Some normalizations of real algebraic varieties

The structure of the statement

The talk is based on the following papers:

• Integral closures in real algebraic geometry, G. Fichou, J-P. Monnier, R. Quarez, arXiv:1810.07556, to appear in Journal of Algebraic Geometry

• Weak and semi normalization in real algebraic geometry, G. Fichou, J-P. Monnier, R. Quarez, arXiv:1706.04467, to appear in Annali della Scuola Normale Superiore di Pisa, Classe di Scienze.

• Central algebraic geometry and seminormality, work in progress.

Central algebraic geometry

Let A be a domain with fraction field $\mathcal{K}(A)$. We recall that the (Zariski) spectrum of A is

$$\operatorname{Spec} A = \{\mathfrak{p} \text{ is a prime ideal of } A\}$$

The spectrum of A is a topological space where the closed subsets are $\mathcal{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subset \mathfrak{p}\}$ for $I \subset A$ an ideal. The maximal spectrum of A is

$$\operatorname{Max} A = \{\mathfrak{m} \text{ is a maximal ideal of } A\}$$

If $A = \mathbb{R}[V]$ is the coordinate ring of an irreducible affine algebraic variety V over $\mathbb{R}$ ($V = \operatorname{Spec} \mathbb{R}[V]$) then by the Nullstellensatz we get

$$\operatorname{Max} \mathbb{R}[V] = V(\mathbb{C})$$

Definition: (real ideal)

Let I be an ideal of A. We say that I is real if for every $a \in A$, for every $b_1, \ldots, b_k \in A$ we have:

$$a^2 + b_1^2 + \cdots + b_k^2 \in I \Rightarrow a \in I$$

For example, $(x-1)$ is real but $(1+x^2)$ is not real in $\mathbb{R}[x]$.
We recall that the real (Zariski) spectrum of A is

$$\text{R-Spec}\, A = \{\mathfrak{p} \text{ is a real prime ideal of } A\}$$

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$$\text{R-Max}\, A = \{\mathfrak{m} \text{ is a real and maximal ideal of } A\}$$

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Central algebraic geometry

Let V be an affine algebraic variety over $\mathbb{R}$, we denote by $V(\mathbb{R})$ the real locus of V (the set of real closed points i.e the set of maximal ideals with residue field equal to $\mathbb{R}$). If I is an ideal of $\mathbb{R}[V]$ then $\mathcal{Z}(I) = \mathcal{V}(I) \cap V(\mathbb{R}) = \{x \in V(\mathbb{R}) \mid I \subset \mathfrak{m}_x\}$ is the real zero set of I. If S is a subset of $V(\mathbb{R})$ then we denote by $\mathcal{I}(S)$ the ideal of $f \in \mathbb{R}[V]$ such that $S \subset \mathcal{Z}(f)$.

Theorem: (Real Nullstellensatz)

Let V be an affine algebraic variety over $\mathbb{R}$. Then:

$$I \subset \mathbb{R}[V] \text{ is a real ideal } \Leftrightarrow I = \mathcal{I}(\mathcal{Z}(I)).$$

In particular,

$$\text{R-Max}\,\mathbb{R}[V] = V(\mathbb{R})$$

Central algebraic geometry

Definition: (central ideal)

Let I be an ideal of A. We say that I is central if for every $a \in A$, for every $b_1, \dots, b_k \in \mathcal{K}(A)$ we have:

$$a^2 + b_1^2 + \cdots + b_k^2 \in I \Rightarrow a \in I$$

Remark

I is central $\Rightarrow$ I is real $\Rightarrow$ I is radical.

Then, the central spectrum of A is

$$\text{C-Spec}\, A = \{\mathfrak{p} \text{ is a central prime ideal of } A\}$$

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Central algebraic geometry

Theorem: (Central Nullstellensatz)

Let V be an affine algebraic variety over $\mathbb{R}$. Then:

$$I \subset \mathbb{R}[V] \text{ is a central ideal } \Leftrightarrow I = \mathcal{I}(\mathcal{V}(I) \cap \overline{V_{reg}(\mathbb{R})}^E) \Leftrightarrow I = \mathcal{I}(\mathcal{Z}(I) \cap \overline{V_{reg}(\mathbb{R})}^E).$$

In particular,

$$\text{C-Max}\,\mathbb{R}[V] = \overline{V_{reg}(\mathbb{R})}^E$$

In the sequel, we denote by $\text{Cent}\,V$ the set $\text{C-Max}\,\mathbb{R}[V] = \overline{V_{reg}(\mathbb{R})}^E$, it is called the central locus of V. We say that V is central if $V(\mathbb{R}) = \text{Cent}\,V$.

Central algebraic geometry

Examples

• Let V be the Whitney umbrella i.e the real algebraic surface with equation $y^2 = zx^2$. Then $\mathfrak{p} = (x, y) \subset \mathbb{R}[V]$ is a central prime ideal since $\mathcal{Z}(\mathfrak{p})$ (the "z"-axis and the stick of the umbrella) meets $\operatorname{Cent} V$ in dimension one (the intersection is half of the stick).

• Let V be the Cartan umbrella i.e the real algebraic surface with equation $x^3 = z(x^2 + y^2)$. Then $\mathfrak{p} = (x, y) \subset \mathbb{R}[V]$ is not a central prime ideal since $\mathcal{Z}(\mathfrak{p})$ (the "z"-axis and the stick of the umbrella) meets $\operatorname{Cent} V$ in a single point. We prove now without the Nullstellensatz that $\mathfrak{p}$ is not central:

We have

$$b = x^2 + y^2 - z^2 = x^2 + y^2 - \frac{x^6}{(x^2+y^2)^2} = \frac{3x^4y^2 + 3x^2y^4 + y^6}{(x^2+y^2)^2} \in (\sum \mathcal{K}(V)^2) \cap \mathbb{R}[V]$$

thus $z^2 + b = x^2 + y^2 \in \mathfrak{p}$ but $z \notin \mathfrak{p}$.

The Whitney Umbrella

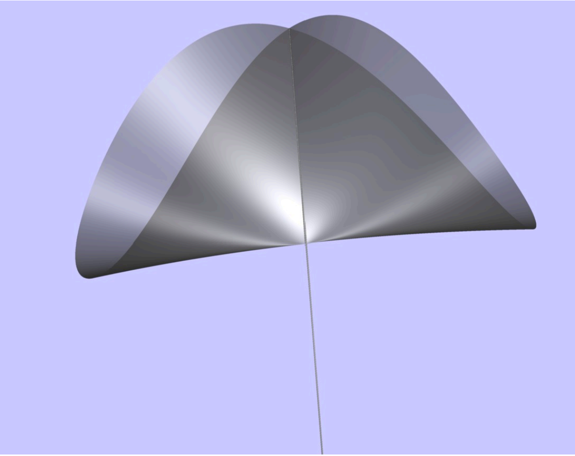


Figure: Whitney Umbrella

The Cartan Umbrella

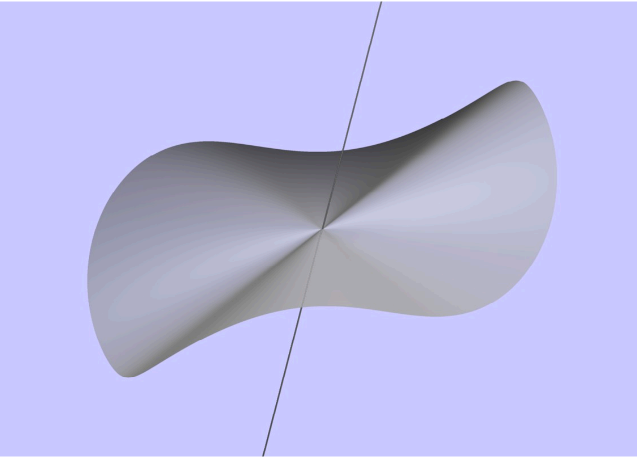


Figure: Cartan Umbrella

Lying-over property

When you normalize an affine algebraic variety the associated extension of coordinate rings is integral (even finite) and birational (same ring of fractions). Indeed if V is an (irreducible) affine algebraic variety over a field k the coordinate ring $k[V']$ of the normalisation V' of V is the integral closure of $k[V]$ in $\mathcal{K}(V)$ i.e

$$k[V'] = k[V]'_{\mathcal{K}(V)}$$

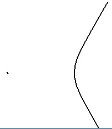
So to understand the normalization of algebraic varieties then we have to understand the properties of integral extensions of rings and we consider domains in order to simplify.

Proposition: (Lying-over)

If $A \to B$ is an integral extension of rings then the associated maps $\operatorname{Spec} B \to \operatorname{Spec} A$ and $\operatorname{Max} B \to \operatorname{Max} A$, $\mathfrak{p} \mapsto \mathfrak{p} \cap A$ are well defined and surjective.

Example: No real lying-over!

If $A \to B$ is an integral extension of rings then the associated map $\operatorname{R-Spec} B \to \operatorname{R-Spec} A$, $\mathfrak{p} \mapsto \mathfrak{p} \cap A$ is well defined but not always surjective. Consider the plane isolated nodal cubic C given with coordinate ring $\mathbb{R}[C] = \mathbb{R}[x,y]/(y^2 - x^2(x-1))$. Let C' be the normalization of C, we have $\mathbb{R}[C'] = \mathbb{R}[x,Y]/(Y^2 - (x-1)) = \mathbb{R}[C][y/x]$ and the finite birational extension $\mathbb{R}[C] \to \mathbb{R}[C']$ is given by sending x to x and y to Yx. Over the real and maximal ideal (x,y) of $\mathbb{R}[C]$ there is a unique maximal ideal of $\mathbb{R}[C']$, namely (x, Y^2+1), and this ideal is not real.



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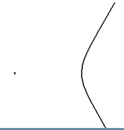

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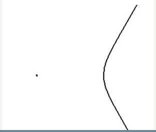

Lying-over property

We have a lying-over property for central ideals. This is a ”central” result in our work!

Theorem: (Central lying-over)

Let $A \to B$ be an integral extension of domains. The maps $\text{C-Spec}\, B \to \text{C-Spec}\, A$ and $\text{C-Max}\, B \to \text{C-Max}\, A$, $\mathfrak{p} \mapsto \mathfrak{p} \cap A$ are well defined. Moreover, if the extension is birational (i.e induces an isomorphism $\mathcal{K}(A) \simeq \mathcal{K}(B)$) then these maps are surjective.

Corollary

Let $\varphi : V \to W$ be a finite and birational morphism between two irreducible affine varieties over $\mathbb{R}$. The maps $\text{C-Spec}\, \mathbb{R}[W] \to \text{C-Spec}\, \mathbb{R}[V]$ and $\text{Cent}\, W \to \text{Cent}\, V$ are well defined and surjective. Moreover, the last one is also closed for the euclidean topology and thus is a quotient map.

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WN and SN of complex algebraic varieties

Let V be an irreducible affine algebraic variety over $\mathbb{C}$. We have $V = \mathrm{Spec}\,\mathbb{C}[V]$, $V(\mathbb{C})$ is a complex algebraic set and

$$\mathbb{C}[V] = \mathbb{C}[x_1,\ldots,x_n]/\mathcal{I}(V) = \mathcal{O}(V(\mathbb{C})) \subset \mathcal{K}(V) = \mathcal{K}(V(\mathbb{C}))$$

The normalization V' of V is the algebraic variety such that $\mathbb{C}[V'] = \mathbb{C}[V]'_{\mathcal{K}(V)}$. The morphism $\pi' : V' \to V$ is finite and birational.

Example

Let V be the nodal plane curve with equation $y^2 - x^2(x+1) = 0$ then $\mathbb{C}[V'] = \mathbb{C}[V][y/x]$ and $(y/x)^2 - (x+1) = 0$.

Universal property for the normalization

Let $\pi : W \to V$ be a morphism between affine algebraic varieties over $\mathbb{C}$. Then, π is finite and birational iff there exists a unique morphism $\psi : V' \to W$ such that $\pi' = \pi \circ \psi$

$$\begin{array}{ccc} W & \xrightarrow{\pi} & V \\ \uparrow & \overset{\pi'}{\nearrow} & \\ V' & & \end{array}$$

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WN and SN of complex algebraic varieties

The weak normalization V^w (WN for short) of V (Andreotti and Bombieri (1969)) is the algebraic variety with coordinate ring

$$\mathbb{C}[V^w] = \{f \in \mathbb{C}[V'] \mid f \text{ is constant on the fibers of } \pi' : V'(\mathbb{C}) \to V(\mathbb{C})\}$$
$$= \{f \in \mathbb{C}[V'] \mid \forall \mathfrak{m} \in \operatorname{Max} \operatorname{Spec} \mathbb{C}[V],\ f \in \mathbb{C}[V]_{\mathfrak{m}} + \operatorname{Rad}(\mathbb{C}[V']_{\mathfrak{m}})\}$$

We have a finite and birational morphism $\pi^w : V^w \to V$.

The seminormalization ${}^+A$ (SN for short) of a domain A with integral closure A' (Traverso (1970)) is

$${}^+A = \{f \in A' \mid \forall \mathfrak{p} \in \operatorname{Spec} A,\ f \in A_{\mathfrak{p}} + \operatorname{Rad}(A'_{\mathfrak{p}})\}$$

Traverso has shown that ${}^+A$ is the biggest subring $B \subset A'$ such that $\operatorname{Spec} B \to \operatorname{Spec} A$, $\mathfrak{p} \mapsto \mathfrak{p} \cap A$ is bijective and equiresidual i.e the extension $k(\mathfrak{p} \cap A) \to k(\mathfrak{p})$ is an isomorphism.

WN=SN for complex varieties!

We have ${}^+\mathbb{C}[V] = \mathbb{C}[V^w]$.

Universal property for the seminormalization

Let $\pi : W \to V$ be a morphism between affine algebraic varieties over $\mathbb{C}$. Then, π is finite and birational and the map $W(\mathbb{C}) \to V(\mathbb{C})$ is bijective iff there exists a unique morphism $\psi : V^w \to W$ such that $\pi^w = \pi \circ \psi$

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WN and SN of complex algebraic varieties

The plane nodal curve with equation $y^2 - x^2(x+1) = 0$ is seminormal, the seminormalization of the cuspidal plane curve with equation $y^2 - x^3 = 0$ is its normalization, the threefolium curve with equation $(x^2 + y^2)^2 - x(x^2 - 3y^2) = 0$ is not seminormal.

The real case doesn't work well!

There exists an irreducible affine variety V over $\mathbb{R}$ such that the set of affine varieties W with a finite and birational morphism $W \to V$ such that $W(\mathbb{R}) \to V(\mathbb{R})$ is a bijection doesn't have a biggest element.

Rational continuous functions

Two founding papers by Kollár and Nowak (Math. Z. 2015) and Fichou, Huisman, Mangolte and M. (Crelle's journal 2016). A talk at ICM Rio 2018 by Kucharz and Kurdyka on the subject.
Let V be an irreducible affine algebraic variety over $\mathbb{R}$ (such that $V(\mathbb{R})$ is Zariski dense in $V(\mathbb{C})$). Then, $V(\mathbb{R})$ is a real algebraic set (a real algebraic variety in the classical sense) and

$$\mathcal{O}(V(\mathbb{R})) = \{f \in \mathcal{K}(V) \mid f \text{ doesn't have poles on } V(\mathbb{R})\}$$

is the ring of regular functions on $V(\mathbb{R})$. For example, $\frac{x}{1+x^2} \in \mathcal{O}(\mathbb{A}^1(\mathbb{R})) \setminus \mathbb{R}[\mathbb{A}^1]$.
Classically (following the complex case), we consider real algebraic varieties provided with the sheaf of regular functions and there are a lot of problems: Not a classical Nullstellensatz, no Theorems A and B of Cartan! It was our motivation to introduce the sheaf of regulous functions on real algebraic varieties, we have now a classical Nullstellensatz and Theorems A and B of Cartan.

Definition: (rational continuous function)

A real function f on $V(\mathbb{R})$ is called a rational continuous function if f is continuous and rational (is regular on a non-empty Zariski open subset of $V(\mathbb{R})$). Let $\mathcal{K}^0(V(\mathbb{R}))$ be the ring of rational continuous functions on $V(\mathbb{R})$.

- The function $\frac{x^3}{x^2+y^2} \in \mathcal{K}^0(\mathbb{A}^2(\mathbb{R}))$.
- Rational continuous=Blow-regular.

Rational continuous functions

Two founding papers by Kollár and Nowak (Math. Z. 2015) and Fichou, Huisman, Mangolte and M. (Crelle's journal 2016). A talk at ICM Rio 2018 by Kucharz and Kurdyka on the subject.
Let V be an irreducible affine algebraic variety over $\mathbb{R}$ (such that $V(\mathbb{R})$ is Zariski dense in $V(\mathbb{C})$). Then, $V(\mathbb{R})$ is a real algebraic set (a real algebraic variety in the classical sense) and

$$\mathcal{O}(V(\mathbb{R})) = \{f \in \mathcal{K}(V) \mid f \text{ doesn't have poles on } V(\mathbb{R})\}$$

is the ring of regular functions on $V(\mathbb{R})$. For example, $\frac{x}{1+x^2} \in \mathcal{O}(\mathbb{A}^1(\mathbb{R})) \setminus \mathbb{R}[\mathbb{A}^1]$.
Classically (following the complex case), we consider real algebraic varieties provided with the sheaf of regular functions and there are a lot of problems: Not a classical Nullstellensatz, no Theorems A and B of Cartan! It was our motivation to introduce the sheaf of regulous functions on real algebraic varieties, we have now a classical Nullstellensatz and Theorems A and B of Cartan.

Definition: (rational continuous function)

A real function f on $V(\mathbb{R})$ is called a rational continuous function if f is continuous and rational (is regular on a non-empty Zariski open subset of $V(\mathbb{R})$). Let $\mathcal{K}^0(V(\mathbb{R}))$ be the ring of rational continuous functions on $V(\mathbb{R})$.

- The function $\frac{x^3}{x^2+y^2} \in \mathcal{K}^0(\mathbb{A}^2(\mathbb{R}))$.
- Rational continuous=Blow-regular.

Regulous functions

Definition: (regulous function)

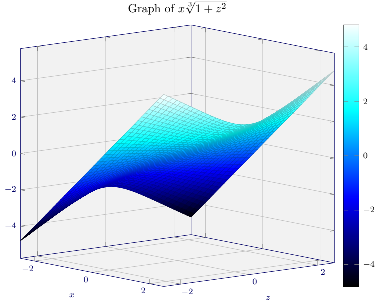
A rational continuous function f on $V(\mathbb{R})$ is called regulous if for any $\mathfrak{p} \in \text{R-Spec}\,\mathbb{R}[V]$ then $f_{|\mathcal{Z}(p)}$ is still rational. Let $\mathcal{R}^0(V(\mathbb{R}))$ be the ring of regulous functions on $V(\mathbb{R})$.

• If $V(\mathbb{R})$ is smooth (Kollár) or $\dim V = 1$ (M.) then $\mathcal{R}^0(V(\mathbb{R})) = \mathcal{K}^0(V(\mathbb{R}))$.
• A regulous function admits a Zariski locally closed stratification where the function is regular on each stratum.
• If for example V is the Cartan or the Whitney umbrella then we do not have inclusions of $\mathcal{R}^0(V(\mathbb{R}))$ and $\mathcal{K}^0(V(\mathbb{R}))$ in $\mathcal{K}(V)$. So, in order to get inclusions in $\mathcal{K}(V)$ it is better to consider regulous and rational functions on the central locus $\text{Cent}\, V$. The definitions are similar, to be regulous functions on $\text{Cent}\, V$ we require for $f \in \mathcal{K}^0(\text{Cent}\, V)$ that for any $\mathfrak{p} \in \text{C-Spec}\,\mathbb{R}[V]$ then $f_{|\mathcal{Z}(p) \cap \text{Cent}\, V}$ is still rational. We get now

$$\mathcal{R}^0(\text{Cent}\, V) \subset \mathcal{K}^0(\text{Cent}\, V) \subset \mathcal{K}(V)$$

The Kollár surface

Let V be the surface given by the equation $y^3 - x^3(1+z^2) = 0$. Then V is central ($V(\mathbb{R}) = \mathrm{Cent}\, V$) and the singular locus of $V(\mathbb{R})$ is $\mathcal{Z}(\mathfrak{p}) = (x,y))$. If we consider the real function $f = y/x$ extended on $\mathcal{Z}(\mathfrak{p})$ by $(1+z^2)^{1/3}$ then $f \in \mathcal{K}^0(V(\mathbb{R})) \setminus \mathcal{R}^0(V(\mathbb{R}))$.



Several normalizations

Let V be an irreducible affine algebraic variety over $\mathbb{R}$ (such that $V(\mathbb{R})$ is Zariski dense in $V(\mathbb{C})$). We have the following sequence of inclusions

$$\mathbb{R}[V] \subset \mathcal{O}(V(\mathbb{R})) \subset \mathcal{R}^0(\mathrm{Cent}\, V) \subset \mathcal{K}^0(\mathrm{Cent}\, V) \subset \mathcal{K}(V)$$

Theorem and definition

- The integral closure $\mathbb{R}[V]'_{\mathcal{K}(V)}$ of $\mathbb{R}[V]$ in $\mathcal{K}(V)$ is the coordinate ring of an affine algebraic variety over $\mathbb{R}$, denoted by V', and called the normalization of V (Noether).
- The integral closure $\mathbb{R}[V]'_{\mathcal{K}^0(\mathrm{Cent}\, V)}$ of $\mathbb{R}[V]$ in $\mathcal{K}^0(\mathrm{Cent}\, V)$ is the coordinate ring of an affine algebraic variety over $\mathbb{R}$, denoted by V^{w_c}, and called the central weak-normalization of V or w_c-normalization of V.
- The integral closure $\mathbb{R}[V]'_{\mathcal{R}^0(\mathrm{Cent}\, V)}$ of $\mathbb{R}[V]$ in $\mathcal{R}^0(\mathrm{Cent}\, V)$ is the coordinate ring of an affine algebraic variety over $\mathbb{R}$, denoted by V^{s_c}, and called the central seminormalization of V or s_c-normalization of V.
- The integral closure $\mathbb{R}[V]'_{\mathcal{O}(V(\mathbb{R}))}$ of $\mathbb{R}[V]$ in $\mathcal{O}(V(\mathbb{R}))$ is the coordinate ring of an affine algebraic variety over $\mathbb{R}$, denoted by V^b, and called the biregular normalization of $V(\mathbb{R})$ or the real biregular normalization of V.

We get finite and birational morphisms $\pi^{w_c} : V^{w_c} \to V$, $\pi^{s_c} : V^{s_c} \to V$ and $\pi^b : V^b \to V$ and a sequence of finite and birational morphisms

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Algebraic constructions

Definition

Let A be a domain with integral closure A'.

• The ring

$$A^{w_c} = \{f \in A' \mid \forall \mathfrak{m} \in \text{C-Max}\, A,\ f \in A_{\mathfrak{m}} + \mathrm{Rad}^{\mathrm{C}}(A'_{\mathfrak{m}})\}$$

is called the central weak normalization of A.

• The ring

$$A^{s_c} = \{f \in A' \mid \forall \mathfrak{p} \in \text{C-Spec}\, A,\ f \in A_{\mathfrak{p}} + \mathrm{Rad}^{\mathrm{C}}(A'_{\mathfrak{p}})\}$$

is called the central seminormalization of A.

With the following theorem, we justify the namings "weak normalization" and "seminormalization" in comparaison with the classical case.

Theorem

Let V be an irreducible and affine algebraic variety over $\mathbb{R}$. We have:

1) $\mathbb{R}[V]^{w_c} = \mathbb{R}[V^{w_c}]$.
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Universal property for the central weak normalization

Theorem

Let $\pi : W \to V$ be a finite and birational morphism between two irreducible affine algebraic varieties over $\mathbb{R}$. The following properties are equivalent:

1) The map $\pi_{|\mathrm{Cent}\, W} : \mathrm{Cent}\, W \to \mathrm{Cent}\, V$ is bijective.
2) The map $\mathcal{K}^0(\mathrm{Cent}\, V) \to \mathcal{K}^0(\mathrm{Cent}\, W)$, $f \mapsto f \circ \pi_{|\mathrm{Cent}\, W}$ is an isomorphism.
3) $\forall g \in \mathbb{R}[W]$ there exists $f \in \mathcal{K}^0(\mathrm{Cent}\, V)$ such that $g = f \circ \pi_{|\mathrm{Cent}\, W}$ on $\mathrm{Cent}\, W$.
4) The map $\pi_{|\mathrm{Cent}\, W} : \mathrm{Cent}\, W \to \mathrm{Cent}\, V$ is bi-rational continuous.

Theorem: (Universal property for the central weak-normalization)

Let $\pi : W \to V$ be a morphism between affine algebraic varieties over $\mathbb{R}$. Then, π is finite and birational and the map $\pi_{|\mathrm{Cent}\, W} : \mathrm{Cent}\, W \to \mathrm{Cent}\, V$ is bijective (or one of the equivalent properties of the previous theorem) iff there exists a unique morphism $\psi : V^{W_C} \to W$ such that $\pi^{W_C} = \pi \circ \psi$

$$\begin{array}{ccc} W & \xrightarrow{\pi} & V \\ \uparrow & \overset{\pi^{W_C}}{\nearrow} & \\ V^{W_C} & & \end{array}$$

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Universal property for the central seminormalization

Remark

Suppose that in the property 3) of the first theorem in the previous slide, we replace rational continuous functions by regulous functions. Namely, let $\pi : W \to V$ be a finite and birational morphism between two irreducible affine algebraic varieties over $\mathbb{R}$ such that $\forall g \in \mathbb{R}[W]$ there exists $f \in \mathcal{R}^0(\mathrm{Cent}\, V)$ such that $g = f \circ \pi_{|\mathrm{Cent}\, W}$ on $\mathrm{Cent}\, W$. Let $\mathfrak{p} \in \mathrm{C\text{-}Spec}\, \mathbb{R}[W]$ and $\mathfrak{q} = (\mathfrak{p} \cap \mathbb{R}[V]) \in \mathrm{C\text{-}Spec}\, \mathbb{R}[V]$. The morphism

$$\mathrm{Spec}(\mathbb{R}[W]/\mathfrak{p}) \to \mathrm{Spec}(\mathbb{R}[V]/\mathfrak{q})$$

is still finite but is not a priori still birational. By the hypothesis and since the regulous functions remains rational by restriction then the morphism $\mathrm{Spec}(\mathbb{R}[W]/\mathfrak{p}) \to \mathrm{Spec}(\mathbb{R}[V]/\mathfrak{q})$ is still birational i.e the extension of the residue fields

$$k(\mathfrak{q}) = \mathcal{K}(\mathbb{R}[V]/\mathfrak{q}) \to k(\mathfrak{p}) = \mathcal{K}(\mathbb{R}[W]/\mathfrak{p})$$

is an isomorphism.

Universal property for the central seminormalization

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Let $\pi : W \to V$ be a finite and birational morphism between two irreducible affine algebraic varieties over $\mathbb{R}$. The following properties are equivalent:

1) The map $\text{C-Spec}\,\mathbb{R}[W] \to \text{C-Spec}\,\mathbb{R}[V]$, $\mathfrak{p} \mapsto (\mathfrak{p} \cap \mathbb{R}[V])$ is bijective and equiresidual.
2) The map $\mathcal{R}^0(\operatorname{Cent} V) \to \mathcal{R}^0(\operatorname{Cent} W)$, $f \mapsto f \circ \pi_{|\operatorname{Cent} W}$ is an isomorphism.
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Let $\pi : W \to V$ be a morphism between affine algebraic varieties over $\mathbb{R}$. Then, π is finite and birational and the map $\pi_{|\operatorname{Cent} W} : \operatorname{Cent} W \to \operatorname{Cent} V$ is bi-regulous (or one of the equivalent properties of the previous theorem) iff there exists a unique morphism $\psi : V^{Sc} \to W$ such that $\pi^{Sc} = \pi \circ \psi$

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Universal property for the biregular normalization

Theorem

Let $\pi : W \to V$ be a finite and birational morphism between two irreducible affine algebraic varieties over $\mathbb{R}$. The following properties are equivalent:

1) The map $\text{R-Spec}\,\mathbb{R}[W] \to \text{R-Spec}\,\mathbb{R}[V]$, $\mathfrak{p} \mapsto (\mathfrak{p} \cap \mathbb{R}[V])$ is bijective and moreover $\mathbb{R}[V]_{\mathfrak{p}\cap\mathbb{R}[V]} \to \mathbb{R}[W]_{\mathfrak{p}}$ is an isomorphism.
2) The map $\mathcal{O}(V(\mathbb{R})) \to \mathcal{O}(W(\mathbb{R}))$, $f \mapsto f \circ \pi_{|W(\mathbb{R})}$ is an isomorphism.
3) The map $\pi_{|W(\mathbb{R})} : W(\mathbb{R}) \to V(\mathbb{R})$ is bi-regular.

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Let $\pi : W \to V$ be a morphism between affine algebraic varieties over $\mathbb{R}$. Then, π is finite and birational and the map $\pi_{|W(\mathbb{R})} : W(\mathbb{R}) \to V(\mathbb{R})$ is bi-regular (or one of the equivalent properties of the previous theorem) iff there exists a unique morphism $\psi : V^b \to W$ such that $\pi^b = \pi \circ \psi$

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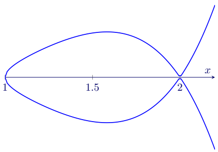
Example 1

Let C be the plane nodal curve with equation $y^2 - x^2(x-1)(x-2)^2(x^2+1)^2 = 0$ i.e $\mathbb{R}[C] = \mathbb{R}[x,y]/(y^2 - x^2(x-1)(x-2)^2(x^2+1)^2)$. We have $\mathrm{Cent}\, C = C(\mathbb{R}) \setminus \{(0,0)\}$. The singularities of C are nodal:

- A real node in $\mathrm{Cent}\, C$ i.e non-isolated in $C(\mathbb{R})$ corresponding to the ideal $(x-2, y) \subset \mathbb{R}[C]$.
- A real node isolated in $C(\mathbb{R})$ corresponding to the ideal $(x, y) \subset \mathbb{R}[C]$.
- Two complex conjugated nodes of $C_{\mathbb{C}}$ corresponding to the ideal $(x^2+1, y) \subset \mathbb{R}[C]$.

Graph of $y^2 = x^2(x-1)(x-2)^2(x^2+1)^2$



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Remark that $f = y/(x^2+1) \in \mathcal{O}(C(\mathbb{R}))$ and $f^2 - x^2(x-1)(x-2)^2 = 0$ so $f \in \mathbb{R}[C]'_{\mathcal{O}(C(\mathbb{R}))}$. We can prove that $\mathbb{R}[C^b] = \mathbb{R}[C][f]$.

Since C is a curve then $\mathcal{K}^0(\operatorname{Cent} C) = \mathcal{R}^0(\operatorname{Cent} C)$ so $C^{w_c} = C^{s_c}$. Let $g = y/(x(x-2)(x^2+1)) \in \mathcal{K}(C) \setminus \mathcal{K}^0(\operatorname{Cent} C)$ and $g^2 - (x-1) = 0$. We have $\mathbb{R}[C'] = \mathbb{R}[C][g]$ and it follows that $C^{w_c} \neq C'$.

Remark that $h = y/(x(x^2+1)) \in \mathcal{K}^0(\operatorname{Cent} C)$ and $h^2 - (x-1)(x-2)^2 = 0$ so $h \in \mathbb{R}[C]'_{\mathcal{K}^0(\operatorname{Cent} C)}$. We can prove $\mathbb{R}[C^{w_c}] = \mathbb{R}[C][h]$.

Example 1

Remark that $f = y/(x^2+1) \in \mathcal{O}(C(\mathbb{R}))$ and $f^2 - x^2(x-1)(x-2)^2 = 0$ so $f \in \mathbb{R}[C]'_{\mathcal{O}(C(\mathbb{R}))}$. We can prove that $\mathbb{R}[C^b] = \mathbb{R}[C][f]$.

Since C is a curve then $\mathcal{K}^0(\operatorname{Cent} C) = \mathcal{R}^0(\operatorname{Cent} C)$ so $C^{w_c} = C^{s_c}$. Let $g = y/(x(x-2)(x^2+1)) \in \mathcal{K}(C) \setminus \mathcal{K}^0(\operatorname{Cent} C)$ and $g^2 - (x-1) = 0$. We have $\mathbb{R}[C'] = \mathbb{R}[C][g]$ and it follows that $C^{w_c} \neq C'$.

Remark that $h = y/(x(x^2+1)) \in \mathcal{K}^0(\operatorname{Cent} C)$ and $h^2 - (x-1)(x-2)^2 = 0$ so $h \in \mathbb{R}[C]'_{\mathcal{K}^0(\operatorname{Cent} C)}$. We can prove $\mathbb{R}[C^{w_c}] = \mathbb{R}[C][h]$.

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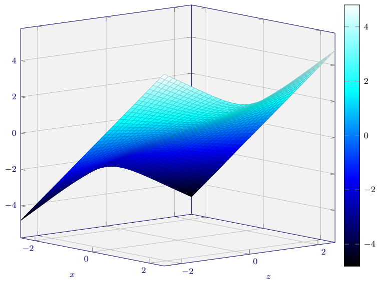
Example 2: The Kollár surface

Let V be the surface given by the equation $y^3 - x^3(1+z^2) = 0$. Then V is central ($V(\mathbb{R}) = \text{Cent}\, V$) and the singular locus of $V(\mathbb{R})$ is $\mathcal{Z}(\mathfrak{p}) = (x, y)$.

Graph of $x\sqrt[3]{1+z^2}$



Example 2: The Kollár surface

If we consider the real function $f = y/x$ extended on $\mathcal{Z}(\mathfrak{p})$ by $(1+z^2)^{1/3}$ then $f \in \mathcal{K}^0(V(\mathbb{R})) \setminus \mathcal{R}^0(V(\mathbb{R}))$. Moreover, $f^3 - (1+x^2) = 0$ and since

$$\mathbb{R}[V'] = \mathbb{R}[V][f] \simeq \mathbb{R}[t,x,y,z]/(t^3 - (1+z^2), yt - x))$$

we have $V' = V^{w_c} \neq V^{s_c}$. Remark that the map C-Spec $\mathbb{R}[V'] \to$ C-Spec $\mathbb{R}[V]$ is bijective but not equiresidual: Let $\mathfrak{q} = (x,y) \subset \mathbb{R}[V']$ be the unique central ideal of $\mathbb{R}[V']$ lying over $\mathfrak{p}$ then $k(\mathfrak{p}) = \mathbb{R}(z)$ and $k(\mathfrak{q}) = \mathbb{R}(z)((1+z^2)^{1/3})$.

Let $h = y^2/x$ extended by 0 on $\mathcal{Z}(\mathfrak{p})$ is continuous on $V(\mathbb{R})$. Since the restriction of h to $\mathcal{Z}(\mathfrak{p})$ is rational then $h \in \mathcal{R}^0(V(\mathbb{R}))$ and since $h^3 - y^3(1+z^2)$ then $h \in \mathbb{R}[V]'_{\mathcal{R}^0(V(\mathbb{R}))}$. We can prove that $\mathbb{R}[V^{s_c}] = \mathbb{R}[V][h]$.

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