# $\mu$-constant families of Newton non-degenerate singularities admit simultaneous embedded desingularization

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# 1. The ideal world: the case of plane curves

A general study of equisingularity in arbitrary dimension was initiated by O. Zariski and R. Thom. See, for example, Zariski's papers [14, 15, 16] as well as his paper [17] entitled "On the elusive concept of equisingularity".

Let $\lambda : (\mathcal{X}, X_0) \to (T, 0)$ be a flat family of plane curve singularities.

**Theorem (Lê, Teissier, Zariski,...)**

*The following conditions are equivalent:*

- *$\mathcal{X}$ is topologically trivial*
- *$\mathcal{X}$ is $\mu$-constant*
- *$\mathcal{X}$ admits a simultaneous resolution.*

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Recall that for an isolated hypersurface singularity $X_0$ defined by an equation $f(x_0, \ldots, x_n) = 0$ in $\mathbb{C}^{n+1}$, the **Milnor number** is defined to be $\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0,\ldots,x_n\}}{\left(\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n}\right)}$.
The Milnor number is known to be a topological invariant. It is equal to the rank of the middle homology group $H_n(X_s)$, where $X_s$ is the manifold obtained by intersecting $f^{-1}(s)$ with a small polydisc $D_\epsilon$ centered at 0, where both $t$ and $\epsilon$ are small enough.

# 2. Isolated hypersurface singularities of arbitrary dimension

**The numerical character $\mu^*$.** Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be an isolated hypersurface singularity of dimension $n$. We define

$$\mu^*(X) := \left(\mu^{(n)}, \mu^{(n-1)}, \ldots, \mu^{(2)}, \mu^{(1)}\right),$$

where $\mu^{(i)}$ is the Milnor number $\mu(X \cap H_i)$ of the intersection $X \cap H_i$ of $X$ with a general linear space of dimension $i$ passing through the singularity. In particular, $\mu^{(1)} = mult_0 X - 1$.

Let $\lambda : (\mathcal{X}, X_0) \to (T,0)$ be a flat family of isolated hypersurface singularities and $s : (T,0) \to (\mathcal{X}, X_0)$ a section of $\lambda$.

# 3. Simultaneous Embedded Resolution

We have the commutative diagram

$$\begin{array}{ccccc} X_0 & \hookrightarrow & \mathcal{X} & \hookrightarrow & \mathbb{C}^{n+1} \times T \\ \downarrow & \square & \downarrow \lambda & \swarrow & \\ 0 & \hookrightarrow & T & & \end{array}$$

Let $X_s := \lambda^{-1}(s)$, $s \in T$. Consider a proper bimeromorphic map $\varphi : \mathbb{C}^{n+1} \times T \to \mathbb{C}^{n+1} \times T$ such that $\mathbb{C}^{n+1} \times T$ is smooth over $T$, and denote by $\widetilde{\mathcal{X}}^{st}$ and $\widetilde{\mathcal{X}}^{t}$ the strict and the total transform of $\mathcal{X}$ in $\mathbb{C}^{n+1} \times T$, respectively. $\varphi : \widetilde{\mathcal{X}}^{st} \to \mathcal{X}$ is a **very weak simult. resolution** if $\widetilde{\mathcal{X}}^{st}_s \to \mathcal{X}_s$ is a resolution of singularities for each $s \in T$. $\varphi$ is a **simult. embedded resolution** if the morphism $\varphi : \widetilde{\mathcal{X}}^{st} \to \mathcal{X}$ is a very weak simult. resolution and $\widetilde{\mathcal{X}}^{t}$ is a normal crossing divisor relative to $T$ (locally analytically a product over $T$).

# 4. Equisingularity criteria.

In this talk, in addition to the notions of topologically trivial and $\mu$-constant, we will be concerned with the following notions of equisingularity.

- (i) $\mathcal{X}$ is $\mu^*$-constant.
- (ii) $\mathcal{X}$ admits a simultaneous embedded resolution.
- (iii) The pair $(\mathcal{X}, s(T))$ satisfies the Whitney conditions.

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# 5. Results known until now.

We have (i) $\mu^*$-constant $\iff$ (iii) Whitney conditions $\implies$ topologically trivial $\implies$ $\mu$-constant.

- The implication (iii) $\implies$ (i) was established by Briançon and Speder in 1976 [4] and (i) $\implies$ (iii) by B. Teissier in 1982 [12].
- (iii) $\implies$ topologically trivial follows from the Thom–Mather stratification theory (late 1960-ies – early seventies [11, 13]).
- The last implication follows since $\mu$ is a topological invariant.
- (i) and (iii) do not follow from topological triviality by an example of Briançon and Speder [3], discussed later in the talk.
- $\mu$-constant implies topologically trivial for dim $X \neq 2$ (Lê–Ramanujam [9], 1976). For dim $X = 2$ this implication is the Lê–Ramanujam conjecture, a major theme of this talk.

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# 6. Newton polyhedra and Newton non-degeneracy.

Let $f = \sum_\alpha c_\alpha x^\alpha$ be a convergent power series, where $\alpha = (x_0, \dots, x_n)$, $c_\alpha \in \mathbb{C}$ and $x^\alpha = x_0^{\alpha_0} \dots x_n^{\alpha_n}$.

## Definition

*The* **Newton polyhedron** $\Delta(f)$ *of* $f$ *is the convex hull in* $\mathbb{R}^{n+1}$ *of the set* $\{\alpha + \mathbb{R}^{n+1}_{\geq 0} \mid c_\alpha \neq 0\}$.

Let $L$ be a face of $\Delta(f)$. The **initial form** of $f$ with respect to $L$ is $in_L f = \sum_{\alpha \in L} c_\alpha x^\alpha$.

## Definition

$f$ *is* **Newton non-degenerate** *if for each compact face* $L$ *of* $\Delta(f)$ *we have* $Sing(\{in_L f = 0\}) \subset \bigcup_{i=0}^{n} \{x_i = 0\}$.

For example, the plane curve $\{y^p + x^q = 0\}$ is Newton non-degenerate while $\{(y^2 + x^3)^2 + x^{10} = 0\}$ is Newton degenerate.

# 7. Statements of the main results.

This work is joint with M. Leyton (Talca, Chile) and H. Mourtada (Paris–Diderot). We relate simult. embedded resolution to the other equisingularity criteria. Here are our main results.

**Theorem**

*Let $\mathcal{X} \to T$ be a flat family of Newton non-degenerate isolated hypersurface singularities. If $\mathcal{X}$ is $\mu$-constant then it has a simult. embedded resolution (in particular, $\mathcal{X}$ is topologically trivial).*

**Remark**

*The Lê–Ramanujam conjecture for Newton non-degenerate singularities was proved by Oulde Abderahmane in 2016 [1] (+Oka).*

On the way to our main theorem we give an elementary solution to a question of Arnold on the monotonicity of Newton numbers for convenient Newton polyhedra [2], dating from 1982.

# 8. Our starting point: a paper by Y. Kawamata.

Our starting point was the paper *An equisingular deformation theory via embedded resolution of singularities* [7]. The author claims that simult. embedded resolution implies $\mu^*$-constant. We constructed a simultaneous embedded resolution of the Birançon–Speder example (which is known not to be $\mu^*$-constant), seemingly contradicting Kawamata's results. Looking more closely, we realized that Kawamata's definition of simultaneous embedded resolution was more restrictive than the one given above. Namely, he requires that the embedded resolution of $\mathcal{X}$ be given by a sequence of blowings up of $\mathbb{C}^{n+1} \times T$ along *regular* centers, contained in the singular locus of $\mathcal{X}$ and its successive strict transforms. In particular, the very first blowing up must be the blowing up along *Sing* $\mathcal{X} = s(T)$.

# 9. Some results from Kawamata's paper.

## Proposition (1)

*Simultaneous embedded resolution implies topological triviality.*

*Proof.* Use partitions of unity. □.

## Corollary (2)

*Simultaneous embedded resolution implies $\mu$-constant.*

Let $\mathcal{H} \subset \mathbb{C}^{n+1} \times T$ be a general family of hyperplans of $\mathbb{C}^{n+1}$ passing through 0 and $H \subset \mathbb{C}^{n+1}$ the special fiber of $\mathcal{H}$. Define the **strict hyperplance section** of $\mathcal{X}$ (resp. $X$) by

$$\mathcal{X}_{\mathcal{H}} = \overline{(\mathcal{X} \cap \mathcal{H}) \setminus \{0 \times T\}} \subset \mathbb{C}^{n+1} \times T \tag{1}$$

$$X_H = \overline{(X \cap H) \setminus \{0\}} \subset \mathbb{C}^{n+1}. \tag{2}$$

## Proposition (3)

*$\mathcal{X}_{\mathcal{H}} \to T$ admits a (Kawamata) simultaneous embedded resolution.*

*Proof.* Trivial. $\square$.

## Corollary (4)

*Kawamata's simultaneous embedded resolution implies $\mu^*$-constant.*

These (incomplete) results of Kawamata inspired us to formulate the following conjecture.

## Conjecture

*Let $\rho : \widetilde{\mathbb{C}^{n+1} \times T} \to \mathbb{C}^{n+1} \times T$ be a proper bimeromorphic map which induces a simultaneous embeded resolution of $\mathcal{X}$ and factors through the blowing up of $s(T)$. Then $\mathcal{X}$ is $\mu^*$-constant.*

# 10. The example of Briançon–Speder.

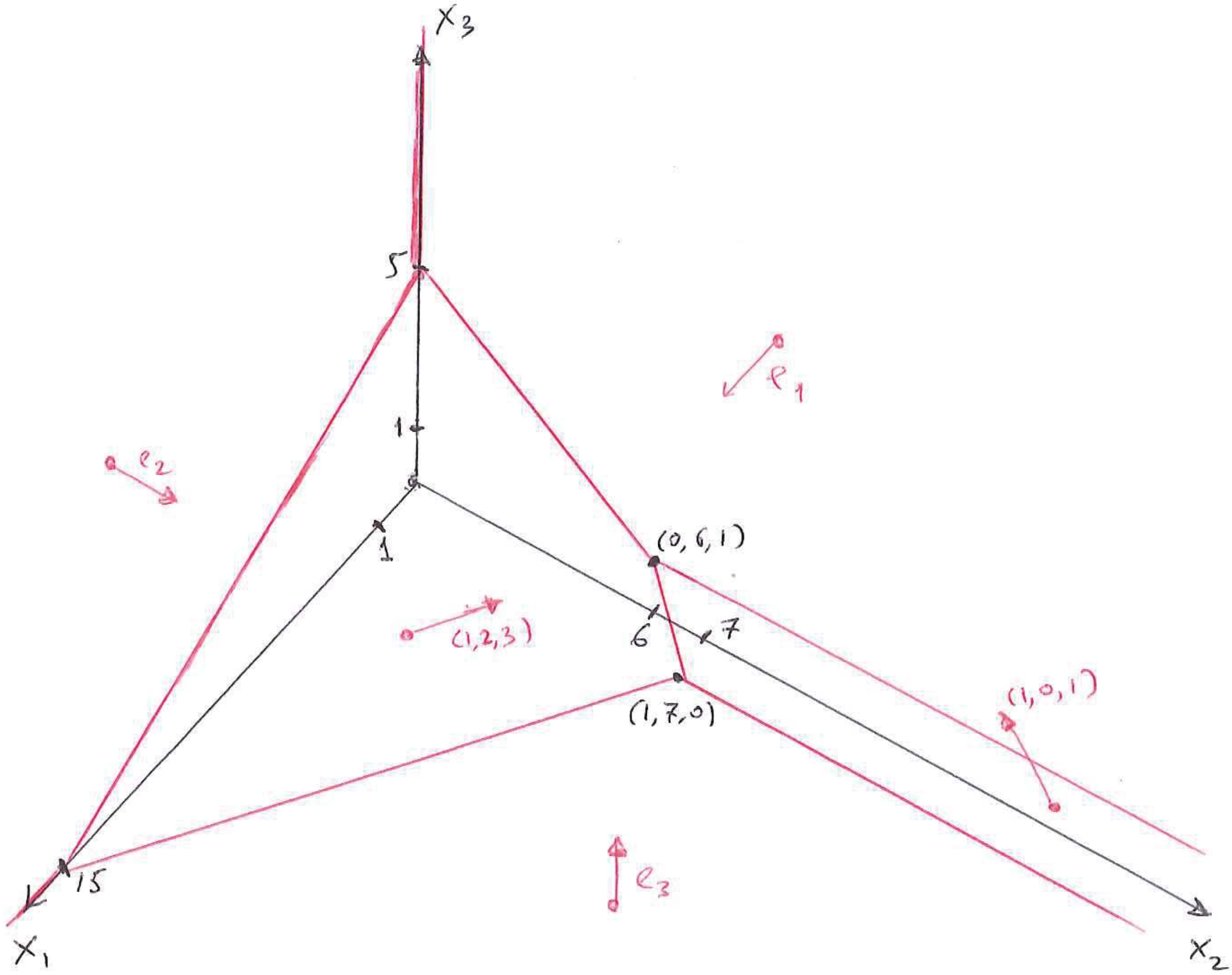
- Consider the family of surfaces defined by

$$F(x,t) = x_3^5 + x_2^7 x_1 + x_1^{15} + t x_2^6 x_3 = 0.$$

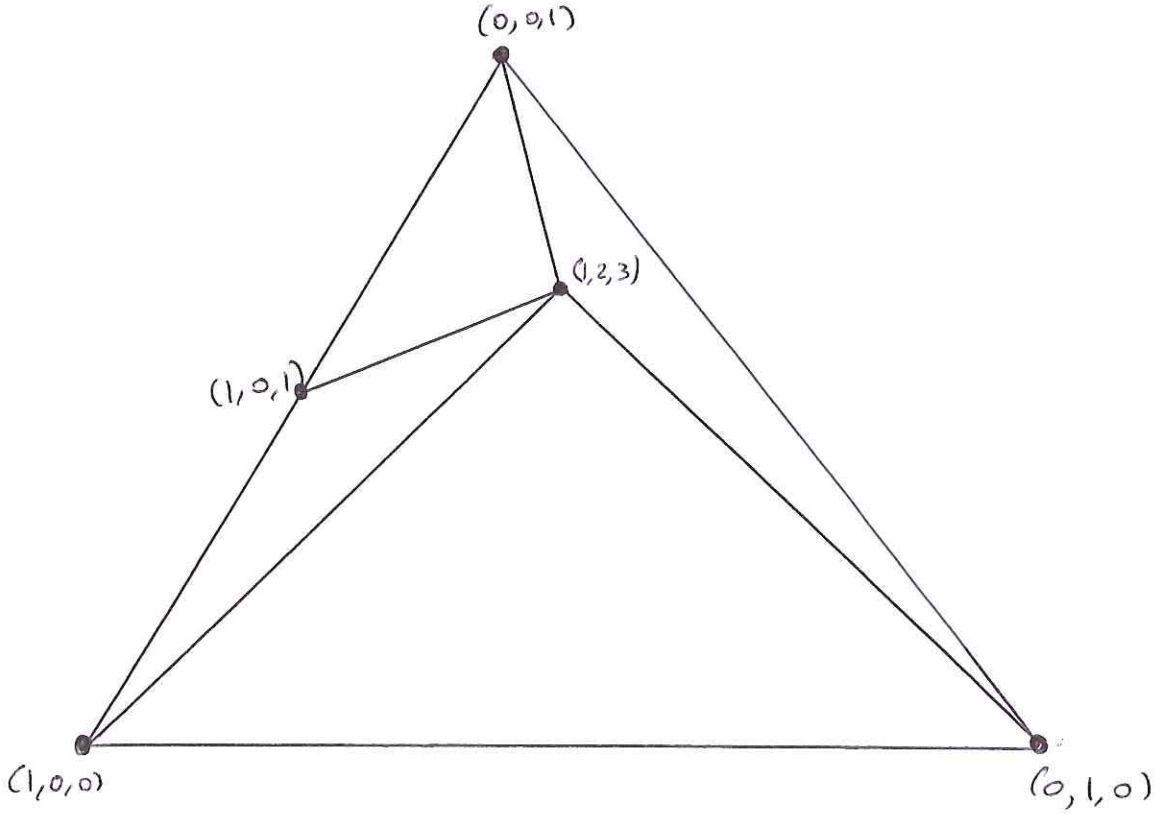
- Each triangle of the subdivision corresponds to a coordinate chart on the resolution of singularities. For example, let us consider the triangle formed by (1,0,0), (1,1,2) and (1,2,3). The transformation law between $(x_1, x_2, x_3)$ and the new coordinates $(y_1, y_2, y_3)$ on the embedded resolution of singularities is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

that is, $x_1 = y_1 y_2 y_3, \quad x_2 = y_2 y_3^2, \quad x_3 = y_2^2 y_3^3.$



$x_3$
5
1
$e_1$
$e_2$
1
(0,6,1)
(1,2,3)
6
7
(1,7,0)
(1,0,1)
15
$e_3$
$x_1$
$x_2$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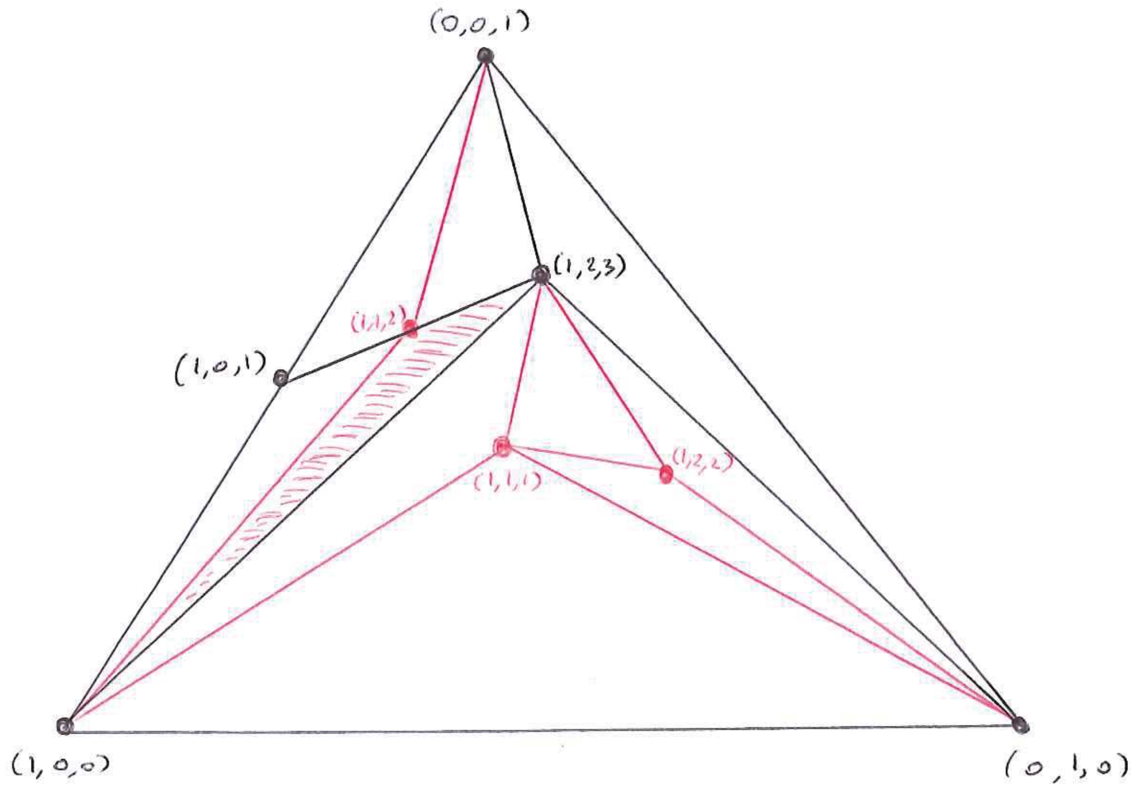
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that is, $\quad x_1 = y_1 y_2 y_3, \quad x_2 = y_2 y_3^2, \quad x_3 = y_2^2 y_3^3.$

Substituting this into $F(x,t)$, we obtain

$$\begin{aligned} F(x,t) &= x_3^5 + x_2^7 x_1 + x_1^{15} + t x_2^6 x_3 = && (3)\\ &= y_2^{10} y_3^{15} + y_1 y_2^8 y_3^{15} + y_1^{15} y_2^{15} y_3^{15} + t y_2^8 y_3^{15} = && (4)\\ &= y_2^8 y_3^{15}\left(y_2^2 + y_1 + y_1^{15} y_2^7 + t\right). && (5)\end{aligned}$$

Thus the preimage of $\mathcal{X}$ in this affine chart of $\widetilde{\mathbb{C}^{n+1} \times T}$ is a normal crossings divisor over $T$. Performing similar computations in all the remaining affine charts of $\widetilde{\mathbb{C}^{n+1} \times T}$, we obtain that $\widetilde{\mathbb{C}^{n+1} \times T}$ is a simultaneous embedded resolution of $\{F(x,t) = 0\}$.

In our paper [10] we generalize this example to the case of an arbitrary Newton non-degenerate isolated hypersurface singularity by means of an involved combinatorial study of Newton polyhedra.

# 11. Convenient Newton polyhedra.

## Definition

*Let $\Delta \subset \mathbb{R}^{n+1}_{\geq 0}$ be a Newton polyhedron (that is, an integral convex polyhedron stable by translations by elements of $\mathbb{R}^{n+1}_{\geq 0}$). $\Delta$ is* **convenient** *if it intersects all of the $(n+1)$ coordinate axes.*

## Remark

*If $f \in \mathbb{C}\{x_0, \dots, x_n\}$ is a convergent power series which defines a Newton non-degenerate isolated hypersurface singularity then so does $\tilde{f} := f + \sum_{i=0}^{n} c_i x_i^e$ for $e \gg 0$ and generic $c_i \in \mathbb{C}$. The polyhedron $\Delta\left(\tilde{f}\right)$ is convenient.*

# 12. Newton number: the definition.

Let $\Delta \subset \mathbb{R}^{n+1}_{\geq 0}$ be a convenient Newton polyhedron not containing 0. Let $\Delta' = \mathbb{R}^{n+1}_{\geq 0} \setminus \Delta$. Since $\Delta$ is convenient, $\Delta'$ is compact. For $0 \leq i \leq n+1$ let $V_i$ denote the $i$-volume of the intersection of $\Delta'$ with the union of the $i$-dimensional coordinate subspaces of $\mathbb{R}^{n+1}_{\geq 0}$.

## Definition

*The* **Newton number** *of* $\Delta$ *is* $\nu(\Delta) = \sum_{i=0}^{n+1} (-1)^{n-i} i! V_i$. *If a power series* $f$ *defines an isolated hypersurface singularity and* $\Delta(f)$ *is convenient, put* $\nu(f) := \nu(\Delta(f))$.

It is known that $\nu(\Delta)$ is a positive integer.

# 13. Kouchnirenko's theorem and its consequences.

Notation: For a convergent power series $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ defining an isolated hypersurface singularity $X$, write $\mu(f)$ for the Milnor number of $X$.

**Theorem (Kouchnirenko [8], 1976)**

*Take an $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ defining an isolated hypersurface singularity $X$ and such that $\Delta(f)$ is convenient. Then $\mu(f) \geq \nu(f)$. If, in addition, $X$ is Newton non-degenerate then $\mu(f) = \nu(f)$.*

Let $\lambda : \mathcal{X} \to T$ be a flat, $\mu$-constant family of Newton non-degenerate isolated hypersurface singularities. Let $f_t := \lambda^{-1}(t)$, where $t$ is a general point of $T$ sufficiently close to 0 and $f_0 := \lambda^{-1}(0)$. Kouchnirenko's theorem implies that

$$\nu(f_t) = \nu(f_0). \tag{6}$$

We have

$$\Delta(f_0) \subset \Delta(f_t). \tag{7}$$

The crux of our work consists of analyzing the inclusion (7) and equality (6) and constructing a (toric) simultaneous embedded resolution of singularities of $\mathcal{X}$.

For example, it follows easily from (6) that all the vertices of $\Delta(f_t) \setminus \Delta(f_0)$ belong to the coordinate hyperplanes.

# 14. Arnold's question.

**Question 1982–16.** The Newton number $\nu(\Delta)$ is (non-strictly) decreasing as $\Delta$ increases in the sense of inclusion. *There is no elementary proof even for Newton polygons in* $\mathbb{R}^2_{\geq 0}$.

There are proofs using the theory of isolated hypersurface singularities (S.K. Lando, J. Gwoździewicz [6] and others...).

S. Brzostowski, T. Krasinski and J. Walewska gave an elementary proof of this (that is, a proof that does not use singularity theory) for Newton polyhedra in $\mathbb{R}^3_{\geq 0}$.

A consequence of our work is an elementary proof of Arnold's conjecture for Newton polyhedra in all dimensions.

- Ould M. Abderrahmane, *On deformation with constant Milnor number and Newton polyhedron*, Math. Z., 284 (1-2) (2016) 167–174.
- V. I. Arnold, *Arnold's problems*, Springer-Verlag, Berlin; PHASIS, Moscow, 2004. Translated and revised edition of the 2000 Russian original, with a preface by V. Philippov, A. Yakivchik and M. Peters.
- J. Briançon and J.-P. Speder, *Families équisingulières de surfaces a singularité isolée*, C.R. Acad. Sc., t. 280. série A (1975) 1013–1016.
- J. Briançon and J.-P. Speder, *Les conditions de Whitney impliquent $\mu^*$-constant*, Annales de l'Institut Fourier, Tome 26 (1976) no. 2, 153–163.

- S. Brzostowski, T. Krasinski and J. Walewska, *Arnold's problem on monotonicity of the Newton number for surface singularities*, J. Math. Soc. Japan, 71 (4) (2019) 1257–1268.
- J. Gwoździewicz, *Note on the Newton number*, Univ. Iagel. Acta Math., 46 (2008) 31–33.
- Y. Kawamata, *An equisingular deformation theory via embedded resolution of singularities*, Publ. RIMS, 16(1) (1980) 233–244.
- A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math., 32 (1976) 1–31
- D. T. Lê and C. P. Ramanujam, *Invariance of Milnors number implies the invariance of topological type*, Amer. J. Math. 98 (1976), 67–78.

- M. Leyton-Álvarez, H. Mourtada and M. Spivakovsky, *Newton non-degenerate $\mu$-constant deformations admit simultaneous embedded resolutions*, preprint, arXiv:2001.10316v3 [math.AG] (2020).
- J. Mather, *Stratifications and mappings,* in "Dynamical Systems", Academic press 1973.
- B. Teissier, *Varietes polaires II. Multiplicites polaires, sections planes et conditions de Whitney*, 314–491 (1982) In: Aroca J.M., Buchweitz R., Giusti M., Merle M. (eds) Algebraic Geometry. Lecture Notes in Mathematics, vol 961. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0071291
- R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc., 75 (1969) 240–284.

- O. Zariski, *Studies in Equisingularity I. Equivalent Singularities of Plane Algebroid Curves*, American J. Math., Vol. 87, No. 2 (1965), 507–536.
- O. Zariski, *Studies in Equisingularity II. Equisingularity in Codimension 1 (and Characteristic Zero)*, American J. Math., Vol. 87, No. 4 (1965), 972–1006.
- O. Zariski, *Studies in Equisingularity III: Saturation of Local Rings and Equisingularity*, American J. Math., Vol. 90, No. 3 (1968), 961–1023.
- O. Zariski, *On the elusive concept of equisingularity*, Symposium in Honor of J. Silvester, Johns Hopkins Univ. Press, Baltimore (1978).

# Thank you