields

Definition

- $\text{Low}_X(p) = \left\{\theta \in \Theta_i : dp \circ \theta = \eta \circ p, \text{ for some } \eta \in \Theta_X\right\}$ *lowerable vector fields with respect to p and X*.
- $Lif_X(p) := \{\eta \in \Theta_X \,|\, \exists \theta \in \Theta_i \text{ such that } dp \circ \theta = \eta \circ p\}$

Derlog and lowerable vector fields

Definition

- $\mathrm{Low}_X(p) = \left\{\theta \in \Theta_i : dp \circ \theta = \eta \circ p, \text{ for some } \eta \in \Theta_X\right\}$ *lowerable vector fields with respect to p and X*.
- $Lif_X(p) := \{\eta \in \Theta_X \,|\exists \theta \in \Theta_i \text{ such that } dp \circ \theta = \eta \circ p\}$

$\mathrm{Low}_X(p)$ is an $\mathcal{O}_i$-submodule of Θ_i

Derlog and lowerable vector fields

Definition

- $\mathrm{Low}_X(p) = \{\theta \in \Theta_i : dp \circ \theta = \eta \circ p, \text{ for some } \eta \in \Theta_X\}$ *lowerable vector fields with respect to p and X*.
- $Lif_X(p) := \{\eta \in \Theta_X \,|\exists \theta \in \Theta_i \text{ such that } dp \circ \theta = \eta \circ p\}$

$\mathrm{Low}_X(p)$ is an $\mathcal{O}_i$-submodule of Θ_i $\mathrm{Lif}_X(p)$ is an $\mathcal{O}_n$-submodule of Θ_n.

Derlog and lowerable vector fields

Definition

- $\mathrm{Low}_X(p) = \{\theta \in \Theta_i : dp \circ \theta = \eta \circ p, \text{ for some } \eta \in \Theta_X\}$ *lowerable vector fields with respect to p and X*.
- $Lif_X(p) := \{\eta \in \Theta_X \,|\exists \theta \in \Theta_i \text{ such that } dp \circ \theta = \eta \circ p\}$

$\mathrm{Low}_X(p)$ is an $\mathcal{O}_i$-submodule of Θ_i $\mathrm{Lif}_X(p)$ is an $\mathcal{O}_n$-submodule of Θ_n.

Proposition

Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a homogeneous ICIS, $n - m \geq 1$ and $X = h^{-1}(0)$. Let $p : \mathbb{C}^i \to \mathbb{C}^n, \; i \in \{m+1, \ldots, n\}, \;\; p \in L_{i,n}$ such that $h \circ p : (\mathbb{C}^i, 0) \to (\mathbb{C}^m, 0)$ is an ICIS of positive dimension. Then

$$\mathrm{Low}_X(p) = \Theta_{p^{-1}(X)}. \tag{9}$$

Thank you!