On subanalytic geometry

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Abstract. These notes constitute a survey on the geometric properties of globally subanalytic sets. We start with their definition and some fundamental results such as Gabrielov's Complement Theorem or existence of cell decompositions. We then give the main basic tools of subanalytic geometry, such as Curve selection Lemma, Łojasiewicz's inequalities, existence of tubular neighborhood, Tamm's theorem (definability of regular points), or existence of regular stratifications (Whitney or Verdier). We then present the developments of the Lipschitz geometry obtained by various authors during the four last decades, giving a proof of existence of metric triangulations, introduced by the author of these notes, definable bi-Lipschitz triviality, Lipchitz conic structure, as well as invariance of the link under definable bi-Lipschitz mappings. The last chapter is devoted to geometric integration theory, studying the Hausdorff measure of globally subanalytic sets, integrals of subanalytic functions, as well as the density of subanalytic sets (the Lelong number). The results of the last two chapters (on Lipschitz geometry and integration theory) recently turned out to be valuable to carry out a satisfying theory of partial differential equations on subanalytic domains possibly singular. Although these applications to analysis go beyond the scope of this survey, these notes aim at providing the material necessary for this purpose in a way which is accessible to both geometers and specialists of PDE's.

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List of Symbols

$\mathbb{R}_+$ set of nonnegative real numbers, page 9

$0_{\mathbb{R}^n}$ origin in $\mathbb{R}^n$, page 9

$\mathbb{N}^*$ positive integers, page 9

$\mathbb{R}^0$ by convention $\mathbb{R}^0 = \{0\}$, page 9

$|.|$ Euclidean norm, page 9

$cl(A)$ closure of the set A, page 9

$int(A)$ interior of the set A, page 9

Γ_F graph of a mapping F, page 9

$f : (X, x_0) \to (Y, y_0)$ germ of mapping at x_0 satisfying $f(x_0) = y_0$, page 9

$[\xi, \zeta]$,(ξ, ζ), $[\xi, \zeta)$, $(\xi, \zeta]$, page 10

$\mathcal{V}_n$ compactification of $\mathbb{R}^n$, page 10

$\mathcal{S}_n$ globally subanalytic subsets of $\mathbb{R}^n$, page 11

$\pi(\mathcal{C})$ projection of a cell decomposition, page 13

$\mathcal{A}_{y_0}$ ring of analytic functions germs at y_0, page 16

$\mathcal{G}(M_*)$ graded ring of formal polynomials associated to an $\mathfrak{M}$-filtration, page 17

$\mathbf{C}(a, \alpha)$ cube centered at a of radius α, page 20

$(\mathcal{H}_n)$ induction hypothesis of the proof of proposition 1.5.4, page 23

$\lesssim$ inequality up to some constant, page 23

$\sim$ equivalent functions, page 23

A_t t-fiber of a set A, page 38

f_t t-fiber of a function, page 39

$\overline{\mathbf{B}}(x,r)$ closed ball of radius r centered at x, page 41

$\mathbf{B}(x,r)$ open ball of radius r centered at x, page 41

$\mathbf{S}(x,r)$ sphere of radius r centered at x, page 41

$\mathbf{S}^{n-1}$ unit sphere of $\mathbb{R}^n$

δA topological boundary of the set A, page 41

$\mathrm{diam}(X)$ diameter of X

$fr(A)$ frontier of the set A, page 41

$e_1, \dots, e_n$ canonical basis of $\mathbb{R}^n$, page 41

$d_x F$ derivative of a mapping, page 41

$\partial_x f$ gradient of a function, page 41

$d(x,A)$ Euclidean distance from x to the set A, page 44

$\dim A$ dimension of A, page 48

A_B restriction to B of the set A, page 49

ρ_Y square of the distance to Y, page 50

X_{reg} regular locus of X, page 51

X_{sing} singular locus of X, page 51

$reg(f)$ regular locus of the mapping f, page 51

$sing(f)$ singular locus of the mapping f

X^k_{reg} $\mathscr{C}^k$-regular locus, page 53

$\mathbb{G}^n_k$ Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$, page 54

$\angle(P,Q)$ angle between two vector subspaces P and Q of $\mathbb{R}^n$, page 54

$\mathcal{S}^+(Y)$ definable positive continuous functions on Y, page 63

π_P orthogonal projection onto $P \in \mathbb{G}^n_k$, page 67

N_λ vector space normal to $\lambda \in \mathbf{S}^{n-1}$

q_λ coordinate of q along λ

Γ_ξ^λ graph of ξ for λ, page 68

$\mathrm{supp}_m(A)$ m-support of the set A, page 68

L_ξ Lipschitz constant of the function ξ, page 68

$\tau(A)$ set of all the tangent spaces at regular points of A, page 68

$d(\lambda, Z)$, $Z \subset \cup_{k=1}^n \mathbb{G}_k^n$, euclidean distance to a subset of $\cup_{k=1}^n \mathbb{G}_k^n$, page 69

$lg(\gamma)$ length of the arc γ, page 70

$d_X(a,b)$ inner metric of X, page 70

$\mathcal{C}_n(R)$, page 69

$\check{X}$, page 89

$|K|$ polyhedron of the simplicial complex K, page 78

$x_0 * A$ cone over A at x_0, page 97

$lk(X, x_0)$ link of X at x_0, page 98

$\mathcal{S}_{n,0}$ germs of subanalytic sets, page 90

ab line segment joining a and b, page 99

$X \approx Y$ definable bi-Lipschitz equivalence of sets or germs, page 105

$\mathcal{H}^k$ k-dimensional Hausdorff measure, page 107

$J_x(f)$ generalized Jacobian of f, page 108

$\gamma_{l,n}$ measure on the Grassmannian, page 109

$\mathrm{card}\, E$ cardinal of E, page 109

N_P^y affine space passing through y and directed by the orthogonal complement of $P \in \mathbb{G}_l^n$, page 109

$K_j^P(A)$ points $x \in P \in \mathbb{G}_l^n$ at which $\pi_P^{-1}(x) \cap A$ has cardinal j, page 111

m_A multiplicity of A, page 111

$|f|_{1,\mathcal{H}^k}$ L^1 norm of f with respect to $\mathcal{H}^k$, page 112

$L^1_{\mathcal{H}^k}(X)$ L^1 space with respect to $\mathcal{H}^k$, page 112 ln logarithm (extended for x negative or 0) , page 112

$\psi(X, x, r)$ local measure of X near x, page 118

$\psi(X, r)$ local measure of X near 0, page 118

θ_X density of X (Lelong number), page 118

$X_{\leq\varepsilon}$ ε-neighborhood of X, page 119

$X_{=\varepsilon}$ level surface $d(x, X) = \varepsilon$, page 119

$d\omega$ exterior differential of a differential form ω, page 127

$\Sigma^{(k)}$ set of k-dimensional strata of Σ, page 128

$\cup\Sigma^{(k)}$ union of all the k-dimensional strata of Σ, page 128

Δ_k oriented standard simplex of $\mathbb{R}^k$, page 129

$C_k(X)$ singular definable k-chains with coefficients in $\mathbb{R}$, page 129

$\partial^i M$ $\mathscr{C}^i$ boundary of M , page 129

Chapter 1

Subanalytic sets and functions

Some notations and conventions. We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. Throughout these notes i, j, k, m, and n stand for integers. We denote by $\mathbb{N}^*$ the set of all the positive integers.

By convention, $\mathbb{R}^0 = \{0\}$. The origin of $\mathbb{R}^n$ is denoted 0 for all n but we will write $0_{\mathbb{R}^n}$ if n is not obvious from the context.

We denote by $|.|$ the Euclidean norm of $\mathbb{R}^n$. Given $A \subset \mathbb{R}^n$, we respectively write $cl(A)$ and $int(A)$ for the closure and interior of A (in this norm). If we say that $M \subset \mathbb{R}^n$ is a manifold, we always mean that it is a submanifold of $\mathbb{R}^n$. The word "smooth" will mean $\mathscr{C}^\infty$.

Given a mapping $F : A \to B$, with $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, we denote by Γ_F **the graph of** F, which is the set $\{(x, y) \in A \times B : y = F(x)\}$. We denote by $F_{|C}$ the restriction of F to C, if $C \subset A$.

When we say **homeomorphism**, we mean an invertible continuous map $h : A \to B$ such that $h^{-1} : B \to A$ is continuous. In particular, we mean that h is an onto map. If it is not onto, we speak about a *homeomorphism onto its image*.

As usual, if U is an open subset of $\mathbb{R}^n$, we say that a $\mathscr{C}^\infty$ function $f : U \to \mathbb{R}$ is **analytic** if for every x, the Taylor series of f at x has a positive radius of convergence. A mapping $F : U \to \mathbb{R}^k$, $x \mapsto (F_1(x), \dots, F_k(x))$ is **analytic** if so is each of its components F_i. More generally, we will say that a mapping g defined on a subset $A \subset \mathbb{R}^n$ is analytic if it coincides with the restriction to A of a mapping f which is analytic on an open neighborhood U of A in $\mathbb{R}^n$.

A **germ** of mapping (resp. set) at $x_0 \in \mathbb{R}^n$ is an equivalence class of the equivalence relation that identifies two mappings (resp. sets) that coincide on a neighborhood of x_0. Given a germ of mapping $f : X \to Y$ at x_0, we shall write $f : (X, x_0) \to (Y, y_0)$ as a shortcut to express that $f(x_0) = y_0$.

Given two functions ζ and ξ on a set $A \subset \mathbb{R}^n$ with $\xi \leq \zeta$ we define the **closed**

interval $[\xi, \zeta]$ as the set:

$$[\xi, \zeta] := \{(x, y) \in A \times \mathbb{R} : \xi(x) \le y \le \zeta(x)\}. \tag{1.0.1}$$

The open and semi-open intervals (ξ, ζ), $(\xi, \zeta]$, and $[\xi, \zeta)$ are defined analogously. We will sometimes admit ξ or ζ to be (identically) $\pm\infty$ (the interval will still be a subset of $A \times \mathbb{R}$ however). When $n = 0$, by convention, the graph of a function ξ on $A = \{0\}$ will be the singleton $\{\xi(0)\} \subset \mathbb{R}$, and the above intervals will stand for the corresponding intervals of $\mathbb{R}$.

1.1 Definitions and basic facts

Definition 1.1.1. A subset $E \subset \mathbb{R}^n$ is called **semi-analytic** if it is *locally* defined by finitely many real analytic equalities and inequalities. Namely, for each $a \in \mathbb{R}^n$, there are a neighborhood U of a as well as real analytic functions f_{ij} and g_{ij} on U, where $i = 1, \dots, r \in \mathbb{N}$, $j = 1, \dots, s_i \in \mathbb{N}$, such that

$$E \cap U = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s_i} \{x \in U : g_{ij}(x) > 0 \text{ and } f_{ij}(x) = 0\}. \tag{1.1.1}$$

Example 1.1.2. In the above definition, a description as displayed in the right-hand-side of (1.1.1) is required near each point of $a \in \mathbb{R}^n$ (and not only near the points of E). It thus can be seen that the graph of $f(x) = \sin \frac{1}{x}$, $x \in (0, 1)$, is not a semi-analytic set, although this function is analytic. Condition (1.1.1) fails at the points of the y-axis that are in the closure of the graph.

Definition 1.1.3. A subset Z of $\mathbb{R}^n$ is **globally semi-analytic** if $\mathcal{V}_n(Z)$ is a semi-analytic subset of $\mathbb{R}^n$, where $\mathcal{V}_n : \mathbb{R}^n \to (-1, 1)^n$ is the homeomorphism defined by

$$\mathcal{V}_n(x_1, \dots, x_n) := (\frac{x_1}{\sqrt{1 + |x|^2}}, \dots, \frac{x_n}{\sqrt{1 + |x|^2}}).$$

Of course, globally semi-analytic sets are semi-analytic. Roughly speaking, we can say that a semi-analytic subset Z of $\mathbb{R}^n$ is globally semi-analytic if it is still semi-analytic after compactifying $\mathbb{R}^n$. Clearly, a bounded subset of $\mathbb{R}^n$ is semi-analytic if and only if it is globally semi-analytic. Unbounded examples are easy to produce:

Example 1.1.4. It is easy to see that semi-algebraic sets, that is to say, sets of type

$$\bigcup_{i=1}^{r} \bigcap_{j=1}^{s_i} \{x \in \mathbb{R}^n : P_{ij}(x) > 0 \text{ and } Q_{ij}(x) = 0\},$$

where P_{ij} and Q_{ij} are n-variable polynomials for all i and j, are all globally semi-analytic. The set $\mathbb{N}$ is an example of a set which is analytic but not globally semi-analytic.

Working with *globally* semi-analytic sets makes it possible to avoid some pathological situations at infinity and provides finiteness properties. The flaw of semi-analytic and globally semi-analytic sets is that these classes of sets are not preserved under linear projections. In other words, the projection of a globally semi-analytic set is not always globally semi-analytic:

Example 1.1.5. (Osgood's example [Osg29]) Define a globally semi-analytic subset of $\mathbb{R}^4$ by:

$$E := \{(x, xy, xe^y, y) : x \in (0,1) \text{ and } y \in (0,1)\}.$$

Let now $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be the projection omitting the last coordinate. If $\pi(E)$ were semi-analytic then there would exist a germ of analytic function (at the origin), not identically zero, vanishing at every point of $\pi(E)$ in this neighborhood. Examining the Taylor expansion of this function at the origin quickly leads to a contradiction [Bie-Mil88, Loj64b, Loj93].

To overcome this problem, we will work with a bigger class of sets: the globally subanalytic sets, which are the projections of globally semi-analytic sets.

Definition 1.1.6. A subset $E \subset \mathbb{R}^n$ is **globally subanalytic** if there exists a globally semi-analytic set $Z \subset \mathbb{R}^{n+p}$, $p \in \mathbb{N}$, such that $E = \pi(Z)$, where $\pi : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is the projection onto the n first coordinates. We shall denote by $\mathcal{S}_n$ the set of globally subanalytic subsets of $\mathbb{R}^n$.

We say that **a mapping** $f : A \to B$ **is globally subanalytic**, $A \in \mathcal{S}_n$, $B \in \mathcal{S}_m$, if its graph is a globally subanalytic subset of $\mathbb{R}^{n+m}$. In the case $B = \mathbb{R}$, we say that f is a **globally subanalytic function**.

Example 1.1.7. Globally semi-analytic sets (see example 1.1.4) provide examples of globally subanalytic sets. The function $\sin x$ is a typical example of a function which is subanalytic but not globally subanalytic. The set E of Example 1.1.5 being globally semi-analytic, its projection $\pi(E)$ (with the notations of the latter example) is globally subanalytic, although not globally semi-analytic.

Basic properties 1.1.8. Below we list some very important properties of globally subanalytic sets and mappings which are direct consequences of their definition.

(1) If $A \in \mathcal{S}_n$ and if $\mu : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, denotes the projection onto the m first coordinates then $\mu(A) \in \mathcal{S}_m$.

(2) If $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_n$ then $A \cup B$ and $A \cap B$ both belong to $\mathcal{S}_n$.

(3) If $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$ then $A \times B \in \mathcal{S}_{n+m}$.

(4) Images and preimages of globally subanalytic sets under globally subanalytic mappings are globally subanalytic.

(5) A mapping $F : A \to \mathbb{R}^p$, $F = (F_1, \dots, F_p)$, $A \in \mathcal{S}_n$, is globally subanalytic if and only if F_i is globally subanalytic for every i.

(6) If $f : A \to B$ and $g : B \to C$ are both globally subanalytic then so is $g \circ f$.

(7) Sums and products of globally subanalytic functions are globally subanalytic.

Proof. (1) is clear from the definition of globally subanalytic sets. To prove (2) take A and B in $\mathcal{S}_n$. By definition of globally subanalytic sets, there exists a globally semi-analytic set $Z \subset \mathbb{R}^{n+p}$ (resp. $Z' \subset \mathbb{R}^{n+p'}$) such that $A = \pi(Z)$ (resp. $B = \pi'(Z')$) where $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ (resp. $\pi' : \mathbb{R}^n \times \mathbb{R}^{p'} \to \mathbb{R}^n$) denotes the projection onto the first factor. Then $A \cup B = \pi''(Y)$, where $\pi'' : \mathbb{R}^{n+p+p'} \to \mathbb{R}^n$ is the obvious projection and

$$Y := \{(x, z, z') \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^{p'} : (x, z) \in Z \text{ or } (x, z') \in Z'\}.$$

Since Z and Z' are globally semi-analytic, the sets $\mathcal{V}_{n+p}(Z)$ and $\mathcal{V}_{n+p'}(Z)$ can be described by inequalities on analytic functions (as in (1.1.1)). Consequently, so does $\mathcal{V}_{n+p+p'}(Y)$, which means that $A \cup B$ is globally subanalytic. Moreover, $A \cap B = \pi''(Y')$ where

$$Y' := \{(x, z, z') \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^{p'} : (x, z) \in Z \text{ and } (x, z') \in Z'\},$$

which entails that $A \cap B$ is globally subanalytic as well. The proof of (3) is similar to the proof of (2) and is left to the reader.

Proof of (4). Let $F : A \to B$ be globally subanalytic, with $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_p$. Observe that $F(C) = \pi_2(\pi_1^{-1}(C) \cap \Gamma_F)$ and $F^{-1}(D) = \pi_1(\pi_2^{-1}(D) \cap \Gamma_F)$ where $\pi_1 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ are the obvious orthogonal projections. Hence, it is enough to consider the case where F is a canonical projection, which follows from (1) and (3).

Proof of (5). Observe that $\Gamma_{F_i} = \mu_i(\Gamma_F)$, where $\mu_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}$, $i = 1, \dots, p$, is the projection defined by $\mu_i(x, v) = (x, v_i)$ if $(x, v) \in \mathbb{R}^n \times \mathbb{R}^p$. By (1), it means that Γ_{F_i} is globally subanalytic if so is F. Conversely, if all the F_i's are globally subanalytic then, in virtue of (3), the Cartesian product of their graphs is globally subanalytic, and hence, so is Γ_F, which can be expressed as a suitable projection of this Cartesian product.

Proof of (6). If g is globally subanalytic then by (3) the map $h : A \times B \to C$ defined by $h(x, y) := g(y)$ is globally subanalytic. Therefore, if f is also globally subanalytic, by (2), so is the set $E := \Gamma_h \cap (\Gamma_f \times C)$. But since $\Gamma_{g \circ f} = \nu(E)$, where $\nu : A \times B \times C \to A \times C$ is the projection omitting the second factor, by (1), the result follows.

Proof of (7). It is easily checked that the mappings $(x, y) \mapsto (x + y)$ and $(x, y) \mapsto x \cdot y$ are globally subanalytic. By (6), the sum and product of globally subanalytic functions are thus globally subanalytic. $\square$

We can summarize by saying that globally subanalytic sets and mappings possess all the most basic properties that one would need to perform geometric constructions. Actually, a very useful one is still missing: the stability under complement. If it was obvious from the definition that the complement of a semi-analytic set is semi-analytic, it is far from being easy to show that the complement of a globally subanalytic set is globally subanalytic. This is nevertheless true and it is generally referred as the Gabrielov's Complement Theorem (Theorem 1.8.8). The proof of this theorem will use almost all the material introduced in this chapter.

1.2 Cell decompositions

Cell decompositions constitute the central tool of these notes.

Definition 1.2.1. Let us define the cell decompositions of $\mathbb{R}^n$ inductively on n. These are finite partitions of $\mathbb{R}^n$ into globally subanalytic sets, called **cells**.

$n = 1$: A **cell decomposition** $\mathcal{C}$ **of** $\mathbb{R}$ is a finite subdivision of $\mathbb{R}$ given by some real numbers $a_1 < ... < a_l$. The **cells of** $\mathcal{C}$ are then the singletons $\{a_i\}$, $0 < i \leq l$, and the intervals (a_i, a_{i+1}), $0 \leq i \leq l$, where $a_0 = -\infty$ and $a_{l+1} = +\infty$.

$n > 1$: A **cell decomposition** $\mathcal{C}$ **of** $\mathbb{R}^n$ is given by a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n-1}$ and, for each cell $D \in \mathcal{D}$, some globally subanalytic functions, analytic on D:

$$\zeta_{D,1} < ... < \zeta_{D,l(D)} : D \to \mathbb{R}.$$

The **cells of** $\mathcal{C}$ are then the $\mathscr{C}^\infty$ manifolds given by the graphs

$$\{(x, \zeta_{D,i}(x)) : x \in D\},\ 1 \leq i \leq l(D),$$

and the **bands**

$$(\zeta_{D,i}, \zeta_{D,i+1}) := \{(x, y) : x \in D \text{ and } \zeta_{D,i}(x) < y < \zeta_{D,i+1}(x)\},$$

for $0 \leq i \leq l(D)$, where $\zeta_{D,0} \equiv -\infty$ and $\zeta_{D,l(D)+1} \equiv +\infty$. The cell D is then called the **basis** of the cells defined as above.

We started this inductive definition at $n = 1$ to make it more explicit. It is convenient to set that a cell decomposition of $\mathbb{R}^0 = \{0\}$ is constituted (exclusively) by $\{0\}$ (we also will adopt the conventions introduced after (1.0.1)).

A cell decomposition is said to be **compatible with finitely many sets** $A_1, \dots, A_k$ if the A_i's are unions of cells. A **refinement** of a cell decomposition $\mathcal{C}$ is a cell decomposition compatible with all the elements of $\mathcal{C}$.

Remark 1.2.2. If $\pi : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is the projection onto the n first coordinates and $\mathcal{C}$ is a cell decomposition of $\mathbb{R}^{n+p}$ then the finite family of sets $\pi(C)$, $C \in \mathcal{C}$, constitutes a cell decomposition of $\mathbb{R}^n$. We will denote it by $\pi(\mathcal{C})$.

It is also worthy of notice that it follows from this inductive definition that every cell is $\mathscr{C}^\infty$ diffeomorphic to $(0,1)^d$ for some d (with $(0,1)^0 = \{0\}$). The following theorem is fundamental to describe the geometry of globally subanalytic sets:

Theorem 1.2.3. *Given $A_1, \dots, A_k$ in $\mathcal{S}_n$, there is a cell decomposition of $\mathbb{R}^n$ compatible with all the A_i's.*

The proof of this theorem is postponed to section 1.7. We start with an easy example and some hints of proofs.

Example 1.2.4. Let $A := \{(x,y) \in \mathbb{R}^2 : f(x,y) := y^2 - x^3 = 0\}$. As f can be factorized $(y - x^{\frac{3}{2}})(y + x^{\frac{3}{2}})$, the set A is composed by the two branches of curves $C_i := \{(x, (-1)^i x^{\frac{3}{2}}) : x > 0\}$, $i = 1, 2$. The sets C_1 and C_2 are cells of $\mathbb{R}^2$ which constitute a cell decomposition compatible with the globally semi-analytic set A.

In this example, the situation is very simple since the set A is described by an equation which is easily factorized. In particular, the cells have a parametrization as a Puiseux series. This example points out the importance of having a nice factorization of the equations defining our set A.

Outline of proof of Theorem 1.2.3 Let $A \in \mathcal{S}_n$ (we restrict ourselves in this outline to the case of one single set A, i.e., we assume $k = 1$ for simplicity). Observe first that, since A is the projection of a globally semi-analytic set $Z \subset \mathbb{R}^{n+p}$, it is enough to construct a cell decomposition of $\mathbb{R}^{n+p}$ compatible with such a set Z (see Remark 1.2.2). In other words, we can assume A to be globally semi-analytic.

We have to find a cell decomposition such that the functions defining the globally semi-analytic set A (see (1.1.1)) are of constant sign on every cell.

The proof will be by induction on n. The basic idea is to proceed in the same way as in Example 1.2.4: we factorize the analytic functions defining the globally semi-analytic set A until we reach an expression which is sufficiently simple to decompose the set A into graphs and bands. The basic idea of this factorization relies on Weierstrass Preparation Theorem and some related finiteness results of algebraic nature that we present below as preliminaries (section 1.3, see Theorem 1.3.2 and Proposition 1.3.11).

Functions that we have put in such a nice factorized form will be said to be *reduced* (Definition 1.5.2). This form, although a bit more complicated than the expression as a Puiseux series obtained in Example 1.2.4, is of the same type.

As we will argue inductively on the number of variables, it will not be possible to stay in the semi-analytic category: making use of the inductive assumptions requires to drop some variables and Example 1.1.5 shows that this forces to exit the semi-analytic category. We thus shall introduce a bigger category of functions than analytic functions: *the $\mathcal{L}$-functions* (see section 1.4).

We shall show that every $\mathcal{L}$-function can be reduced (Proposition 1.5.4), from which it will follow that we can find a cell decomposition such that finitely many given $\mathcal{L}$-functions have constant sign on every cell (Lemma 1.6.2). The reader is invited to glance at the proof of this lemma which unravels the narrow links between reducibility of $\mathcal{L}$-functions (Proposition 1.5.4) and existence of what we call $\mathcal{L}$-cell decompositions (Lemma 1.6.2), accounting for the fact that we prove this lemma and this proposition simultaneously in a joint induction. To complete this outline, we detail separately in section 1.5 the strategy of the proof of this proposition, which is the main technical difficulty of the proof of Theorem 1.2.3 (although our cell decompositions will rather be provided by the closely related Lemma 1.6.2).

1.3 Preliminaries on analytic functions

1.3.1 The Weierstrass Preparation Theorem

We give a proof this theorem that relies on basic facts of complex analysis. This is the only place in these notes where complex numbers will be involved.

Definition 1.3.1. Let $U \subset \mathbb{R}^{m-1} \times \mathbb{R}$ be an open set and let $(u_0, z_0) \in U$. An analytic function $\psi(u, z)$ on U is **z-regular** at (u_0, z_0) if the function $\varphi : [0, \varepsilon) \to \mathbb{R}$, $\varepsilon > 0$ small, defined by $\varphi(t) := \psi(u_0, z_0 + t)$ is not identically zero. It is said to be **z-regular of order** d at (u_0, z_0) if $\varphi(t) = at^d + \dots$ with $a \neq 0$.

Theorem 1.3.2. *(Weierstrass Preparation Theorem) Let ψ be an analytic function on a neighborhood of $(u_0, z_0) \in \mathbb{R}^{m-1} \times \mathbb{R}$. If $\psi(u, z)$ is z-regular of order d at (u_0, z_0) then there exists a neighborhood of (u_0, z_0) on which ψ has a representation*

$$\psi(u, z) = W(u, z) \cdot f(u, z),$$

where W is an analytic function near (u_0, z_0) satisfying $W(u_0, z_0) \neq 0$, and

$$f(u, z) = z^d + a_1(u)z^{d-1} + \dots + a_d(u)$$

is a unit polynomial of degree d with analytic coefficients.

Proof. We may assume $(u_0, z_0) = (0_{\mathbb{R}^{m-1}}, 0_{\mathbb{R}})$ and $\psi(0, 0) = 0$. The Taylor series of ψ being convergent near the origin, this function extends to a holomorphic function (still denoted ψ) on a neighborhood of this point in $\mathbb{C}^{m-1} \times \mathbb{C}$.

Let γ be a small circle around 0 in $\mathbb{C}$ (oriented counterclockwise) and let for $u \in \mathbb{C}^{m-1}$ close to the origin:

$$c_0(u) := \frac{1}{2i\pi} \int_\gamma \frac{\frac{\partial \psi}{\partial z}(u, z)}{\psi(u, z)} dz.$$

By the residue theorem, if γ is sufficiently small, c_0 is a continuous integer valued function on a small neighborhood of the origin in $\mathbb{C}^{m-1}$. It is thus equal to some constant d on a neighborhood of the origin. For every $u \in \mathbb{C}^{m-1}$ close to $0_{\mathbb{C}^{m-1}}$, let $b_1(u), \dots, b_d(u)$ be the (complex) roots of the function $z \mapsto \psi(u,z)$ (repeated with multiplicity) that lie inside the disk delimited by γ. Let then for $j = 1, \dots, d$

$$c_j := b_1^j + \cdots + b_d^j ,$$

and observe that, again due to the residue theorem, we have

$$c_j(u) = \frac{1}{2\pi i} \int_\gamma z^j \frac{\frac{\partial \psi}{\partial z}(u,z)}{\psi(u,z)} dz,$$

which shows that the c_j's are holomorphic functions on a neighborhood of the origin in $\mathbb{C}^{m-1}$, real valued on $\mathbb{R}^{m-1}$ (the b_j's are pairwise conjugate). We now set for $(u,z) \in \mathbb{C}^{m-1} \times \mathbb{C}$:

$$f(u,z) := \prod_{j=1}^{d} (z - b_j(u)) = z^d + a_1(u) z^{d-1} + \cdots + a_d(u),$$

for some $a_1, \dots, a_d$. Since the a_i's are polynomial functions of the c_j's (by Girard-Newton's identities), by the above, these are holomorphic functions. We then set $W(u,z) := \frac{\psi(u,z)}{f(u,z)}$ and notice that this function is real valued on a neighborhood of the origin in $\mathbb{R}^m$. Because for every u near 0, the functions $z \mapsto \psi(u,z)$ and $z \mapsto f(u,z)$ have the same zeros with the same multiplicities, W is holomorphic with respect to z and nonzero at the origin. By Cauchy formula, we thus have on a neighborhood of the origin

$$W(u,z) = \int_\gamma \frac{W(u,\zeta)}{z - \zeta} d\zeta. \tag{1.3.1}$$

where γ is as above. Since f and ψ are both holomorphic functions, $W(u,\zeta)$ is also holomorphic with respect to u on the complement of the zeros of f, which, by (1.3.1), means that it is holomorphic everywhere on a neighborhood of the origin. □

Given $y_0 \in \mathbb{R}^m$, we write $\mathcal{A}_{y_0}$ for the ring of analytic function-germs at y_0.

Corollary 1.3.3. *The ring $\mathcal{A}_{y_0}$ is Noetherian.*

Proof. Set $y_0 = (u_0, z_0) \in \mathbb{R}^{m-1} \times \mathbb{R}$. Arguing inductively, we can suppose that $\mathcal{A}_{u_0}$ is Noetherian, which, by Hilbert's Basis Theorem, means that so is the ring

$$\mathcal{A}_{u_0}[z] := \{ f \in \mathcal{A}_{y_0} : f(u,z) = \sum_{i=0}^{d} a_i(u) z^i, \quad \text{for some } \; a_i \in \mathcal{A}_{u_0}, d \in \mathbb{N} \}.$$

Let $\mathcal{I}$ be an ideal of $\mathcal{A}_{y_0}$. Thanks to Weierstrass Preparation Theorem, we know that $\mathcal{I}$ is generated by $\mathfrak{I} := \mathcal{I} \cap \mathcal{A}_{u_0}[z]$. As $\mathfrak{I}$ is an ideal of $\mathcal{A}_{u_0}[z]$, which is a Noetherian ring, it is finitely generated. □

1.3.2 Some finiteness properties

Algebraic machinery. We recall that a ring R is **local** if it has only one maximal ideal. It is well-known that the maximal ideal of a local ring is constituted by all the non invertible elements of this ring.

Throughout this section R stands for a Noetherian local ring and $\mathfrak{M}$ for its maximal ideal.

Definition 1.3.4. Let M be a finitely generated R-module. We say that a decreasing sequence $M_* = \{M_i\}_{i\in\mathbb{N}}$ of submodules of M is an **$\mathfrak{M}$-filtration of** M if $\mathfrak{M}M_i \subset M_{i+1}$ for all $i \geq 0$.

An $\mathfrak{M}$-filtration $\{M_i\}_{i\in\mathbb{N}}$ of M is **$\mathfrak{M}$-stable** if $\mathfrak{M}M_i = M_{i+1}$ for each integer i sufficiently large.

Given an $\mathfrak{M}$-filtration M_*, we denote by $\mathcal{G}(M_*)$, the set of formal polynomials whose T^i-coefficient lies in M_i, that is to say

$$\mathcal{G}(M_*) = \{\sum_{i=0}^{d} a_i T^i : a_i \in M_i, d \in \mathbb{N}\}.$$

In particular, since we can regard $\mathfrak{M}$ as an R-module, $\mathfrak{M}_* := (\mathfrak{M}^i)_{i\in\mathbb{N}}$ is an $\mathfrak{M}$-filtration of $\mathfrak{M}$ and

$$\mathcal{G}(\mathfrak{M}_*) = \{\sum_{i=0}^{d} a_i T^i : a_i \in \mathfrak{M}^i, d \in \mathbb{N}\}.$$

Clearly, $\mathcal{G}(M_*)$ is a $\mathcal{G}(\mathfrak{M}_*)$-module.

Remark 1.3.5. We recall that a module is said to be Noetherian if every submodule is finitely generated, and that a finitely generated module over a Noetherian ring is always Noetherian. In particular, in the above definition, all the M_i's are finitely generated.

Lemma 1.3.6. *Let M be a finitely generated R-module and let $M_* = \{M_i\}_{i\in\mathbb{N}}$ be an $\mathfrak{M}$-filtration of M. The $\mathcal{G}(\mathfrak{M}_*)$-module $\mathcal{G}(M_*)$ is finitely generated if and only if M_* is $\mathfrak{M}$-stable.*

Proof. Assume that the $\mathcal{G}(\mathfrak{M}_*)$-module $\mathcal{G}(M_*)$ is finitely generated, say by $f_1, \dots f_p$, with for each j, $f_j := \sum_{k=0}^{d_j} a_{j,k} T^k$, where $a_{j,k} \in M_k$. Set for $i \in \mathbb{N}$

$$Z_i := \{g \in \mathcal{G}(M_*) : g = cT^i, \text{ for some } c \in M_i\}.$$

Note that the elements of $a_{j,k}T^k$, $k \leq d_j$, $j \leq p$, also generate $\mathcal{G}(M_*)$. Since M_* is an $\mathfrak{M}$-filtration, this implies that for $i \geq \max\{d_j : j = 1, , \dots, p\}$ we have $\mathcal{G}(\mathfrak{M}_*)Z_i \supset Z_{i+1}$, which means that $\mathfrak{M}M_i \supset M_{i+1}$, as required.

Conversely, if M_* is $\mathfrak{M}$-stable then there is d such that $\mathfrak{M}M_i = M_{i+1}$ for each $i \geq d$, which means that $\mathfrak{M}^\kappa M_i = M_{i+\kappa}$ for every such i and every positive integer κ. This establishes that $\mathcal{G}(M_*)$ is generated by the union of the respective generators of $Z_1, \ldots, Z_d$, which, since every M_i is finitely generated, is a finite set. □

Lemma 1.3.7. *Let M be a finitely generated R-module and let $M_* = \{M_i\}_{i\in\mathbb{N}}$ be an $\mathfrak{M}$-stable $\mathfrak{M}$-filtration of M. For every submodule N of M, the filtration $N_* := \{N \cap M_i\}_{i\in\mathbb{N}}$ of N is $\mathfrak{M}$-stable.*

Proof. Since M_* is $\mathfrak{M}$-stable, by Lemma 1.3.6, $\mathcal{G}(M_*)$ is a finitely generated $\mathcal{G}(\mathfrak{M}_*)$-module. Notice that $\mathcal{G}(\mathfrak{M}_*)$, regarded as an R-algebra, is finitely generated. Since R is Noetherian, by Hilbert's Basis Theorem, this implies that $\mathcal{G}(\mathfrak{M}_*)$ is Noetherian as well, which implies in turn that $\mathcal{G}(M_*)$ is a Noetherian $\mathcal{G}(\mathfrak{M}_*)$-module (see Remark 1.3.5). This establishes that $\mathcal{G}(N_*)$, which is a submodule of $\mathcal{G}(M_*)$, is a finitely generated $\mathcal{G}(\mathfrak{M}_*)$ module, which, by Lemma 1.3.6, yields that N_* is $\mathfrak{M}$-stable. □

Lemma 1.3.8. *(Artin-Rees) Let M be a finitely generated R-module and let N be a submodule. For all i large enough,*

$$N \cap \mathfrak{M}^{i+1}M = \mathfrak{M}(N \cap \mathfrak{M}^i M).$$

In other words, the $\mathfrak{M}$-filtration $\{N \cap \mathfrak{M}^i M\}_{i\in\mathbb{N}}$ of N is $\mathfrak{M}$-stable.

Proof. It suffices to apply Lemma 1.3.7 to the $\mathfrak{M}$-stable $\mathfrak{M}$-filtration $\{\mathfrak{M}^i M\}_{i\in\mathbb{N}}$. □

This leads us to the following famous result which is sometimes rephrased by saying that ideals are closed in the $\mathfrak{M}$-adic topology.

Theorem 1.3.9. *(Krull's intersection theorem) Every ideal $\mathcal{I}$ of R satisfies*

$$\bigcap_{i\in\mathbb{N}} (\mathcal{I} + \mathfrak{M}^i) = \mathcal{I}.$$

Proof. We start with the case $\mathcal{I} = 0$. Applying Lemma 1.3.8 to $N := \bigcap_{i\in\mathbb{N}} \mathfrak{M}^i$ and $M = R$, we get:

$$N = \mathfrak{M}N.$$

Assume that $N \neq 0$ and take a minimal system of generators $f_1, \ldots, f_p$. We have:

$$N = \mathfrak{M}N = \mathfrak{M}f_1 \oplus \cdots \oplus \mathfrak{M}f_p.$$

In particular $f_1 = \sum_{i=1}^p x_i f_i$ with $x_i \in \mathfrak{M}$ for all i, which implies that $(1 - x_1)f_1 = \sum_{i=2}^p x_i f_i$. As R is local, $(1 - x_1)$ is invertible, which, thanks to the latter equality, means that $f_2, \ldots, f_p$ span N, in contradiction with our minimality assumption on the system of generators. This yields $\bigcap_{i\in\mathbb{N}} \mathfrak{M}^i = \{0\}$, as required.

The general case can now be deduced from this particular case by quotienting by the ideal $\mathcal{I}$. Namely, set $A := R/\mathcal{I}$ and let $q : R \to A$ be the quotient map. The ring A is local and its maximal ideal is $\overline{\mathfrak{M}} := q(\mathfrak{M})$. By the above, we thus have

$$\bigcap_{i\in\mathbb{N}} \mathcal{I} + \mathfrak{M}^i = \bigcap_{i\in\mathbb{N}} q^{-1}(\overline{\mathfrak{M}}^i) = q^{-1}(\bigcap_{i\in\mathbb{N}} \overline{\mathfrak{M}}^i) = q^{-1}(\{0\}) = \mathcal{I}.$$

□

Application to rings of analytic function-germs. Let $y_0 \in \mathbb{R}^m$. We have seen that the ring $\mathcal{A}_{y_0}$ is Noetherian. This ring is moreover local and its maximal ideal is the ideal $\mathfrak{M}_{y_0}$ constituted by all the elements of $\mathcal{A}_{y_0}$ that vanish at the origin. The following corollary of the above theorem, sometimes referred as Krull's Theorem, will be useful to us.

Corollary 1.3.10. *Let $h : \mathcal{A}_{y_0}^d \to \mathcal{A}_{y_0}$ be an $\mathcal{A}_{y_0}$-linear mapping and let $a \in \mathcal{A}_{y_0}$. If the equation*

$$h(f_1, \dots, f_d) = a$$

has solutions $f_1, \dots, f_d$ in the ring of formal power series $\mathbb{R}[[Y_1, \dots, Y_m]]$ then it has solutions $g_1, \dots, g_d$ in the ring $\mathcal{A}_{y_0}$.

Proof. Let $f_1, \dots, f_d$ be some formal power series which are solutions and let, for every $j \le d$ and $i \in \mathbb{N}$, $f_{j,i}$ denote the polynomial of degree i obtained by truncating the formal series f_j at the order i. Clearly $(a - h(f_{1,i}, \dots, f_{d,i})) \in \mathfrak{M}_{y_0}^{i+1}$ which means that $a \in \mathcal{I} + \mathfrak{M}^i$ for all i, where $\mathcal{I}$ is the ideal $h(\mathcal{A}_{y_0}^d)$. The result thus follows from Theorem 1.3.9. □

Artin-Rees' Lemma can be established for a finitely generated module M, which entails that Krull's intersection theorem can actually be proved not only for an ideal, but for every finitely generated R-module. Consequently, the above corollary is still valid for a linear mapping $h : \mathcal{A}_{y_0}^d \to \mathcal{A}_{y_0}^p$, i.e., for a *linear system of equations*. Furthermore, it is not difficult to see that we can require the g_i's to coincide with the f_i's at any prescribed order. The just above corollary is however enough for our purpose, as we shall only need the following consequence in the proof of Proposition 1.5.4 (which is crucial to establish Theorem 1.2.3).

Proposition 1.3.11. *Let $y_0 = (u_0, z_0) \in \mathbb{R}^{m-1} \times \mathbb{R}$. For each $\psi \in \mathcal{A}_{y_0}$, there exist $A_0, \dots, A_d \in \mathcal{A}_{y_0}$ satisfying $A_i(y_0) \neq 0$ for all i, as well as some $(m-1)$-variable analytic function-germs $c_0, \dots, c_d \in \mathcal{A}_{u_0}$, such that for all (u, z) near y_0:*

$$\psi(u, z) = \sum_{i=0}^{d} c_i(u)(z - z_0)^i A_i(u, z).$$

Proof. We can assume that y_0 is the origin. Write then for (u, z) close to the origin

$$\psi(u, z) = \sum_{i \in \mathbb{N}} c_i(u) z^i,$$

with c_i $(m-1)$-variable analytic function-germ for every i. Since the ring of germs of analytic functions is Noetherian, the ideal generated by all the c_i's is generated by the germs at 0 of a finite family $c_0, \dots, c_d$, $d \in \mathbb{N}$. For every $j > d$, there thus exist $(d+1)$ germs of analytic functions $b_{0,j}, \dots, b_{d,j}$ such that

$$c_j(u) = \sum_{i=0}^{d} b_{i,j}(u) c_i(u).$$

Hence, as formal power series in the indeterminates $(U, Z) = (U_1, \dots, U_{m-1}, Z)$ we have:

$$\begin{aligned} \psi(U, Z) &= \sum_{i=0}^{d} c_i(U) Z^i + \sum_{j>d} \sum_{i=0}^{d} b_{i,j}(U) c_i(U) Z^j \\ &= \sum_{i=0}^{d} c_i(U) Z^i (1 + Z f_i(U, Z)), \end{aligned}$$

for some formal power series f_i, $i = 0, \dots, d$. Let us consider the linear equation of formal power series:

$$\psi(U, Z) = \sum_{i=0}^{d} c_i(U) Z^i (1 + Z F_i),$$

in the unknowns $F_0, \dots, F_d$. This equation has a solution $F_i = f_i$, $i = 0, \dots, d$, in the ring of formal power series. By Corollary 1.3.10, it must have a solution g_i, $i = 0, \dots, d$, in the ring of convergent power series. It is then enough to set $A_i := 1 + z g_i$. □

1.4 $\mathcal{L}$-functions and $\mathcal{L}$-cells

As we said in the outline of proof of Theorem 1.2.3, we will have to work with another class of functions, *the* $\mathcal{L}$*-functions*, that we introduce in this section. It will turn out that every globally subanalytic function is piecewise given by $\mathcal{L}$-functions (Proposition 1.8.1).

The **cube** of radius $\varepsilon > 0$ and centered at $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ is the set:

$$\mathbf{C}(a, \varepsilon) := \{(x_1, \dots, x_n) \in \mathbb{R}^n : \forall i,\ |x_i - a_i| \leq \varepsilon\}.$$

A **restricted analytic function** is a function $\psi : C \to \mathbb{R}$, with C cube of $\mathbb{R}^n$, which can be extended analytically in an open neighborhood of C. Let $\mathcal{R}_{an}$ be the set of all the restricted analytic functions (of all the cubes).

Let now $\mathcal{L}$ be the set of functions obtained by adding the functions $x \mapsto x^\lambda$, $\lambda \in \mathbb{Q}$ (the power functions), to the family $\mathcal{R}_{an}$. We then introduce the **$\mathcal{L}$-functions** inductively on what we will call the complexity, as follows.

(i) If A is a globally subanalytic set then the restriction to A of an element of $\mathcal{L}$ is an $\mathcal{L}$-function.

(ii) If ψ_1 and ψ_2 are $\mathcal{L}$-functions then so are $(\psi_1 + \psi_2)$ and $\psi_1 \cdot \psi_2$.

(iii) Any function $\Psi : A \to \mathbb{R}$ of type $x \mapsto \psi(\phi_1(x), \dots, \phi_k(x))$, with ψ as well as $\phi_1, \dots, \phi_k$ $\mathcal{L}$-functions and A globally subanalytic set, is an $\mathcal{L}$-function.

Roughly speaking, the $\mathcal{L}$-functions are the functions that coincide with the restriction to a globally subanalytic set of a function given by finite sums, products, and composites of elements of $\mathcal{L}$. The minimal number of operations (sum, product, composition with an element of $\mathcal{L}$) needed to generate an $\mathcal{L}$-function f is called the **complexity** of the function f. A mapping whose components are $\mathcal{L}$-functions is called an **$\mathcal{L}$-mapping**.

Since, by Properties 1.1.8, the class of globally subanalytic mappings is closed under sums, products, and compositions, it directly follows from the above definition that $\mathcal{L}$-mappings are globally subanalytic. Proposition 1.8.1 below can be seen as a partial converse of this fact.

The class of $\mathcal{L}$-functions is much bigger than the class of restricted analytic functions, as shown by the following examples.

Example 1.4.1. Since every polynomial is an $\mathcal{L}$-function, so is for instance the function $\frac{x-3}{\sqrt{x^2+y^4}}$ (on its domain). The function e^x, defined on $\mathbb{R}$, although analytic, is not an $\mathcal{L}$-function. Its restriction to any bounded interval is however an $\mathcal{L}$-function, which entails for instance that so is the function $\mathbb{R}^2 \setminus \{(0,0)\} \ni (x,y) \mapsto e^{-\frac{x^2}{x^2+y^2}}$.

We then define the **$\mathcal{L}$-cells** of $\mathbb{R}^n$ by induction on n. Let C be a cell of $\mathbb{R}^n$ and denote by B its basis. If $n = 0$ then C is always an $\mathcal{L}$-cell. If $n > 0$, we say that C is an $\mathcal{L}$-cell if so is B and if in addition one of the following properties holds:

(i) C is the graph of some $\mathcal{L}$-function $\xi : B \to \mathbb{R}$.

(ii) C is a band (ξ_1, ξ_2), $\xi_1 < \xi_2$, where ξ_1 is either $-\infty$ or an $\mathcal{L}$-function on B, and ξ_2 is either $+\infty$ or an $\mathcal{L}$-function on B.

A cell decomposition of $\mathbb{R}^n$ consisting exclusively of $\mathcal{L}$-cells is called an **$\mathcal{L}$-cell decomposition**.

1.5 Reduced functions

Roughly speaking, the reduced functions will be the functions that have a nice form, up to an $\mathcal{L}$-unit, which requires to define $\mathcal{L}$-units.

Definition 1.5.1. Let $C \subset \mathbb{R}^n$ denote an $\mathcal{L}$-cell, B its basis, and let $\theta : B \to \mathbb{R}$ be an $\mathcal{L}$-function satisfying $\Gamma_\theta \cap C = \emptyset$.

An **$\mathcal{L}$-unit** (in the variable y) of C is a function U on C that can be written $\psi(V(x))$ with, for $x := (\tilde{x}, x_n) \in C \subset \mathbb{R}^{n-1} \times \mathbb{R}$:

$$V(x) = (b_1(\tilde{x}), \ldots, b_k(\tilde{x}), u(\tilde{x})\, y^{\frac{1}{s}}, v(\tilde{x})\, y^{-\frac{1}{s}}), \quad y := |x_n - \theta(\tilde{x})|,$$

where $s \in \mathbb{N}^*$, $b_1, \ldots, b_k, u, v$ are analytic $\mathcal{L}$-functions on B such that $V(C)$ is relatively compact, and where ψ is an analytic function on a neighborhood of $cl(V(C))$ nowhere vanishing on this set.

We can say that an $\mathcal{L}$-unit is a Puiseux series in y and $\frac{1}{y}$ which has coefficients that are analytic $(n-1)$-variable $\mathcal{L}$-functions on the basis of C, and which is bounded away from zero and infinity.

Definition 1.5.2. Let $E \subset \mathbb{R}^n$ and let $C \subset E$ be an $\mathcal{L}$-cell of basis B. A function $\xi : E \to \mathbb{R}$ is **reduced** on C if we can find analytic $\mathcal{L}$-functions $a : B \to \mathbb{R}$ and $\theta : B \to \mathbb{R}$, as well as $r \in \mathbb{Q}$, such that $\Gamma_\theta \cap C = \emptyset$ and

$$\xi(\tilde{x}, x_n) = a(\tilde{x}) \cdot y^r \cdot U(\tilde{x}, y), \quad y := |x_n - \theta(\tilde{x})|, \tag{1.5.1}$$

for all $x = (\tilde{x}, x_n) \in C \subset \mathbb{R}^{n-1} \times \mathbb{R}$, where U is an $\mathcal{L}$-unit of C in the variable y.

The function θ is then called the **$\mathcal{L}$-translation of the reduction**. If r is nonnegative, we say that the reduction is **nondegenerate**.

A function $\xi : E \to \mathbb{R}$ is **reducible** if there is an $\mathcal{L}$-cell decomposition compatible with E such that ξ is reduced on every cell $C \subset E$ of this cell decomposition.

In short, a reduced function is merely, up to a product with an $\mathcal{L}$-unit, a power of $|x_n - \theta(\tilde{x})|$ times an $(n-1)$-variable analytic $\mathcal{L}$-function a.

Remark 1.5.3. Let a function ξ be reduced with $\mathcal{L}$-translation θ on an $\mathcal{L}$-cell C and let θ' be an $\mathcal{L}$-function on the basis of C. It easily follows from the above definitions that if $(x_n - \theta)$ is reduced on C *with $\mathcal{L}$-translation* θ' then so is ξ.

The main difficulty of this chapter is to show the following fact.

Proposition 1.5.4. *If C is an $\mathcal{L}$-cell then every $\mathcal{L}$-function on C is reducible.*

The proof of this result occupies the whole of section 1.6. To motivate the preliminaries that we shall carry out, we sketch the main strategy of the proof.

Strategy of the proof of Proposition 1.5.4. The proof is carried out by induction on the number of variables of the function. $\mathcal{L}$-functions are explicitly known: they are finite sums, products, and composites of power functions and restricted analytic functions. Arguing also by induction on the complexity of the function, we just have to show that each of these operations (sum, product, power, composition with a restricted analytic function) preserves reducible functions. The main issue is indeed to show that composition with a restricted analytic function preserves reduced functions (Proposition 1.6.15). The difficulty in studying $\mathcal{L}$-functions is that they involve negative powers. The strategy is to split the proof of Proposition 1.6.15 into two steps: we will first show, relying on Weierstrass' Preparation Theorem, that we can reduce n-variable $\mathcal{L}$-functions that are analytic with respect to the last variable x_n (Proposition 1.6.12) and then show that we can handle n-variable $\mathcal{L}$-functions that are analytic functions of both x_n and $\frac{c(\tilde{x})}{x_n}$, with c $\mathcal{L}$-function (Proposition 1.6.14).

1.6 Reduction of $\mathcal{L}$-functions

All this section is devoted to the proof of Proposition 1.5.4, which will be carried out by induction on the dimension of the ambient space. More precisely, we shall establish the following facts by induction on n:

($\mathcal{H}_n$) If C is an $\mathcal{L}$-cell of $\mathbb{R}^n$ then every $\mathcal{L}$-function $\xi : C \to \mathbb{R}$ is reducible.

The assertion ($\mathcal{H}_0$) is vacuous. Fix $n \geq 1$ and assume that ($\mathcal{H}_i$) holds true for all $i < n$.

Note. All the propositions and lemmas of this section will also be proved by induction n. Hence, we will assume that all of them are true when n is replaced by $(n-1)$ and simply check them for this fixed value of n (the case $n = 0$ always being trivial).

Definition 1.6.1. We say that two functions $f_1 : A \to \mathbb{R}$ and $f_2 : A \to \mathbb{R}$ are **comparable** if either $f_1(x) \leq f_2(x)$ for all $x \in A$ or $f_2(x) \leq f_1(x)$ for all $x \in A$. We say that a finite collection of functions $f_i : A \to \mathbb{R}$, $i = 1, \ldots, k$, is **totally ordered** if the f_i's are all pairwise comparable with each other.

Given $x \in \mathbb{R}$, let $sign(x) := 1$ if x is positive and $sign(x) := -1$ whenever x is negative, and set $sign(0) := 0$. We say that a function $\xi : A \to \mathbb{R}$ **has constant sign** on $B \subset A$ if the function $sign(\xi(x))$ is constant on B.[1]

Given two real valued functions f and g on a set X, we write $f \lesssim g$ if there exists a positive real number C such that $f \leq Cg$ on X. We write $f \sim g$ (and say that f is **equivalent** to g) if $f \lesssim g$ and $g \lesssim f$.

[1] A nonnegative function is thus not always of constant sign (since it can vanish).

1.6.1 A few consequences of $(\mathcal{H}_{n-1})$

The proof of Proposition 1.5.4 requires the following lemmas. As we mentioned, we shall just prove these lemmas for our fixed value of n.

Lemma 1.6.2. *Let $C_1, \dots, C_k$ be $\mathcal{L}$-cells of $\mathbb{R}^n$ and let $\xi_i : C_i \to \mathbb{R}$, $i = 1, \dots, k$ be reduced $\mathcal{L}$-functions. There is an $\mathcal{L}$-cell decomposition of $\mathbb{R}^n$ compatible with the C_i's and such that for every i the function ξ_i has constant sign on every cell $D \subset C_i$ of this $\mathcal{L}$-cell decomposition.*

Proof. For every $i \leq k$, since ξ_i is reduced, there are $(n-1)$-variable $\mathcal{L}$-functions a_i and θ_i on B_i (here B_i stands for the basis of C_i) such that:

$$\xi_i(\tilde{x}, x_n) = a_i(\tilde{x}) \cdot y_i^{r_i} \cdot U_i(\tilde{x}, y_i), \quad y_i := |x_n - \theta_i(\tilde{x})|,$$

where $U_i(x, y_i)$ is an $\mathcal{L}$-unit in the variable y_i and $r_i \in \mathbb{Q}$. Let also $\zeta_1, \dots, \zeta_p$ be all the $(n-1)$-variable $\mathcal{L}$-functions defining the cells $C_1, \dots, C_k$ (see Definition 1.2.1).

By $(\mathcal{H}_{n-1})$, the a_i's are reducible. Moreover, by induction on n, this lemma holds true if n is replaced with $(n-1)$. We therefore can find an $\mathcal{L}$-cell decomposition $\mathcal{C}$ of $\mathbb{R}^{n-1}$ compatible with the B_i's and such that for every $i \leq k$, the $(n-1)$-variable function a_i has constant sign on every cell (included in B_i). For the same reason, we may also assume that the family constituted by the respective restrictions of the functions θ_i, $i \leq k$, together with the functions ζ_i, $i \leq p$, to each cell of $\mathcal{C}$ (on which these functions are defined), is totally ordered.

The cells of the desired $\mathcal{L}$-cell decomposition $\mathcal{C}'$ of $\mathbb{R}^n$ are now given by the graphs and bands defined by the respective restrictions of the functions θ_i, $i = 1, \dots, k$, and ζ_i, $i = 1, \dots, p$, to the cells of $\mathcal{C}$ on which they are defined.

Fix some $i \leq k$ and let D be a cell of $\mathcal{C}'$ included in C_i. On D, since the functions y_i, U_i (by definition of $\mathcal{L}$-units), and a_i have constant sign, we see that ξ_i has constant sign as well. $\square$

Remark 1.6.3. This lemma entails that, given an $\mathcal{L}$-cell C of $\mathbb{R}^n$ of basis B and an $\mathcal{L}$-function $\phi : B \to \mathbb{R}$, we can find an $\mathcal{L}$-cell decomposition of $\mathbb{R}^n$ compatible with C such that on every cell $E \subset C$ we have for all $(\tilde{x}, x_n) \in E$ either $|x_n| \leq |\phi(\tilde{x})|$ or $|x_n| \geq |\phi(\tilde{x})|$. Indeed, it is enough to apply Lemma 1.6.2 to the reduced $\mathcal{L}$-functions ϕ, x_n, $(x_n + \phi)$, and $(x_n - \phi)$.

Remark 1.6.4. Applying the above lemma to the case where the C_i's are the cells of two given cell $\mathcal{L}$-decompositions of $\mathbb{R}^n$, we can conclude that two $\mathcal{L}$-cell decompositions of $\mathbb{R}^n$ have a common refinement.

Remark 1.6.5. A reduced function on a cell is analytic on this cell. By $(\mathcal{H}_{n-1})$ and the preceding remark, it means that, up to a refinement of the cell decomposition, we can always assume that some given $(n-1)$-variable $\mathcal{L}$-functions are analytic on every cell.

Lemma 1.6.6. *Let C be an $\mathcal{L}$-cell of $\mathbb{R}^n$, ϕ be an $\mathcal{L}$-function on the basis B of C, $p \in \mathbb{Z}$, and define a function on C by $\xi(x) := \phi(\tilde{x})x_n^p$, where $x = (\tilde{x}, x_n)$. Given two real numbers $a < b$, there is an $\mathcal{L}$-cell decomposition compatible with $\xi^{-1}([a, b])$.*

Proof. By Lemma 1.6.2, there is an $\mathcal{L}$-cell decomposition compatible with C such that ϕ and x_n have constant sign on every cell. If $E \subset C$ is a cell of this cell decomposition, it is easily checked that $\xi_{|E}^{-1}([a, b])$ can be described by sign conditions on reduced functions. The result thus follows after applying again Lemma 1.6.2. □

Lemma 1.6.7. *Given finitely many reduced functions $\xi_1, \dots, \xi_k$ on an $\mathcal{L}$-cell C of $\mathbb{R}^n$, we can find an $\mathcal{L}$-cell decomposition compatible with C such that on every cell $E \subset C$, all these functions are reduced with the same $\mathcal{L}$-translation.*

Proof. Let $\theta_1, \dots, \theta_k$ denote the respective $\mathcal{L}$-translations of the reductions of the ξ_i's. As a consequence of Lemma 1.6.2 (see Remark 1.6.3), there is an $\mathcal{L}$-cell decomposition of $\mathbb{R}^n$ compatible with C such that on each cell $D \subset C$, the functions $|x_n - \theta_i|, |\theta_i - \theta_j|$, $i < j \le k$, are comparable with each other (see Definition 1.6.1) and the functions $(x_n - \theta_i)$, $(\theta_i - \theta_j)$, $i < j \le k$, have constant sign. Fix a cell D and choose j such that on D we have for all $i \le k$:

$$|x_n - \theta_j| \le |x_n - \theta_i|. \tag{1.6.1}$$

We are going to show that for all i, the function $(x_n - \theta_i)$ is reduced on D with $\mathcal{L}$-translation θ_j, so that the statement of the lemma will then follow from Remark 1.5.3. Fix $i \le k$. The proof now breaks down into two cases.

<u>Case 1</u>: $|x_n - \theta_j| \le |\theta_j - \theta_i|$ on D.

If there is $\tilde{x}$ in the basis of D such that $\theta_j(\tilde{x}) = \theta_i(\tilde{x})$ then $x_n \equiv \theta_j \equiv \theta_i$ on D (since $(\theta_i - \theta_j)$ has constant sign), and the result is trivial. Otherwise, either the two functions $(x_n - \theta_j)$ and $(\theta_j - \theta_i)$ have the same sign or $|\dfrac{x_n - \theta_j}{\theta_j - \theta_i}| \le \dfrac{1}{2}$ (by (1.6.1)). It means that $(1 + \dfrac{x_n - \theta_j}{\theta_j - \theta_i})$ is an $\mathcal{L}$-unit in the variable $(x_n - \theta_j)$ and hence, the function $(x_n - \theta_i)$ can be reduced by

$$x_n - \theta_i = (\theta_j - \theta_i)(1 + \frac{x_n - \theta_j}{\theta_j - \theta_i}). \tag{1.6.2}$$

<u>Case 2</u>: $|x_n - \theta_j| \ge |\theta_j - \theta_i|$ on D.

If there is $x = (\tilde{x}, x_n) \in C$ such that $x_n = \theta_j(\tilde{x})$ then $x_n \equiv \theta_j$ on C, and the result is trivial. Otherwise, by (1.6.1), $(x_n - \theta_j)$ and $(\theta_j - \theta_i)$ are of the same sign, and hence the function $(1+\dfrac{\theta_j-\theta_i}{x_n-\theta_j})$ is an $\mathcal{L}$-unit. As a matter of fact, the function $(x_n-\theta_i)$ can be reduced by writing

$$x_n-\theta_i=(x_n-\theta_j)(1+\frac{\theta_j-\theta_i}{x_n-\theta_j}). \tag{1.6.3}$$

□

Remark 1.6.8. A direct consequence of Remark 1.6.4 and Lemma 1.6.7 is that the product of two reducible functions $\xi_1 : C \to \mathbb{R}$ and $\xi_2 : C \to \mathbb{R}$, where C is an $\mathcal{L}$-cell of $\mathbb{R}^n$, is reducible. The quotient, if well-defined, is also reducible.

Lemma 1.6.9. *Given two reduced functions f and g on an $\mathcal{L}$-cell C of $\mathbb{R}^n$ and a positive constant ε, there is an $\mathcal{L}$-cell decomposition compatible with C such that on every cell $E \subset C$ either $|f| \le \varepsilon|g|$, or $|g| \le \varepsilon|f|$, or $|f| \sim |g|$.*

Proof. By Lemma 1.6.7, we may assume that f and g are reduced on the cells of an $\mathcal{L}$-cell decomposition with *the same* $\mathcal{L}$-translation on every cell. Since it is enough to establish the lemma for arbitrarily small values of $\varepsilon > 0$ and because $\mathcal{L}$-units are bounded away from zero and infinity, it is enough to prove the result for some functions of the form $f(x) = a(\tilde{x})|x_n - \theta(\tilde{x})|^r$ and $g(x) = b(\tilde{x})|x_n - \theta(\tilde{x})|^s$, with a, θ, and b $\mathcal{L}$-functions on the basis B of an $\mathcal{L}$-cell C, r and s in $\mathbb{Q}$, and $x = (\tilde{x}, x_n) \in C$.

By Lemma 1.6.2, we may assume that a, $(x_n-\theta)$, and b have constant sign on C. The inequality $|f| \le \varepsilon|g|$ on a cell C now amounts to inequalities of type $x_n \le \phi(\tilde{x})$ or $x_n \ge \phi(\tilde{x})$ with ϕ $\mathcal{L}$-function on its basis B. But, again by Lemma 1.6.2, given any $\mathcal{L}$-function ϕ on B, there is a refinement of our $\mathcal{L}$-cell decomposition such that the function $(x_n - \phi)$ has constant sign on every cell. □

Lemma 1.6.10. *Given an n-variable reducible function ξ, we may always assume, up to a refinement of the $\mathcal{L}$-cell decomposition, that on every cell either $x_n \sim \theta(\tilde{x})$ or $\theta(\tilde{x}) \equiv 0$, where θ is the $\mathcal{L}$-translation of the reduction on the cell.*

Proof. Take a cell decomposition such that ξ is reduced on every cell. By Lemma 1.6.9 and Remark 1.6.4, up to a refinement of the $\mathcal{L}$-cell decomposition we may assume that on every cell either $|x_n| \sim |\theta(\tilde{x})|$ or $|x_n| \le \frac{1}{2}|\theta(\tilde{x})|$ or $|\theta(\tilde{x})| \le \frac{1}{2}|x_n|$. By Lemma 1.6.2, x_n and θ may be assumed to be of constant sign on every cell, and we will assume them to be nonzero (since otherwise we are done). If on a cell $|\theta(\tilde{x})| \le \frac{1}{2}|x_n|$ then writing

$$x_n - \theta(\tilde{x}) = x_n(1 - \frac{\theta(\tilde{x})}{x_n}), \tag{1.6.4}$$

we see that, since $(1 - \frac{\theta(\tilde{x})}{x_n})$ is an $\mathcal{L}$-unit, we can assume the $\mathcal{L}$-translation of the reduction to be identically 0.

Similarly, in the case where $|x_n| \le \frac{1}{2}|\theta(\tilde{x})|$, writing $x_n - \theta(\tilde{x}) = \theta(\tilde{x})(\frac{x_n}{\theta(\tilde{x})} - 1)$ also immediately reduces to the case where the $\mathcal{L}$-translation of the reduction is identically 0.

Finally, assume that $|x_n| \sim |\theta(\tilde{x})|$. If these two functions are of the same sign, then $x_n \sim \theta(\tilde{x})$ and we are done. Otherwise, by (1.6.4), we see that since $(1 - \frac{\theta(\tilde{x})}{x_n})$ is an $\mathcal{L}$-unit when x_n and θ have opposite signs and are equivalent, we can assume the $\mathcal{L}$-translation of the reduction to be identically 0. $\square$

Lemma 1.6.11. *Let $C \subset \mathbb{R}^n$ be an $\mathcal{L}$-cell of basis B and $c : B \to \mathbb{R}$ an $\mathcal{L}$-function. Let $p \in \mathbb{Z}$ be such that the $\mathcal{L}$-mapping*

$$H(x) := (\tilde{x}, c(\tilde{x})|x_n|^{1/p}), \quad x = (\tilde{x}, x_n) \in C,$$

is well-defined on C. If an $\mathcal{L}$-function $\zeta : H(C) \to \mathbb{R}$ is reducible then so is $\xi := \zeta \circ H : C \to \mathbb{R}$.

Proof. By Lemma 1.6.2, we can assume that x_n and c are of constant (nonzero) sign on C (we will assume that they are positive for simplicity). Observe that $H(C)$ is then an $\mathcal{L}$-cell and that the inverse image under H of an $\mathcal{L}$-cell included in $H(C)$ is an $\mathcal{L}$-cell. If ζ is reducible then ξ may be written on every cell $E \subset C$ of a suitable $\mathcal{L}$-cell decomposition compatible with C:

$$\xi(x) = \zeta(H(x)) = b(\tilde{x}) \cdot y(x)^r \cdot U(\tilde{x}, y(x)), \quad y(x) := |c(\tilde{x}) \cdot x_n^{1/p} - \theta(\tilde{x})|,$$

where $U(\tilde{x}, y(x))$ is an $\mathcal{L}$-unit in the variable y on E, $r \in \mathbb{Q}$, and θ and b are $\mathcal{L}$-functions on the basis of E. It suffices to show that y is reducible.

By Lemma 1.6.2, refining the $\mathcal{L}$-cell decomposition if necessary, we may assume that θ has constant sign on every cell included in C. If $\theta \equiv 0$ on a cell then the result is clear on this cell. Moreover, possibly rewriting y as $x_n^{1/p} \cdot |c(\tilde{x}) - \theta(\tilde{x}) \cdot x_n^{-1/p}|$, we see that it suffices to address the case where p is positive (see Remark 1.6.8). As c nowhere vanishes, y can be factorized $c(\tilde{x})|x_n^{1/p} - \frac{\theta(\tilde{x})}{c(\tilde{x})}|$, which means that, up to a change of θ, we may assume that $c \equiv 1$.

Thanks to Lemma 1.6.10, we can suppose that $x_n \sim \theta(\tilde{x})$ on $H(E)$. As we assume $c \equiv 1$, it means that we have on E

$$x_n \sim \theta(\tilde{x})^p. \tag{1.6.5}$$

Write then

$$y = x_n^{1/p} - \theta(\tilde{x}) = \frac{x_n - \theta(\tilde{x})^p}{x_n^{\frac{p-1}{p}} + x_n^{\frac{p-2}{p}}\theta(\tilde{x}) + \cdots + \theta(\tilde{x})^{p-1}}. \tag{1.6.6}$$

Let $D(x)$ denote the denominator of this fraction. As D is a sum of positive terms which are all $\sim$ to $\theta(\tilde{x})^{p-1}$ (by (1.6.5)), we clearly have $D(x) \sim \theta(\tilde{x})^{p-1}$. Hence, the function $W(x) := \theta(\tilde{x})^{1-p}D(x)$ is bounded away from zero and infinity. It is therefore an $\mathcal{L}$-unit, which means in particular that it is reduced. Therefore, by Lemma 1.6.7 (see Remark 1.6.8) and (1.6.6), y is reducible. $\square$

1.6.2 Reduction of $\mathcal{L}$-functions

The first step of the reduction process deals with functions which, roughly speaking, are "analytic in the last variable":

Proposition 1.6.12. *Let $C \subset \mathbb{R}^n$ be an $\mathcal{L}$-cell of basis B and let $\phi_1, \dots, \phi_k$ be $\mathcal{L}$-functions on B. Set*

$$\Phi(x) := (\phi_1(\tilde{x}), \dots, \phi_k(\tilde{x}), x_n), \quad x = (\tilde{x}, x_n) \in C,$$

and let ψ be an analytic function on a neighborhood of $cl(\Phi(C))$. If $\Phi(C)$ is bounded then the function $\xi := \psi \circ \Phi$ admits a nondegenerate reduction.

Proof. **Step 1.** Reduction to the case where $\psi(u_1, , \dots, u_k, z)$ is z-regular on a neighborhood of $cl(\Phi(C))$.

By Proposition 1.3.11, for every $y_0 = (u_0, z_0) \in cl(\Phi(C)) \subset \mathbb{R}^k \times \mathbb{R}$, there is $\varepsilon_{y_0} > 0$ and $d \in \mathbb{N}$ for which ψ has the following form on the cube $\mathbf{C}(y_0, \varepsilon_{y_o}) \subset \mathbb{R}^k \times \mathbb{R}$:

$$\psi(u, z) = \sum_{i=0}^{d} c_i(u)(z - z_0)^i A_i(u, z), \tag{1.6.7}$$

where the c_i's are analytic functions on $\mathbf{C}(u_0, \varepsilon_{y_0})$ and the A_i's are analytic functions on $\mathbf{C}(y_0, \varepsilon_{y_0})$ nowhere zero on this set.

As $cl(\Phi(C))$ is compact, we can extract a finite covering by such cubes. By Remark 1.6.4 and Lemma 1.6.6, there is an $\mathcal{L}$-cell decomposition compatible with the inverse images under Φ of the elements of this finite covering.

Fix a cell $E \subset C$ (to show that ξ is reducible, we may focus on one single cell, by Remark 1.6.4). By construction, $\Phi(E)$ fits in some cube $\mathbf{C}(y_0, \varepsilon_{y_0})$, with $y_0 = (u_0, z_0) \in cl(\Phi(E))$, on which (1.6.7) holds. Define some $\mathcal{L}$-functions on the basis D of E by:

$$g_i(\tilde{x}) := c_i(\phi_1(\tilde{x}) \dots, \phi_k(\tilde{x})), \quad i = 0, \dots, d.$$

By $(\mathcal{H}_{n-1})$ and Lemma 1.6.2, up to a refinement of the $\mathcal{L}$-cell decomposition, we can assume that the family $|g_0|, \dots, |g_d|$ is totally ordered on D, and that if some g_i vanishes on D then it is identically zero on this set.

Let $m \leq d$ be the smallest integer such that $|g_m| = \max_{i \leq d} |g_i|$ on D. If all the g_i's are zero on D, the function ξ is reduced on E, since identically zero. We thus will assume that g_m is nowhere zero.

Define now a function φ by setting for $(t, u, z) \in \mathbb{R}^{d+1} \times \mathbf{C}(y_0, \varepsilon_{y_0})$:

$$\varphi(t, u, z) := \sum_{i=0}^{d} t_i \, (z - z_0)^i \, A_i(u, z).$$

A straightforward computation of partial derivatives shows that this analytic function is z-regular of order at most m at any (t, u, z), with $t \in [0,1]^{d+1}$ satisfying $t_m = 1$ and $(u, z) \in \mathbf{C}(y_0, \varepsilon_{y_0})$ (if ε_{y_0} was chosen small enough). To complete Step 1, we are going to show that it suffices to work with φ instead of ψ.

As g_m nowhere vanishes on D, by (1.6.7), for every $x \in E$ we have $\xi(x) = g_m(\tilde{x}) \cdot \varphi(\Theta(x))$, where Θ is the bounded $\mathcal{L}$-mapping

$$\Theta(x) := (g_0(\tilde{x})/g_m(\tilde{x}), \dots, g_d(\tilde{x})/g_m(\tilde{x}), \Phi(x)).$$

As g_m is an $(n-1)$-variable function, it is enough to reduce $\varphi \circ \Theta$. As $\varphi(u, z)$ is z-regular at any point of $cl(\Theta(E))$, this completes Step 1.

Step 2. Proof in the case where $\psi(u_1, \dots, u_k, z)$ is z-regular near $cl(\Phi(C))$.

Let d be the greatest order of z-regularity of ψ near $cl(\Phi(C))$ and let us argue by induction on d (the case $d = 0$ being trivial).

By Weierstrass Preparation Theorem, given a point of $cl(\Phi(C))$, there is a cube centered at this point such that the function ψ is, up to a unit, a polynomial with analytic coefficients. As $cl(\Phi(C))$ is compact, we can extract a finite covering by such cubes. By Lemma 1.6.6 (and Remark 1.6.4), there is an $\mathcal{L}$-cell decomposition $\mathcal{E}$ of $\mathbb{R}^n$ compatible with C and the respective preimages under Φ of all these cubes. Fix a cell $E \in \mathcal{E}$ included in C. Since we can argue up to a unit and up to a translation, we will assume that $cl(\Phi(E))$ fits in a cube centered at the origin on which ψ coincides with a $(k+1)$-variable polynomial with analytic coefficients:

$$\psi(u, z) = z^d + a_1(u)z^{d-1} + \cdots + a_d(u), \quad (u, z) \in \mathbb{R}^k \times \mathbb{R}.$$

If we make the change of variable $z \mapsto z - \frac{a_1(u)}{d}$, the coefficient of z^{d-1} of this polynomial becomes zero. Consequently, since it is enough to show that the function $\xi(\tilde{x}, x_n + \frac{a_1(\phi(\tilde{x}))}{d})$, where $\phi = (\phi_1, \dots, \phi_k)$, is reducible, we will assume that $a_1 \equiv 0$. For simplicity, set for $\tilde{x}$ in the basis of E

$$b_i(\tilde{x}) := a_i(\phi_1(\tilde{x}), \dots, \phi_k(\tilde{x})), \quad i = 2, \dots, d.$$

By Lemma 1.6.2 and $(\mathcal{H}_{n-1})$, refining the $\mathcal{L}$-cell decomposition if necessary, we can assume the b_i's to have constant sign and the family of $\mathcal{L}$-functions $|b_i|^{\frac{1}{i}}$, $i = 2, \dots, d$, to be totally ordered on E. Let j be such that on E:

$$|b_j|^{1/j} = \max_{2 \le i \le d} |b_i|^{1/i}. \tag{1.6.8}$$

If all the b_i's are identically zero, then ξ is indeed already reduced. Since b_j is of constant sign on E, we will therefore assume below that it nowhere vanishes on E. Up to one more refinement of the $\mathcal{L}$-cell decomposition (see Remark 1.6.3), we may assume that one of the following two cases occurs on E:

Case I: $|x_n| \leq 2|b_j(\tilde{x})|^{\frac{1}{j}}$, for $(\tilde{x}, x_n) \in E$. (1.6.9)

In this case, we first are going to change ξ for a function $\hat{\xi} = \hat{\psi} \circ \hat{\Phi}$, with $\hat{\psi}$ z-regular of order less than d (and $\hat{\Phi}$ as in (1.6.10) below). Let

$$\hat{\xi}(\tilde{x}, x_n) := |b_j(\tilde{x})|^{-d/j} \cdot \xi(\tilde{x}, |b_j(\tilde{x})|^{1/j} \cdot x_n).$$

This function is defined on the cell E' which is the image of E under the mapping $(\tilde{x}, x_n) \mapsto (\tilde{x}, |b_j(\tilde{x})|^{-1/j} \cdot x_n)$. It is clearly enough to show that $\hat{\xi}$ is reducible. Observe that $\hat{\xi}$ is nothing but the composite of the d-variable polynomial function

$$\hat{\psi}(v_2, \dots, v_d, z) := z^d + v_2 z^{d-2} + \dots + v_d$$

with the $\mathcal{L}$-mapping on E':

$$\hat{\Phi}(x) = \left(\frac{b_2(\tilde{x})}{|b_j(\tilde{x})|^{2/j}}, \dots, \frac{b_d(\tilde{x})}{|b_j(\tilde{x})|^{d/j}}, x_n \right). \tag{1.6.10}$$

By (1.6.8) and (1.6.9), the set $\hat{\Phi}(E')$ is bounded ((1.6.9) entails that the coordinate x_n of $x \in E'$, is bounded away from infinity on E'). As $\hat{\psi}(v, z)$ is z-regular of order at most $(d-1)$ at every $(v, z) \in cl(\hat{\Phi}(E'))$ (since $v_j \neq 0$ for all $(v, z) = (v_2, \dots, v_d, z)$ in this set and $\frac{\partial^{d-1}\hat{\psi}}{\partial z^{d-1}}(v, z) = d!\, z$), the result follows by induction on d.

Case II: $|x_n| \geq 2|b_j(\tilde{x})|^{\frac{1}{j}}$, for $(\tilde{x}, x_n) \in E$. (1.6.11)

In this case, ξ can be easily reduced as follows. By (1.6.8) and (1.6.11), $|\frac{b_k(\tilde{x})}{x_n^k}| \leq \frac{1}{2^k}$ on E, for every k. As a matter of fact,

$$|\frac{b_2(\tilde{x})}{x_n^2} + \dots + \frac{b_d(\tilde{x})}{x_n^d}| < \frac{1}{2},$$

so that the equality

$$\xi(\tilde{x}, x_n) = x_n^d \left(1 + \frac{b_2(\tilde{x})}{x_n^2} + \dots + \frac{b_d(\tilde{x})}{x_n^d} \right)$$

reduces ξ on the given cell. $\square$

We now are going to deal with $\mathcal{L}$-functions that are analytic in both x_n and $\frac{c(\tilde{x})}{x_n}$, where c is an $\mathcal{L}$-function (see Proposition 1.6.14 below). The strategy is to split the considered function into two functions, one analytic in x_n and one analytic in $\frac{c(\tilde{x})}{x_n}$, in order to apply Proposition 1.6.12. To this end, the following lemma will be needed.

Lemma 1.6.13. *Let ψ be an analytic function on a neighborhood of $(a, 0, 0)$ in $\mathbb{R}^{k+2}$, $a \in \mathbb{R}^k$. There exist $\varepsilon > 0$ and two analytic functions ψ_1 and ψ_2 on some compact cubes such that*

$$\psi(u, z, \frac{c}{z}) = z\psi_1(u, c, z) + \psi_2(u, c, \frac{c}{z}), \tag{1.6.12}$$

for every (u, c, z) satisfying $(u, z, \frac{c}{z}) \in \mathbf{C}(a, \varepsilon) \times [-\varepsilon, \varepsilon]^2$.

Proof. If $\sum b_{m,i,j}(u-a)^m z^i t^j$ denotes the Taylor expansion at $(a, 0, 0)$ of the function $\psi(u, z, t)$, it suffices to set:

$$\psi_1(u, c, z) := \sum_{\substack{0 \le j < i \\ m \in \mathbb{N}^k}} b_{m,i,j}(u-a)^m c^j z^{i-j-1} \text{ and } \psi_2(u, c, t) := \sum_{\substack{0 \le i \le j \\ m \in \mathbb{N}^k}} b_{m,i,j}(u-a)^m c^i t^{j-i}.$$

□

Proposition 1.6.14. *Let $C \subset \mathbb{R}^n$ be an $\mathcal{L}$-cell of basis B and let $c, \phi_1, \ldots, \phi_k$ be $\mathcal{L}$-functions on B. Set*

$$\Phi(x) := \big(\phi_1(\tilde{x}), \ldots, \phi_k(\tilde{x}), x_n, \frac{c(\tilde{x})}{x_n}\big), \quad x = (\tilde{x}, x_n) \in C,$$

and let ψ be an analytic function on a neighborhood of $cl(\Phi(C))$. If $\Phi(C)$ is bounded then the function $\xi := \psi \circ \Phi$ is reducible.

Proof. **Step 1.** We show that the restriction of ξ to an $\mathcal{L}$-cell C on which $\frac{x_n}{c(\tilde{x})}$ is bounded (assuming that $c(\tilde{x})$ nowhere vanishes on C) is reducible.

On such a cell C, as by assumption $\Phi(C)$ is bounded, we have $|c(\tilde{x})| \sim |x_n|$. Consequently, as x_n is bounded on C, so is c, as well as the $\mathcal{L}$-mapping

$$\Phi'(x) := \big(\phi_1(\tilde{x}), \ldots, \phi_k(\tilde{x}), c(\tilde{x}), \frac{x_n}{c(\tilde{x})}\big), \quad x = (\tilde{x}, x_n) \in C.$$

Since $\frac{x_n}{c(\tilde{x})}$ is bounded away from zero on C, the function

$$\psi'(u_1, \ldots, u_m, z, t) := \psi(u_1, \ldots, u_m, zt, \frac{1}{t})$$

is analytic on a neighborhood of $cl(\Phi'(C))$ ($1/t$ is analytic on the complement of the origin). By Lemma 1.6.2, we may assume that x_n is of constant sign on C. As $\xi = \psi' \circ \Phi'$, thanks to Lemma 1.6.11, if we set $H(\tilde{x}, x_n) := (\tilde{x}, \frac{x_n}{c(\tilde{x})})$, it suffices to show that $\zeta := \psi' \circ \Phi' \circ H^{-1}$ is reducible. Since on $H(C)$:

$$\Phi' \circ H^{-1}(x) = (\phi_1(\tilde{x}), \ldots, \phi_k(\tilde{x}), c(\tilde{x}), x_n),$$

the result follows from Proposition 1.6.12.

Step 2. We show the proposition in its full generality.

For $\tilde{x}$ in B, let $\phi(\tilde{x}) := (\phi_1(\tilde{x}), \dots, \phi_k(\tilde{x}))$. By Lemma 1.6.13, for every $a \in cl(\phi(B))$, there are $\varepsilon > 0$ and two analytic functions ψ_1 and ψ_2 satisfying (1.6.12) on $\mathbf{C}(a,\varepsilon) \times [-\varepsilon,\varepsilon]^2$ (for our function ψ). As $cl(\phi(B))$ is compact, it can be covered by finitely many such cubes $\mathbf{C}(a_i,\varepsilon)$, $i = 1, \dots, l$.

By Lemma 1.6.6 (and Remark 1.6.4), we can find an $\mathcal{L}$-cell decomposition $\mathcal{C}$ compatible with the sets $\phi^{-1}(\mathbf{C}(a_i,\varepsilon))$, $i = 1, \dots, l$. Refining $\mathcal{C}$ if necessary, we can assume it to be compatible with the respective inverse images of $[-\varepsilon,\varepsilon]$ under the functions $c(\tilde{x})$, x_n, and $\frac{c(\tilde{x})}{x_n}$. By Lemma 1.6.2, we also can suppose that $c(\tilde{x})$ and x_n are of constant sign on every cell of $\mathcal{C}$ included in C. Fix $E \in \mathcal{C}$ included in C.

If $|x_n| \geq \varepsilon$ on E, by Proposition 1.6.12, we are done since $t \mapsto \frac{1}{t}$ is analytic on the complement of the origin.

If $|\frac{c(\tilde{x})}{x_n}| \geq \varepsilon$ on E then $|\frac{x_n}{c(\tilde{x})}|$ is bounded and the result directly follows from Step 1.

If $|c(\tilde{x})| \geq \varepsilon$ on E then (since x_n is bounded on C by assumption) $|\frac{x_n}{c(\tilde{x})}|$ is still bounded and the result also follows from Step 1.

We thus can assume that $|c|$, $|x_n|$, and $|\frac{c(\tilde{x})}{x_n}|$ are all smaller than ε on E. For $x = (\tilde{x}, x_n) \in C$, let

$$\Phi_1(\tilde{x}, x_n) := (\phi_1(\tilde{x}), \dots, \phi_k(\tilde{x}), c(\tilde{x}), x_n),$$

as well as

$$\Phi_2(\tilde{x}, x_n) := (\phi_1(\tilde{x}), \dots, \phi_k(\tilde{x}), c(\tilde{x}), \frac{c(\tilde{x})}{x_n}).$$

By construction, the basis of E is comprised in $\phi^{-1}(\mathbf{C}(a_i,\varepsilon))$ for some $i \leq l$, which means that $E \subset \Phi_j^{-1}(\mathbf{C}(a_i,\varepsilon) \times [-\varepsilon,\varepsilon]^2)$, for $j = 1, 2$. As a matter of fact, (1.6.12) holds on $\Phi(E)$.

By Proposition 1.6.12 and Remark 1.6.8, the function $\xi_1(x) := x_n \cdot \psi_1 \circ \Phi_1(x)$ is reducible. We claim that $\xi_2 := \psi_2 \circ \Phi_2$ is reducible as well (note that by (1.6.12) we have $\xi = \xi_1 + \xi_2$). Indeed, let $H(x) := (\tilde{x}, \frac{c(\tilde{x})}{x_n})$, and define a bounded $\mathcal{L}$-mapping on $H(E)$ by:

$$\Phi_2'(\tilde{x}, x_n) := (\phi_1(\tilde{x}), \dots, \phi_k(\tilde{x}), c(\tilde{x}), x_n).$$

By Proposition 1.6.12, $\xi_2' := \psi_2 \circ \Phi_2'$, defined on $H(E)$, is reducible, so that, by Lemma 1.6.11, $\xi_2 = \xi_2' \circ H$ is reducible, as claimed.

We are ready to check that ξ is reducible. By Lemma 1.6.2, up to a refinement of $\mathcal{C}$, we may assume that ξ_1 and ξ_2 are of constant sign on E (recall that so are x_n and $c(\tilde{x})$). By Lemma 1.6.9, refining again the obtained cell decomposition of $\mathbb{R}^n$, we may assume in addition that one of the two following situations occurs:

First Case. $|\xi_1| \leq \frac{1}{2}|\xi_2|$ or $|\xi_2| \leq \frac{1}{2}|\xi_1|$ on E.

For simplicity, we will assume that the first inequality holds. It means that $(1+\frac{\xi_1}{\xi_2})$ is bounded away from zero and infinity (if $\xi_2 \equiv 0$ on E, the result is clear). Since, thanks to Lemma 1.6.7, ξ_1 and ξ_2 can be reduced with the same $\mathcal{L}$-translation on every cell, the function $(1+\frac{\xi_1}{\xi_2})$ induces an $\mathcal{L}$-unit on every cell included in E of some $\mathcal{L}$-cell decomposition. Hence, it suffices to rewrite ξ as $\xi_2 \cdot (1+\frac{\xi_1}{\xi_2})$.

Second case: $|\xi_1| \sim |\xi_2|$ on E.

As it will make no difference, we will assume that ξ_1, ξ_2, $c(\tilde{x})$, and x_n are positive on E (if one of them is zero the result is clear). We first establish the following

Claim. x_n is $\sim$ on E to an $\mathcal{L}$-function $b(\tilde{x})$.

To check this claim, observe that Proposition 1.6.12 actually ensures that both $\psi_1 \circ \Phi_1$ and ξ_2' admit *nondegenerate* reductions. We thus have (recall that $\xi_1 = x_n \psi_1 \circ \Phi_1$)

$$\xi_1(\tilde{x}, x_n) \sim x_n \cdot a_1(\tilde{x})|x_n - \theta_1(\tilde{x})|^r \quad \text{and} \quad \xi_2(\tilde{x}, x_n) \sim a_2(\tilde{x})|\frac{c(\tilde{x})}{x_n} - \theta_2(\tilde{x})|^s,$$

for some $\mathcal{L}$-functions a_1, a_2, θ_1, and θ_2 on the basis E' of E and some *nonnegative* rational numbers r and s. If $x_n \sim \theta_1(\tilde{x})$ or $\frac{c(\tilde{x})}{x_n} \sim \theta_2(\tilde{x})$ on E then the claim clearly holds true. Otherwise, by Lemma 1.6.10, we may assume $\theta_1 = \theta_2 = 0$, so that

$$a_1(\tilde{x})x_n^{r+1} \sim \xi_1(x) \sim \xi_2(x) \sim a_2(\tilde{x})\frac{c(\tilde{x})^s}{x_n^s},$$

which entails that $x_n \sim b(\tilde{x}) := (\frac{a_2(\tilde{x})c(\tilde{x})^s}{a_1(\tilde{x})})^{\frac{1}{s+r+1}}$ (here $(s+r+1)$ is nonzero for s and r are both nonnegative), yielding the claim.

Let now

$$\psi'(u_1, \dots, u_k, w, z, t) := \psi(u_1, \dots, u_k, w, zt),$$

and

$$\Phi'(\tilde{x}, x_n) := (\phi_1(\tilde{x}), \dots, \phi_k(\tilde{x}),\, x_n\,, \frac{c(\tilde{x})}{b(\tilde{x})}, \frac{b(\tilde{x})}{x_n}).$$

Since $b(\tilde{x}) \sim x_n$ and $\Phi(E)$ is bounded, the set $\Phi'(E)$ is bounded as well. By Step 1, as $\frac{b(\tilde{x})}{x_n}$ is bounded below away from zero on the cell E, the function $\psi' \circ \Phi'$ must be reducible. As $\xi = \psi \circ \Phi = \psi' \circ \Phi'$, we are done. $\square$

Proposition 1.6.15. *Let $g_1, \dots, g_m$ be reducible functions on an $\mathcal{L}$-cell $C \subset \mathbb{R}^n$. Set*

$$G(x) := (g_1(x), \dots, g_m(x)),$$

and let f be a function which is analytic on a neighborhood of $cl(G(C))$. If $G(C)$ is bounded then $\xi := f \circ G$ is reducible.

Proof. By Remark 1.6.4, we can assume that all the g_i's are reduced on the cells of one single $\mathcal{L}$-cell decomposition $\mathcal{C}$ compatible with C. Moreover, by Lemma 1.6.7, we can assume that on every cell of $\mathcal{C}$, the g_i's are reduced with the same $\mathcal{L}$-translation θ, which, up to a change $(\tilde{x}, x_n) \mapsto (\tilde{x}, x_n + \theta(\tilde{x}))$ can be assumed to be zero. The function ξ may therefore be written on a given cell $E \in \mathcal{C}$, $E \subset C$, $\psi \circ \Phi$ with ψ restricted analytic function and Φ bounded mapping of type

$$\Phi(x) = (\phi(\tilde{x}), \dots \phi_k(\tilde{x}), b(\tilde{x}) \cdot |x_n|^{1/s}, c(\tilde{x}) \cdot |x_n|^{-1/s}), \quad x = (\tilde{x}, x_n) \in E \subset \mathbb{R}^n,$$

where $b, c, \phi_1, \dots, \phi_k$ are $\mathcal{L}$-functions and $s \in \mathbb{N}$ (since the g_i's are reduced). For such a cell E, by Lemma 1.6.2, we can assume x_n to be of constant sign on every cell, and, by Lemma 1.6.11, we may assume that $s = 1$ and $b \equiv 1$, so that, by Proposition 1.6.14, the function $\xi_{|E}$ must be reducible. □

We are now ready to carry out the induction step of Proposition 1.5.4:

proof of $(\mathcal{H}_n)$. By definition, any $\mathcal{L}$-function may be expressed as a finite sum, product, and composite of restricted analytic functions and power functions. Arguing by induction on the complexity of the expression of ξ, it is enough to show that each of these operations (sum, product, power, composition with a restricted analytic function) preserves reducible functions.

The power of a reduced function is clearly reduced. We have seen that the product of two reduced functions is also reduced (see Remark 1.6.8). By Proposition 1.6.15, composition with a restricted analytic function preserves reducible functions.

It remains to show that the sum of two reducible functions $\xi_1 : C \to \mathbb{R}$ and $\xi_2 : C \to \mathbb{R}$ is reducible, when C is an $\mathcal{L}$-cell of $\mathbb{R}^n$. Indeed, by Remark 1.6.4, there is an $\mathcal{L}$-cell decomposition such that ξ_1 and ξ_2 are reduced on every cell $E \subset C$. By Lemma 1.6.2, up to a refinement, we can assume that ξ_1 and ξ_2 have constant sign (taking values in $\{-1, 0, 1\}$) on every cell. By Lemma 1.6.9, we can also assume that for every cell $E \subset C$ there is a constant $M > 0$ such either $|\xi_1| \le M|\xi_2|$ or $|\xi_2| \le M|\xi_1|$ on E. We will suppose for simplicity that $|\xi_1| \le M|\xi_2|$ on a fixed cell $E \subset C$. Then, writing $\xi = \xi_2 \cdot (1 + \frac{\xi_1}{\xi_2})$, by Remark 1.6.8, we see that it suffices to show that $(1 + \frac{\xi_1}{\xi_2})$ is reducible (if ξ_2 is identically zero on E we are done). But since $G(x) := \frac{\xi_1(x)}{\xi_2(x)}$ is bounded and reducible (again due to Remark 1.6.8), this follows from Proposition 1.6.15 (applied to the one variable function $f(y) := 1 + y$). □

1.7 Existence of cell decompositions

We are now ready to establish Theorem 1.2.3. The cell decomposition that we are going to construct will indeed be an $\mathcal{L}$-cell decomposition. Observe first that thanks to Remark 1.6.4, we can assume that $l = 1$, i.e., it is enough to construct an $\mathcal{L}$-cell decomposition compatible with one single given set $A \in \mathcal{S}_n$.

Reduction to the case where A is globally semi-analytic. Since A is globally subanalytic, there exists a globally semi-analytic subset $Z \subset \mathbb{R}^m$, $m > n$, such that $\pi(Z) = A$, where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is the projection onto the n first coordinates. As the images under π of the cells of a cell decomposition of $\mathbb{R}^m$ constitute a cell decomposition of $\mathbb{R}^n$ (see Remark 1.2.2), it is enough to construct an $\mathcal{L}$-cell decomposition of $\mathbb{R}^m$ which is compatible with the globally semi-analytic set Z.

Proof in the case where A is globally semi-analytic. By definition of globally semi-analytic sets, for every $z \in [-1,1]^n$, there are some analytic functions f_{ij}, g_{ij}, $i = 1,\dots,r, j = 1,\dots,s_i$, on a neighborhood U_z of z such that

$$\mathcal{V}_n(A) \cap U_z = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s_i} \{x \in U_z : g_{ij}(x) > 0 \text{ and } f_{ij}(x) = 0\}. \tag{1.7.1}$$

Let, for each $z \in [-1,1]^n$, V_z be a cube containing z and included in U_z. Since $[-1,1]^n$ is compact, it may be covered by finitely many such cubes. By Remark 1.6.4, this means that it is enough to construct an $\mathcal{L}$-cell decomposition of $\mathbb{R}^n$ compatible with $A \cap W_z$ for every $z \in [-1,1]^n$, where $W_z := \mathcal{V}_n^{-1}(V_z)$.

Fix for this purpose $z \in cl(\mathcal{V}_n(A))$. As V_z can be described by sign conditions on analytic functions and $\mathcal{V}_n$ is an $\mathcal{L}$-mapping, by Lemma 1.6.2 (and Proposition 1.5.4), there is a cell decomposition $\mathcal{D}$ compatible with W_z. Moreover, by (1.7.1), we see that

$$A \cap W_z = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s_i} \{x \in W_z : g_{ij}(\mathcal{V}_n(x)) > 0 \text{ and } f_{ij}(\mathcal{V}_n(x)) = 0\}. \tag{1.7.2}$$

As $\xi_{ij}(x) := f_{ij}(\mathcal{V}_n(x))$ and $\zeta_{ij}(x) := g_{ij}(\mathcal{V}_n(x))$ are $\mathcal{L}$-functions, again thanks to Lemma 1.6.2, there is a refinement $\mathcal{E}$ of $\mathcal{D}$ such that the ξ_{ij}'s and the ζ_{ij}'s have constant sign on every cell included in W_z, which, by (1.7.2), entails that $A \cap W_z$ is a union of cells of this cell decomposition, as required.

Remark 1.7.1. The cell decomposition that we have constructed is indeed an $\mathcal{L}$-cell decomposition.

1.8 The Preparation Theorem and Gabrielov's Complement Theorem

In this section, we gather some consequences of Theorem 1.2.3 and Proposition 1.5.4. The first thing we establish in this section is that globally subanalytic functions are piecewise given by $\mathcal{L}$-functions. This gives a very precise description of globally subanalytic functions and will lead us to the Preparation Theorem.

Proposition 1.8.1. *Let $\xi : E \to \mathbb{R}$ be a globally subanalytic function, $E \in \mathcal{S}_n$. There is an $\mathcal{L}$-cell decomposition $\mathcal{C}$ of $\mathbb{R}^n$ compatible with E, such that for every cell $C \subset E$ of $\mathcal{C}$ the function $\xi_{|C}$ coincides with an $\mathcal{L}$-function.*

Proof. The graph of ξ being a globally subanalytic set, by Theorem 1.2.3 (see Remark 1.7.1), there is an $\mathcal{L}$-cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n+1}$ compatible with Γ_ξ. This $\mathcal{L}$-cell decomposition induces an $\mathcal{L}$-cell decomposition $\mathcal{C}$ of $\mathbb{R}^n$ (see Remark 1.2.2). Let $C \in \mathcal{C}$ with $C \subset E$. There is an $\mathcal{L}$-cell $D \in \mathcal{D}$ included in Γ_ξ which projects onto C. This $\mathcal{L}$-cell cannot be a band since it is a subset of Γ_ξ. It is thus the graph of an $\mathcal{L}$-function $\zeta : C \to \mathbb{R}$ which coincides with $\xi_{|C}$. □

Theorem 1.8.2. *(The Preparation Theorem) Every globally subanalytic function is reducible.*

Proof. It is a consequence of Propositions 1.5.4 and 1.8.1 (see Remark 1.6.4). □

Remark 1.8.3. A reducible function induces analytic functions on the cells of some cell decomposition. Theorem 1.8.2 thus entails that a globally subanalytic function is analytic on the cells of a suitable cell decomposition.

In the case $n = 1$, the Preparation Theorem (Theorem 1.8.2) yields the famous Puiseux Lemma for globally subanalytic functions:

Proposition 1.8.4. *(Puiseux Lemma) Let $f : (0, \eta) \to \mathbb{R}$ be a globally subanalytic function, with η positive real number. There exist $\varepsilon \in (0, \eta)$, $m \in \mathbb{Z}$, and $p \in \mathbb{N}^*$ such that f has a convergent Puiseux expansion on $(0, \varepsilon)$:*

$$f(t) = \sum_{i=m}^{+\infty} a_i t^{\frac{i}{p}}, \quad a_i \in \mathbb{R},\ \forall i \geq m.$$

Proof. By the Preparation Theorem (Theorem 1.8.2), there is a right-hand-side neighborhood of 0 on which f is reduced. By definition, the germs of reduced one-variable functions are germs of Puiseux series. □

The following two propositions may be considered as *Puiseux Lemmas with parameters.*

Definition 1.8.5. Let $A \in \mathcal{S}_n$. A **globally subanalytic partition** of A is a *finite* partition of this set into globally subanalytic sets. A definable partition is **compatible** with a set if this set is union of some elements of the partition.

Proposition 1.8.6. *Let $A \in \mathcal{S}_n$ and let f be a continuous globally subanalytic function on a neighborhood U of $A \times \{0\}$ in $A \times \mathbb{R}_+$. There exist a globally subanalytic partition of A into $\mathscr{C}^\infty$ manifolds and a positive integer p such that for every element C of this partition, $f(x, t^p)$ is analytic on a neighborhood of $C \times \{0\}$ in $C \times \mathbb{R}$.*

Proof. Apply the Preparation Theorem to $f : U \to \mathbb{R}$. This provides a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n+1}$ compatible with U such that $f(x,t)$ is reduced on every cell $D \subset U$, that is to say, we can find $\mathcal{L}$-functions θ and a on $C := \pi(D)$ (where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the canonical projection), a bounded $\mathcal{L}$-mapping W on D of type:

$$W(x,t) := (u(x), v(x)(t-\theta(x))^{\frac{1}{s}}, w(x)(t-\theta(x))^{-\frac{1}{s}})$$

with $u : C \to \mathbb{R}^d$ $\mathcal{L}$-mapping, $v : C \to \mathbb{R}$ and $w : C \to \mathbb{R}$ $\mathcal{L}$-functions, $s \in \mathbb{N}^*$, as well as a function ψ, analytic and nowhere vanishing on a neighborhood of $cl(W(D))$ and such that for some $r \in \mathbb{Q}$:

$$f(x,t) = a(x)(t-\theta(x))^r \psi(W(x,t)). \tag{1.8.1}$$

By Theorem 1.2.3, we may assume that our cell decomposition is compatible with $A \times \{0\}$, and consequently, that the cell decomposition $\pi(\mathcal{D})$ of $\mathbb{R}^n$ (see Remark 1.2.2) is compatible with A.

Let $C \in \pi(\mathcal{D})$ be a cell included in A. Refining the cell decomposition, we may assume that either $\theta \equiv 0$ or that θ never vanishes on C. We may also assume that a is nonzero on C (if $a \equiv 0$ the result is clear).

Since $\mathcal{D}$ is compatible with $A \times \{0\}$, there is a unique cell D of $\mathcal{D}$ which is a band $(0,\xi)$, with $\xi : C \to \mathbb{R}$ positive globally subanalytic function (as the integer p in the statement of the proposition can be chosen even, it is enough to deal with the values of f at the positive values of t). Let $\theta' := \min(|\theta|, \xi)$.

If $\theta(x)$ is nonzero on C then for every $x \in C$ the function $t \mapsto (t-\theta(x))^r$ induces on $[0, \frac{|\theta'(x)|}{2}]$ an analytic function. The function f therefore extends in this case to a function which is analytic with respect to t on a neighborhood of $C \times \{0\}$ and the proposition is clear (in this case).

If $\theta \equiv 0$ then $W(x,t)$ is a Puiseux series in t and $\frac{1}{t}$, analytic in x, and hence, by (1.8.1), so is f. Observe that, as f is locally bounded (it is continuous), we have $r \geq 0$, and since $w(x) \cdot t^{-\frac{1}{s}}$ is bounded (by definition of reduced functions, W is a bounded mapping), we then see that $w \equiv 0$. As a matter of fact, (1.8.1) indeed gives the desired expansion in this case. $\square$

In the case where f does not necessarily extend continuously to $A \times \{0\}$, we have the following result.

Proposition 1.8.7. *Let $A \in \mathcal{S}_n$ and let $f : (0,\zeta) \to \mathbb{R}$ be a globally subanalytic function, with ζ positive globally subanalytic function on A. There is a globally subanalytic partition of A into $\mathscr{C}^\infty$ manifolds such that for every element C of this partition, $f(x,t)$ coincides with a Puiseux series with analytic coefficients:*

$$f(x,t) = \sum_{i \geq k} a_i(x) t^{\frac{i}{p}}, \qquad k \in \mathbb{Z}, \;\; p \in \mathbb{N}^*,$$

on $(0, \xi)$*, where* ξ *is a positive continuous globally subanalytic function on* C *satisfying* $\xi \leq \zeta_{|C}$*.*

Proof. Apply the Preparation Theorem (Theorem 1.8.2) to the function $f : (0, \zeta) \to \mathbb{R}$. This provides a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n+1}$ compatible with $(0, \zeta)$ such that (1.8.1) holds on every cell comprised in $(0, \zeta)$. We may assume that our cell decomposition is compatible with $A \times \{0\}$. Take a cell C of $\pi(\mathcal{D})$ (where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the canonical projection) included in A. As in the proof of the preceding theorem, there is a unique cell D of $\mathcal{D}$ which is a band $(0, \xi)$, with $\xi : C \to \mathbb{R}_+$ globally subanalytic function.

It now follows from (1.8.1) that for any integer $k > |r|$, the function $t^k f(x, t)$ goes to zero as $(x, t) \in D$ tends to a point of $C \times \{0\}$, which entails that it extends continuously at every point of $C \times \{0\}$. It thus suffices to apply Proposition 1.8.6 to this function, for such k. $\square$

Theorem 1.8.8. *(Gabrielov's Complement Theorem) If* $A \in \mathcal{S}_n$ *then* $\mathbb{R}^n \setminus A \in \mathcal{S}_n$*.*

Proof. Let $A \in \mathcal{S}_n$. By Theorem 1.2.3, there is a cell decomposition of $\mathbb{R}^n$ compatible with A. The complement of A being a finite union of cells of this cell decomposition, it is a globally subanalytic set, in virtue of Property 1.1.8 (2). $\square$

We now give three finiteness results.

Corollary 1.8.9. *Globally subanalytic sets have only finitely many connected components. They are globally subanalytic.*

Proof. Cells of $\mathbb{R}^n$ are connected. As, by Theorem 1.2.3, every set $A \in \mathcal{S}_n$ is the union of finitely many cells, it has at most finitely many connected components, which are unions of cells. $\square$

It is natural to regard a set $A \in \mathcal{S}_{m+n}$ as a family of subsets of $\mathbb{R}^n$ parametrized by $\mathbb{R}^m$. Let us make it more precise.

Definition 1.8.10. Let $A \in \mathcal{S}_{m+n}$. We define for t in $\mathbb{R}^m$, the **fiber** of A at t as:

$$A_t := \{x \in \mathbb{R}^n : (t, x) \in A\}.$$

We thus get a family $(A_t)_{t \in \mathbb{R}^m}$ of globally subanalytic subsets of $\mathbb{R}^n$. Any family constructed in this way is said to be a **globally subanalytic family of sets**.

Note. A globally subanalytic family of sets $(A_t)_{t \in \mathbb{R}^m}$ is not only a collection of globally subanalytic subsets of $\mathbb{R}^n$. We demand that the set

$$A = \bigcup_{t \in \mathbb{R}^m} \{t\} \times A_t$$

be itself globally subanalytic.

Remark 1.8.11. Let $A \in \mathcal{S}_{m+n}$ and let $\mathcal{C}$ be a cell decomposition of $\mathbb{R}^{m+n}$ compatible with A. For $t \in \mathbb{R}^m$, let then $\mathcal{C}_t := \{C_t : C \in \mathcal{C}\}$. It follows from the definition of cell decompositions that for every $t \in \mathbb{R}^m$, $\mathcal{C}_t$ is a cell decomposition of $\mathbb{R}^n$ compatible with A_t.

We now have the following parameterized version of Corollary 1.8.9:

Corollary 1.8.12. *(Uniform finiteness) Let $A \in \mathcal{S}_{m+n}$. The number of connected components of A_t is bounded independently of $t \in \mathbb{R}^m$.*

Proof. The same proof as for Corollary 1.8.9 applies (see Remark 1.8.11). □

We introduce in a similar way the globally subanalytic families of functions and mappings.

Definition 1.8.13. A **globally subanalytic family of mappings** is a family of mappings $f_t : A_t \to B_t$, $t \in \mathbb{R}^m$, with $A \in \mathcal{S}_{m+n}$ and $B \in \mathcal{S}_{m+k}$, such that the mapping $f : A \to B$, $(t, x) \mapsto (t, f_t(x))$ is globally subanalytic.

In the case $B_t = \mathbb{R}$, for all $t \in \mathbb{R}^m$, we call such a family a **globally subanalytic family of functions**. We shall sometimes (abusively) regard a function $f : A \to \mathbb{R}$, $A \in \mathcal{S}_{m+n}$, as a family of functions $f_t : A_t \to \mathbb{R}$, $t \in \mathbb{R}^m$, setting $f_t(x) := f(t, x)$.

Observe that it follows from the definitions that $\Gamma_{f_t} = (\Gamma_f)_t$. Here is an important property of globally subanalytic families of mappings:

Corollary 1.8.14. *Let $f_t : A_t \to B_t, t \in \mathbb{R}^m$, be a globally subanalytic family of mappings, with $A \in \mathcal{S}_{m+n}$, $B \in \mathcal{S}_{m+k}$. The number of connected components of $f_t^{-1}(b)$ is bounded by a constant independent of (t, b) in B.*

Proof. Apply Corollary 1.8.12 to the family $(f_t^{-1}(b))_{(t,b)\in B}$ (by Property 1.1.8 (4) it is a globally subanalytic family). □

Historical notes. The first deep insight into real semi-analytic geometry was achieved by S. Łojasiewicz [Loj59, Loj64a, Loj64b] (see also [Den-Sta07] for a similar content). Subanalytic sets were introduced by A. Gabrielov [Gab68] (rewritten in [Gab96]) who showed his Complement Theorem. The description of subanalytic sets in terms of convergent series with negative rational powers that we provide in sections 1.4-1.8 is due to several people and it is not easy to quote all the references. The first major contribution seems to be H. Hironaka's resolution of singularities [Hir73], which lead him to establish the rectilinearization and uniformization theorems (see also [Bie-Mil88, Bie-Mil90]), closely related to the Preparation Theorem. As well-known, easier proofs of resolution of singularities appeared later [Bie-Mil97, Wło05, Kol07]. J. Denef and L. Van den Dries [Den-vdD88] established a quantifier elimination result, showing existence of what we call $\mathcal{L}$-cell decompositions (Theorem 1.2.3, see Remark 1.7.1), as well as the fact that globally subanalytic functions are piecewise given by $\mathcal{L}$-functions (Proposition 1.8.1). A few years later, A. Parusiński, relying on rectilinearization procedures and Hironaka's local flattening, proved the Preparation Theorem (Theorem 1.8.2) [Par94a, Par94b], that also yields quantifier elimination. The proofs of Proposition 1.5.4 and Theorems 1.2.3 and 1.8.2 that are presented here nevertheless follow very closely the slightly more recent proof of the Preparation Theorem given in the article of J.-M. Lion and J.-P. Rolin [Lio-Rol97], which is inspired from the proof of the classical Puiseux Lemma and Denef and van den Dries' article [Den-vdD88], from which Proposition 1.3.11 is directly taken. Puiseux Lemma with parameters (Proposition 1.8.7) is due to W. Pawłucki [Paw84].

Chapter 2

Basic results of subanalytic geometry

We describe some basic properties of globally subanalytic sets which are consequences of the results of the previous chapter. We start with the very useful quantifier elimination principle (Theorem 2.1.4), which is a well-known logic principle that provides a convenient way to check that a set is globally subanalytic. We then establish the famous Łojasiewicz's inequalities as well as "*definable choice*" (Proposition 2.2.1) and Curve Selection Lemma (Lemma 2.2.3), which will be of service many times in the next chapters. We then shift our interest to the description of the geometric properties of globally subanalytic sets, showing existence of globally subanalytic tubular neighborhoods for globally subanalytic manifolds (sections 2.3 and 2.4), and establishing the famous Tamm's theorem, asserting that the regular locus of a globally subanalytic set is globally subanalytic (Theorem 2.5.4). We also show that globally subanalytic sets and mappings can be stratified (with regularity conditions, see section 2.6), which will be of service in the next two chapters.

Given $r \geq 0$ and $x \in \mathbb{R}^n$, $\mathbf{B}(x,r)$ (resp. $\overline{\mathbf{B}}(x,r)$) will stand for the open (resp. closed) ball of radius r centered at x and $\mathbf{S}(x,r)$ for the corresponding sphere. Balls and spheres will be taken with respect to the Euclidean norm $|.|$. The unit sphere of $\mathbb{R}^n$ centered at the origin is denoted $\mathbf{S}^{n-1}$ for simplicity.

We define the **topological boundary** of A, by setting $\delta A := cl(A) \setminus int(A)$ as well as the **frontier** of A by setting $fr(A) := cl(A) \setminus A$.

We denote by $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$ (for all n, we will make it more precise when it is not obvious from the context in which $\mathbb{R}^n$ lies e_i).

We denote by d_xF the derivative of a differentiable map F and by $\partial_x f$ the gradient of a differentiable function f.

2.1 Quantifier elimination

We give a brief introduction to quantifier elimination. This model-theoretic principle will provide an efficient tool to check that a set is globally subanalytic. These facts are not proper to the theory of globally subanalytic sets and play a central role in the theory of o-minimal structures [Cos00, vdD98], as well as in even larger frameworks. To motivate our purpose, we start with a proposition.

Proposition 2.1.1. *If $A \in \mathcal{S}_n$ then $cl(A)$ and $int(A)$ also belong to $\mathcal{S}_n$.*

Proof. Observe first that the closure of A may be defined as:

$$\{x \in \mathbb{R}^n : \forall \varepsilon > 0, \exists y \in A, |x-y|^2 < \varepsilon\}. \tag{2.1.1}$$

This set coincides with the set:

$$\mathbb{R}^n \setminus \mu\left(\mathbb{R}^n \times (0,+\infty) \setminus \pi(B)\right),$$

where

$$B = \{(x,\varepsilon,y) \in \mathbb{R}^n \times \mathbb{R} \times A : |x-y|^2 < \varepsilon\},$$

and where $\mu : \mathbb{R}^{n+1} \to \mathbb{R}^n$ (resp. $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$) is the projection omitting the last coordinate (resp. n last coordinates). It follows from Property 1.1.8 (4) that B is globally subanalytic, so that Property 1.1.8 (1) and Gabrielov's Complement Theorem establish that $cl(A)$ and $int(A)$ are globally subanalytic. □

The above proposition shows how stability under projections is useful to establish that a set is globally subanalytic. It also emphasizes that it can be tedious to prove that a set is globally subanalytic by describing it in terms of projections of globally semi-analytic sets. The following basic logic principle makes it possible to get rid of the technical difficulties.

Definition 2.1.2. We define the **$\mathcal{S}$-formulas** inductively as follows.

(i) If $A \in \mathcal{S}_n$ then the formula $\Phi(x)$:="$x \in A$" is an $\mathcal{S}$-formula.

(ii) If $\Phi(x)$, where $x = (x_1, \dots, x_n)$, is an $\mathcal{S}$-formula then "*not* $\Phi(x)$" is an $\mathcal{S}$-formula.

(iii) If $\Phi(x)$ and $\Psi(x)$ are $\mathcal{S}$-formulas, where $x = (x_1, \dots, x_n)$, then "Φ *and* Ψ" and "Φ *or* Ψ" are also $\mathcal{S}$-formulas.

(iv) If $\Phi(x,y)$ is an $\mathcal{S}$-formula, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_p)$, then "$\exists y \in \mathbb{R}^p,\ \Phi(x,y)$" and "$\forall y \in \mathbb{R}^p,\ \Phi(x,y)$" are $\mathcal{S}$-formulas.

Roughly speaking, the $\mathcal{S}$-formulas are first order mathematical sentences involving globally subanalytic sets. The point (i) defines the most elementary formulas and the other axioms explain how to build new sentences from these sentences. The minimal number of steps needed to generate a formula is called **the complexity** of the formula. The above definition of $\mathcal{S}$-formulas is thus by induction on the complexity of the sentence.

Remarks 2.1.3. ⋄ By (i), if f is globally subanalytic then the sentences $f(x) > 0$ and $f(x) = 0$ are equivalent to $\mathcal{S}$-formulas (see Property 1.1.8 (4)). We will thus regard them as $\mathcal{S}$-formulas.

⋄ It is important to note that the variable y in (iv) has to range over the whole of $\mathbb{R}^p$, i.e., we cannot write "$\exists y \in \mathbb{N}^p$". Thanks to (i), we can nevertheless write "$\exists y \in A$", if $A \in \mathcal{S}_p$.

⋄ We restrict ourselves to what is called by logicians *first order formulas*, in the sense that the quantified variables cannot be functions or sets: they have to stand for real numbers. The sentences starting like "$\exists$ a globally subanalytic function..." or "$\exists$ a globally subanalytic set..." are *not* $\mathcal{S}$-formulas.

⋄ The formulas depend on finitely many variables $x = (x_1, \dots, x_n)$. These are called **the free variables**. The free variables of $\Phi(x)$ are the variables which are not quantified in the assertion $\Phi(x)$. The value of the assertion (true or false) of course depends on the chosen value for $x \in \mathbb{R}^n$.

Theorem 2.1.4. *If $\Phi(x)$ is an $\mathcal{S}$-formula, $x = (x_1, \dots, x_n)$, then the set*

$$E_\Phi := \{x \in \mathbb{R}^n : \Phi(x) \text{ holds true } \}$$

belongs to $\mathcal{S}_n$.

Proof. We prove it by induction on the complexity of the formula. If $\Phi(x)$ is the formula "$x \in A$", for some $A \in \mathcal{S}_n$, then $E_\Phi = A$ is a globally subanalytic set. We thus have to show that conditions $(ii - iv)$ of Definition 2.1.2 also produce globally subanalytic sets.

Indeed, if $\Phi(x)$ is an $\mathcal{S}$-formula then $\Phi' :=$"*not* Φ" defines the complement of E_Φ in $\mathbb{R}^n$, which is a globally subanalytic set, by Theorem 1.8.8 (and induction on the complexity). Thus, (ii) provides assertions which only give rise to globally subanalytic sets. Similarly, since $\mathcal{S}_n$ is stable under finite union and intersection (see Property 1.1.8 (2)), (iii) only gives rise to globally subanalytic sets.

For (iv), we will proceed as in the proof of Proposition 2.1.1. Let $\Phi(x, y)$ be an $\mathcal{S}$-formula, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_p)$, and let $\Psi(x)$ be the formula "$\exists y \in \mathbb{R}^p$, $\Phi(x, y)$". We have:

$$E_\Psi = \pi(E_\Phi),$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the canonical projection. By Property 1.1.8 (1), the set E_Ψ is globally subanalytic since so is E_Φ, by induction on the complexity. Finally, as the formula "$\forall y \in \mathbb{R}^p,\ \Phi(x, y)$" amounts to "*not* $(\exists y \in \mathbb{R}^p,$ *not* $\Phi(x, y))$", it defines a globally subanalytic set as well. □

By way of conclusion, let us give the following useful facts. These are striking examples of how the above theorem is convenient to establish that a set is globally subanalytic.

Proposition 2.1.5. *(1) If $A \in \mathcal{S}_{m+n}$ then the set $B := \{t \in \mathbb{R}^m : A_t$ is closed $\}$ belongs to $\mathcal{S}_m$.*

(2) If $A \in \mathcal{S}_n$ then the function $\mathbb{R}^n \ni x \mapsto d(x, A) := \inf\{|x - y| : y \in A\}$ is globally subanalytic.

(3) If $f : U \to \mathbb{R}^p$ is globally subanalytic, $U \subset \mathbb{R}^n$ open, and $k \in \mathbb{N}$ then the set of points of U at which f is $\mathscr{C}^k$ is globally subanalytic.

Proof. For (1), the set B could be described by a formula similar as in (2.1.1). It is left to the reader to write the corresponding formulas in either of the other cases. □

It is however not easy to prove that the set of points at which a globally subanalytic function is $\mathscr{C}^\infty$ is globally subanalytic (see (3) of the above proposition). We shall nevertheless establish that this is true (Corollary 2.5.6).

From now on, in order to shorten the statements, globally subanalytic sets (resp. mappings, families, partitions) will generally be called **definable** sets (resp. mappings, families, partitions). This terminology is usual to logicians or "o-minimal geometers". It is motivated by the fact that Theorem 2.1.4 yields that all what can be defined by an $\mathcal{S}$-formula is globally subanalytic.

2.2 Curve selection Lemma and Łojasiewicz's inequalities

Curve Selection Lemma comes down from the following useful result.

Proposition 2.2.1. *(Definable choice) Let $A \in \mathcal{S}_{m+n}$ and let $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be the canonical projection. There exists a definable mapping $f : B \to \mathbb{R}^n$, where $B := \pi(A)$, such that $\Gamma_f \subset A$.*

Proof. We prove it by induction on n. Assume first that $n = 1$. Taking a cell decomposition adapted to A if necessary, it is enough to address the case where A is a cell. If A is a graph over a cell of $\mathbb{R}^m$, the result is trivial. If A is a band (ζ, ζ')

(see Definition 1.2.1), with ζ and ζ' not infinite, then take $f := \frac{\zeta+\zeta'}{2}$. If for instance $\zeta' = +\infty$ and $\zeta > -\infty$ take $f = \zeta + 1$. If ζ and ζ' are both infinite, we set $f \equiv 0$. This completes the proof in the case $n = 1$.

Assume the result true for $(n-1)$. Let $A \in \mathcal{S}_{m+n}$ and let $\mu : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n-1}$ be the projection omitting the last coordinate. Applying the induction hypothesis to $A' := \mu(A)$, we get a definable mapping $g : B \to \mathbb{R}^{n-1}$ with $\Gamma_g \subset A'$. Applying the case $n = 1$ to A, we get a definable mapping $h : A' \to \mathbb{R}$ satisfying $\Gamma_h \subset A$. It suffices to set $f(x) := h(x, g(x))$. □

Remark 2.2.2. Combining the latter proposition with Theorem 2.1.4 provides the following version of definable choice. Let $\Phi(x, y)$ be an $\mathcal{S}$-formula, with $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ free variables, and assume that there are $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, such that for every $x \in A$ there is $y \in B$ for which $\Phi(x, y)$ holds. Then there is a definable mapping $f : A \to B$ such that $\Phi(x, f(x))$ holds for all $x \in A$.

Lemma 2.2.3. *(Curve Selection Lemma) Let $A \in \mathcal{S}_n$ and let $x_0 \in cl(A)$. There exists an analytic arc $\gamma : [0, \varepsilon) \to \mathbb{R}^n$ such that $\gamma(0) = x_0$ and $\gamma((0, \varepsilon)) \subset A$.*

Proof. Applying Proposition 2.2.1 (with $m := 1$) to the definable set

$$A' := \{(r, x) \in (0, +\infty) \times A : |x - x_0| < r\},$$

we get a definable map $\gamma : (0, +\infty) \to \mathbb{R}^n$ satisfying $\gamma(r) \in A \cap \mathbf{B}(x_0, r)$, for all $r > 0$. This yields that there is a definable arc in A tending to x_0. Existence of an analytic parametrization then follows from Puiseux Lemma (Proposition 1.8.4). □

Proposition 2.2.4. *Any two points of a connected definable set X may be joint by a continuous definable arc in X.*

Proof. Let $X \in \mathcal{S}_n$ be a connected set. By Theorem 1.2.3, X is a finite union of cells $C_1, \dots, C_k$ of $\mathbb{R}^n$. Let $x \in X$ and let E be the set of points of X that can be joint to x by means of a continuous definable curve in X. As any two points of a cell may be joint by a continuous definable arc, E must be the union of some of the C_i's, which entails that it is a definable set. By Curve Selection Lemma (Lemma 2.2.3), E is a closed subset of X. Moreover, again due to Curve Selection Lemma, $X \setminus E$ is closed in X as well. As X is connected and E is nonempty (it contains x), $X = E$. □

This leads us to the famous Łojasiewicz's inequalities.

Theorem 2.2.5. *(Łojasiewicz's inequality) Let f and g be two definable functions on a definable set A. Assume that f is bounded and that*

$$\lim_{t \to 0} f(\gamma(t)) = 0, \tag{2.2.1}$$

for every definable arc $\gamma : (0, \varepsilon) \to A$ *such that* $\lim_{t\to 0} g(\gamma(t)) = 0$. *Then there exist* $N \in \mathbb{N}^*$ *and* $C \in \mathbb{R}$ *such that for any* $x \in A$*:*

$$|f(x)|^N \le C|g(x)|.$$

Proof. Possibly replacing f and g with their respective absolute values we may assume that these functions are nonnegative. For $t \in g(A)$, let

$$\varphi(t) := \sup_{x \in g^{-1}(t)} f(x).$$

By Theorem 2.1.4, the function φ is definable. Thanks to assumption (2.2.1) (and definable choice, see Remark 2.2.2), we see that φ tends to zero as t goes to zero. By Puiseux Lemma (Proposition 1.8.4), there is a real number a and a positive rational number α such that for t positive small enough

$$\varphi(t) = at^\alpha + \dots,$$

which implies that there must be a constant C such that $\varphi(t) \le Ct^\alpha$, for t positive small enough. Therefore, for $g(x)$ positive small enough we can write:

$$f(x) \le \varphi(g(x)) \le Cg(x)^\alpha,$$

which means that, if we choose an integer $N \ge \frac{1}{\alpha}$ then the desired inequality holds on $g^{-1}([0, \varepsilon))$, $\varepsilon > 0$ small enough.

On $g^{-1}([\varepsilon, +\infty))$, as g is bounded below away from zero and f is bounded, the desired inequality will continue to hold if C is chosen large enough. $\square$

Corollary 2.2.6. *If two continuous definable functions* f *and* g *on a compact definable set* A *satisfy*

$$g^{-1}(0) \subset f^{-1}(0), \tag{2.2.2}$$

then there exist $N \in \mathbb{N}^*$ *and* $C \in \mathbb{R}$ *such that for any* $x \in A$*:*

$$|f(x)|^N \le C|g(x)|.$$

Proof. Since f is continuous on a compact set, it is a bounded function. Moreover, if $\gamma : (0, \varepsilon) \to A$ is a definable arc such that $\lim_{t\to 0} g(\gamma(t)) = 0$ then, setting $a = \lim_{t\to 0} \gamma(t)$ (which exists since A is compact and γ is definable, see Proposition 1.8.4), we get $g(a) = 0$, which, via hypothesis (2.2.2), entails that $f(a) = 0$. Hence, $\lim_{t\to 0} f(\gamma(t)) = f(a) = 0$, and the conclusion follows from Theorem 2.2.5. $\square$

Corollary 2.2.7. *Let* $\xi : A \to \mathbb{R}$ *be a definable function. If* ξ *is bounded on every bounded subset of* A *then there are* $C > 0$ *and* $N \in \mathbb{N}$ *such that for all* $x \in A$*:*

$$|\xi(x)| \le C(1 + |x|)^N.$$

Proof. Apply Theorem 2.2.5 to $f(x) := \frac{1}{1+|x|}$ and $g(x) := \frac{1}{1+|\xi(x)|}$. $\square$

We are going to derive from Theorem 2.2.5 another famous estimate which is also called Łojasiewicz's inequality (Corollary 2.2.9). This one will compare a $\mathscr{C}^1$ definable function with its gradient.

Lemma 2.2.8. *Let $f : M \to \mathbb{R}$ be a $\mathscr{C}^1$ definable function, with M definable $\mathscr{C}^1$ submanifold of $\mathbb{R}^n$, and let $a \in cl(M)$. If f extends continuously at a then there is a positive constant C such that for all $x \in M$ sufficiently close to a*

$$|f(x) - f(a)| \leq C|x-a| \cdot |\partial_x f|.$$

Proof. Without loss of generality, we can assume that $a = 0_{\mathbb{R}^n}$, $f(a) = 0$. Thanks to Curve Selection Lemma, it suffices to show the desired inequality along a definable (non constant) arc $\gamma : (0, \varepsilon) \to M$, with $\gamma(s)$ tending to the origin as s goes to zero. By Puiseux Lemma (Proposition 1.8.4), the arcs $\gamma(s)$ and $\partial_{\gamma(s)} f$ admit Puiseux expansions, say

$$\gamma(s) = bs^k + \dots \quad \text{ and } \quad \partial_{\gamma(s)} f = cs^l + \dots, \quad \text{ with } k, l \in \mathbb{Q}$$

(with $c = 0$ if and only if $\partial_{\gamma(s)} f \equiv 0$, and $b \neq 0$). It means that $\gamma'(s) = kbs^{k-1} + \dots$, so that:

$$|f(\gamma(r))| = |\int_0^r \partial_{\gamma(s)} f \cdot \gamma'(s) ds| \lesssim |c|\, r^{k+l} \lesssim |\partial_{\gamma(r)} f| \cdot |\gamma(r)|,$$

yielding the desired estimate along γ. $\square$

Corollary 2.2.9. *Let $f : M \to \mathbb{R}$ be a $\mathscr{C}^1$ definable function with M definable $\mathscr{C}^1$ submanifold of $\mathbb{R}^n$ and let $x_0 \in cl(M)$. If f extends continuously at x_0 then there are $\rho \in (0,1) \cap \mathbb{Q}$ and $C > 0$ such that for all $x \in M$ sufficiently close to x_0*

$$|f(x) - f(x_0)|^\rho \leq C|\partial_x f|.$$

Proof. Lemma 2.2.8 yields that there is $\eta > 0$ such that $\partial_x f = 0$ entails $f(x) = f(x_0)$, for all $x \in \mathbf{B}(x_0, \eta) \cap M$. We thus will check the desired inequality on the set

$$V := \{x \in M : \partial_x f \neq 0 \text{ and } |x - x_0| < \eta\}.$$

Define a function g on this set by setting $g(x) := \frac{|f(x) - f(x_0)|}{|\partial_x f|}$, and observe that, due to Lemma 2.2.8, this function must be bounded in the vicinity of x_0. In order to apply Theorem 2.2.5, we first show that $g(\gamma(t))$ tends to zero for every definable arc $\gamma : (0, \varepsilon) \to V$ such that $f(\gamma(t))$ tends to $f(x_0)$.

Such an arc γ being bounded, it must have an endpoint $a \in cl(V)$ (as $t \to 0$). Moreover, since $\gamma(t)$ and $f(\gamma(t))$ are Puiseux arcs, we have for $t > 0$ small

$$|f(\gamma(t)) - f(x_0)| \leq C|\gamma(t) - a|^\alpha,$$

for some positive rational number α and some constant C. The arc γ is thus either constant (in which case the needed fact is clear) or included in the manifold

$$M' := \{x \in V : |f(x) - f(x_0)| < 2C|x - a|^\alpha\}.$$

Clearly, $f(a) := f(x_0)$ extends $f_{|M'}$ continuously at a. Applying Lemma 2.2.8 to $f_{|M'}$ yields $|g(x)| \lesssim |x - a|$ for $x \in M'$ near a, which shows that $\lim_{t \to 0} g(\gamma(t)) = 0$.

Hence, by Theorem 2.2.5, there are $C > 0$ and $N \in \mathbb{N}^*$ such that $|g(x)|^N \leq C|f(x) - f(x_0)|$ for x in V close to x_0, which implies that for such x:

$$|f(x) - f(x_0)|^{1-\frac{1}{N}} \leq C'|\partial_x f|,$$

for some constant C'. □

Remark 2.2.10. When f fails to extend continuously at x_0, it is possible to give an inequality which involves the so-called asymptotic critical values of f [Kur98, aVa19].

2.3 Closure and dimension

We define the **dimension** of $A \in \mathcal{S}_n$ as

$$\dim A = \max\{\dim C : C \in \mathcal{C}, C \subset A\},$$

where $\mathcal{C}$ is a cell decomposition of $\mathbb{R}^n$ compatible with A and $\dim C$ denotes the dimension of C as a manifold (cells are analytic manifolds by definition). By convention, the dimension of the empty set is -1.

Proposition 2.3.1. *(1) $\dim A$ is independent of the chosen cell decomposition.*

(2) If $F : A \to B$ is definable and if $E \subset A$ is also definable then $\dim F(E) \leq \dim E$.

(3) If $A \in \mathcal{S}_n$ is nowhere dense then $\dim A < n$.

Proof. (1) comes from the fact that cell decompositions have common refinements and (3) is obvious from the definitions. Since F is smooth on the cells of a suitable cell decomposition, (2) is also clear. □

Lemma 2.3.2. *Given $A \in \mathcal{S}_{m+n}$, there is a dense definable subset B of $\mathbb{R}^m$ such that $cl(A)_t = cl(A_t)$ for any $t \in B$.*

Proof. Since the set

$$E := \{t \in \mathbb{R}^m : cl(A_t) \neq cl(A)_t\}$$

may be described with an $\mathcal{S}$-formula, by Theorem 2.1.4, it is definable. We have to show that $\dim E < m$. Note that, for any $t \in \mathbb{R}^m$ we have $cl(A_t) \subset cl(A)_t$, since $cl(A)_t$ is closed and contains A_t.

Suppose that E is of dimension m and take a cell decomposition of $\mathbb{R}^m$ compatible with E. Let C be a cell of dimension m included in E. For each $t \in C$, there are $r_t > 0$ and $a_t \in cl(A)_t$ such that $\mathbf{B}(a_t, r_t)$ does not meet A_t. By Definable Choice (Proposition 2.2.1, see Remark 2.2.2), we can assume that r_t and a_t are definable functions of t and, by Theorem 1.8.2, up to a refinement of the cell decomposition, we can assume that they are continuous on C. Let

$$U := \{(t,x) \in C \times \mathbb{R}^n : x \in \mathbf{B}(a_t, r_t)\}.$$

Since C is open in $\mathbb{R}^m$ and because a_t and r_t are continuous with respect to t, the set U is an open subset of $\mathbb{R}^{m+n}$. As U intersects $cl(A)$ (at the points (t, a_t)) and is disjoint from A, this is a contradiction, which yields that $\dim E < m$. $\square$

Given $A \in \mathcal{S}_{m+n}$ and $B \in \mathcal{S}_m$, we define the **restriction of A to B** as:

$$A_B := A \cap (B \times \mathbb{R}^n). \tag{2.3.1}$$

Lemma 2.3.3. *Given $A \in \mathcal{S}_{m+n}$, there is a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathcal{P}$ we have for any $t \in B$:*

$$cl(A_B)_t = cl(A_t).$$

Proof. We prove the lemma by induction on m. The result being clear for $m = 0$, assume it to be true for $(m-1)$, $m \geq 1$. Let B be the set provided by Lemma 2.3.2 (applied to A) and take a cell decomposition $\mathcal{C}$ of $\mathbb{R}^m$ compatible with B. It is enough to establish the lemma for the sets A_C, $C \in \mathcal{C}$. In fact, if $C \subset B$, this comes from Lemma 2.3.2. Otherwise, as B is definable and dense in $\mathbb{R}^m$, $\dim C < m$. In this case, up to a definable homeomorphism, we may assume that $C \subset \mathbb{R}^{m'} \times \{0_{\mathbb{R}^{m-m'}}\}$, where $m' = \dim C < m$, and the result follows from the induction hypothesis. $\square$

Proposition 2.3.4. *For any $A \in \mathcal{S}_n$, we have* $\dim fr(A) < \dim A$.

Proof. Let $k := \dim A$ and assume that $m := \dim fr(A)$ is not smaller than k. Take a cell decomposition compatible with $fr(A)$ and let D be a cell of dimension m included in $fr(A)$. Up to a homeomorphism, we may assume that D is open in $\mathbb{R}^m \times \{0_{\mathbb{R}^{n-m}}\}$. Taking a suitable cell decomposition if necessary, we can assume that A is a cell, which means that $\dim A_t = k - m \leq 0$ for every $(t,0) \in D$. Hence, A_t is finite or empty and consequently must be closed in $\mathbb{R}^{n-m}$. But, by Lemma 2.3.2, we know that for almost every $(t,0) \in D$, $cl(A_t)$ contains D_t which is disjoint from A_t. This is a contradiction. $\square$

Remark 2.3.5. As a matter of fact, $\delta A = cl(A) \setminus int(A)$ has empty interior in $\mathbb{R}^n$ for all $A \in \mathcal{S}_n$.

Proposition 2.3.6. *Let $A \in \mathcal{S}_{m+n}$ and let $f : A \to \mathbb{R}$ be a definable function. If f_t is continuous for every $t \in \mathbb{R}^m$ then there is a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathcal{P}$ the function f is continuous on A_B.*

Proof. Possibly replacing f with $\frac{f}{f^2+1}$, we can assume that f is bounded. By Lemma 2.3.3 (applied to Γ_f), there is a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathcal{P}$ we have for all $t \in B$

$$cl(\Gamma_{f_t}) = cl(\Gamma_f \cap (B \times \mathbb{R}^{n+1}))_t. \tag{2.3.2}$$

Since f_t is continuous for all t, the set Γ_{f_t} is closed in $A_t \times \mathbb{R}$ which means (by (2.3.2)) that $(\Gamma_f)_B$ is closed in $B \times \mathbb{R}^{n+1}$, if $B \in \mathcal{P}$. As f is bounded, this implies that it induces a continuous function on A_B. □

2.4 Orthogonal retraction onto a manifold

Given $Y \in \mathcal{S}_n$, let us define a function $\rho_Y : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho_Y(x) := d(x, Y)^2 = \inf_{y \in Y} |x - y|^2.$$

Proposition 2.4.1. *Let Y be a $\mathscr{C}^p$ definable submanifold of $\mathbb{R}^n$, $p \geq 2$ (possibly infinite). There are a definable open neighborhood U of Y and a definable $\mathscr{C}^{p-1}$ retraction $\pi_Y : U \to Y$ such that for all $x \in U$ we have*

$$\rho_Y(x) = |x - \pi_Y(x)|^2.$$

Moreover, for $x \in U$ we have $\partial_x \rho_Y = 2(x - \pi_Y(x))$, which is always orthogonal to $T_{\pi_Y(x)}Y$. The triplet (U, π_Y, ρ_Y) will be called **a tubular neighborhood** *of Y.*

Proof. For $(x, y) \in \mathbb{R}^n \times Y$ let $\mu(x, y) := |x - y|^2$. If $x \in \mathbb{R}^n$ is a point such that $d(x, Y)$ is smaller than $d(x, fr(Y))$ then there must be a point z in Y such that $\rho_Y(x) = \mu(x, z)$. As z realizes the minimum of $\mu_x : Y \to \mathbb{R}$, we must have $\partial_z \mu_x = 0$. Moreover, in the case where $x \in Y$, a simple computation shows that the differential of the mapping $Y \ni y \mapsto \partial_y \mu_x \in T_y Y$ has rank $k := \dim Y$ at $x = z$.

By the implicit function theorem, we deduce that every point x_0 in Y has a neighborhood W_{x_0} in $\mathbb{R}^n$ and a $\mathscr{C}^{p-1}$ mapping $\pi_{x_0} : W_{x_0} \to Y$ for which $\partial_y \mu_x = 0$ amounts to $y = \pi_{x_0}(x)$. Clearly, by construction, if x is sufficiently close to x_0 then $\pi_{x_0}(x)$ is the unique point for which $\rho_Y(x) = |x - \pi_{x_0}(x)|^2$. The map-germs π_{x_0} thus clearly glue together into a definable retraction on a neighborhood of Y. The last sentence comes down from an easy computation of derivative. □

Proposition 2.4.2. *Let $Y \in \mathcal{S}_n$ be locally closed. The function ρ_Y is $\mathscr{C}^\infty$ on a neighborhood of Y if and only if Y is a $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$.*

Proof. The if part follows from the preceding proposition. We prove the only if part by way of contradiction, assuming that ρ_Y is smooth and Y is singular. Observe that all the points of Y are critical points of ρ_Y, since this function is nonnegative and vanishes identically on Y. Let y be a singular point of Y and let Z be a $\mathscr{C}^\infty$ manifold of minimal dimension containing a neighborhood of y in Y.

As Z is nonsingular, it cannot coincide with Y near y, which means that there is a sequence z_m in $Z \setminus Y$ converging to y. For each m large, let $y_m \in Y$ be such that $\rho_Y(z_m) = |z_m - y_m|^2$, and set $u := \lim \frac{z_m - y_m}{|z_m - y_m|} \in T_y Z$ (extracting a subsequence if necessary, we may assume that this limit exists), as well as, for $x \in Z$, $\lambda(x) := d_x\rho_Y(u)$. Clearly, $\rho_Y(x) = |x - y_m|^2$, for all x on the line segment joining z_m and y_m. A straightforward computation of derivative in the direction u thus shows that $d_y\lambda(u) = 2$, which implies that $\lambda(x)$ is a smooth submersion on Z near y. Therefore, its zero locus is a submanifold of Z which contains Y (since all the points of Y are critical points of ρ_Y) and of lower dimension than Z, in contradiction with the minimality assumption on the dimension of Z. □

2.5 The regular locus of a definable set

Definition 2.5.1. Given $X \in \mathcal{S}_n$, we denote by X_{reg} the set of points at which X is an analytic manifold (of dimension $\dim X$ or smaller), and by X_{sing} its complement in X. We call X_{reg} the **regular locus of** X and X_{sing} the **singular locus of** X.

Given a definable mapping $f : X \to \mathbb{R}^k$, let $reg(f)$ be the set constituted by the points of X_{reg} at which f is analytic and let $sing(f)$ be its complement in X.

We shall establish that X_{reg} and $reg(f)$ are definable (Theorem 2.5.4 and Corollary 2.5.6).

Proposition 2.5.2. *Let $U \in \mathcal{S}_n$ be an open set and let $f : U \to \mathbb{R}$ be a definable function. There exists an integer k such that the following assertion holds true for every $x \in U$:*

$(*)$ *If f is $\mathscr{C}^k$ on a neighborhood of x then f is analytic on a neighborhood of x.*

Proof. By Theorem 2.1.4, the set U' of points at which f is continuous is definable. Possibly replacing U with U', we thus may assume that f is continuous.

We shall show by downward induction on d that for every $d \leq n$, there is a definable subset $E_d \subset U$ with $\dim E_d < d$ such that the property $(*)$ holds for the restriction of f to $U \setminus cl(E_d)$. The result clearly follows from the case $d = 0$.

By Theorem 1.8.2 (see Remark 1.8.3), there is a cell decomposition of $\mathbb{R}^n$ compatible with U such that f is analytic on every cell of dimension n included in U. Hence, for $d = n$, the result is clear.

Choose $d < n$, assume the result to be true for $(d+1)$, and take a cell decomposition $\mathcal{E}$ compatible with U and E_{d+1}, where $E_{d+1} \subset U$ is provided by the induction hypothesis. Let X be a cell of dimension d included in E_{d+1} (if $\dim E_{d+1} < d$, we are done).

For $x \in U$, let $k_f(x)$ be the greatest integer k such that f is $\mathscr{C}^k$ on a neighborhood of x, with $k_f(x) = \infty$ if f is $\mathscr{C}^\infty$ around x. We are going to show that there is a definable subset $F \subset X$ of dimension $< d$ such that k_f takes only finitely many values (in $\mathbb{N} \cup \{\infty\}$) on $X \setminus F$. As we will also prove that if k_f is infinite at $x \in X \setminus F$ then f is analytic on a neighborhood of x, this will complete the induction step.

Up to an analytic diffeomorphism, we may assume that $X \subset \mathbb{R}^d \times \{0_{\mathbb{R}^{n-d}}\}$ (we will sometimes regard X as a subset of $\mathbb{R}^d$). Apply Proposition 1.8.6 to the definable function

$$\tilde{f} : X \times \mathbf{S}^{n-d-1} \times [0, \varepsilon] \to \mathbb{R}, \quad \tilde{f}(x, u, r) := f(x, ru),$$

and let $\mathcal{C}$ be a cell decomposition compatible with the partition of $X \times \mathbf{S}^{n-d-1}$ provided by this proposition. For every $W \in \mathcal{C}$ included in $X \times \mathbf{S}^{n-d-1}$, $\tilde{f}(x, u, r^p)$ extends to an analytic function on a neighborhood of $W \times \{0\}$ in $W \times \mathbb{R}$, for suitable p. We may assume that p is the same for all $W \in \mathcal{C}$ (taking the product of all the corresponding values of p). Let $\mathcal{D} := \pi(\mathcal{C})$, where $\pi : \mathbb{R}^n \to \mathbb{R}^d$ stands for the projection onto the d first coordinates (see Remark 1.2.2).

Fix a d-dimensional cell $C \in \mathcal{D}$ which is included in X. We are going to prove that k_f takes only finitely many values on $C \times \{0_{\mathbb{R}^{n-d}}\}$ (the desired subset F may thus be defined as the union of the cells of dimension $< d$ contained in X).

Let $W_1, \dots, W_l$ be the family of all the cells of $\mathcal{C}$ that are included in $C \times \mathbf{S}^{n-d-1}$. For every j, there exist some definable functions $(a_i)_{i \in \mathbb{N}}$, analytic on W_j, such that for $(x, u) \in W_j \subset C \times \mathbf{S}^{n-d-1}$ and $r > 0$ small enough we have the convergent expansion:

$$\tilde{f}(x, u, r) = \sum_{i \in \mathbb{N}} a_i(x, u)\, r^{\frac{i}{p}}. \tag{2.5.1}$$

As the W_j's cover $C \times \mathbf{S}^{n-d-1}$, we shall regard each a_i as a function defined on $C \times \mathbf{S}^{n-d-1}$, gluing together (discontinuously) the respective $a_i : W_j \to \mathbb{R}$ obtained on each W_j. We distinguish three cases:

First case. a_{i,x_0} is not identically zero for some i not divisible by p and some $x_0 \in C$.

There is $u \in \mathbf{S}^{n-d-1}$ such that $a_{i,x_0}(u) \neq 0$. Let ν be such that $(x_0, u) \in W_\nu$. Observe now that, as a_i is analytic on W_ν, it is nonzero on an open dense subset of W_ν (since W_ν is connected). As a matter of fact, by (2.5.1), there is $j \in \mathbb{N}$ such that $\frac{\partial^j \tilde{f}}{\partial r^j}(x, u, r)$ is unbounded as r goes to zero for almost all $(x, u) \in W_\nu$. Consequently, f is not $\mathscr{C}^j$ at x, for any $x \in C = \pi(W_\nu)$, which means that $k_f(x) \leq j$ for all $x \in C$.

Second case. There exist i divisible by p and $x_0 \in C$ such that a_{i,x_0} is neither identically zero nor the restriction of a homogeneous polynomial of degree $\frac{i}{p}$.

We claim that in this case f fails to be $\mathscr{C}^{\frac{i}{p}}$ at every point of C, i.e., $k_f(x) < \frac{i}{p}$ for all $x \in C$. We proceed by way of contradiction: if f were a $\mathscr{C}^{\frac{i}{p}}$ function on a neighborhood of some point of C then $u \mapsto \frac{d^j f(x,ru)}{dr^j}|_{r=0}$ would be either identically zero or a homogeneous polynomial of degree j, for all $j \leq \frac{i}{p}$ and x close to this point. By (2.5.1), it means that for such x, $a_{i,x}$ would be either 0 or the restriction of a homogeneous polynomial of degree $\frac{i}{p}$. As a_i is analytic on the W_j's which are connected, this would imply that a_{i,x_0} coincides with this polynomial, leading to a contradiction. Hence, $k_f(x) < \frac{i}{p}$ for every $x \in C$.

Third case. Negation of the two above cases: for all $x \in C$ we assume that for all i not divisible by p, $a_{i,x} \equiv 0$, and that for each i divisible by p, $a_{i,x}$ is either zero or a homogeneous polynomial of homogeneous degree $\frac{i}{p}$.

In this case, by (2.5.1)

$$f(x,y) = \tilde{f}(x, \frac{y}{|y|}, |y|) = \sum_{l \in \mathbb{N}} a_{lp}(x, \frac{y}{|y|})|y|^l = \sum_{l \in \mathbb{N}} a_{lp}(x,y),$$

where each $a_{lp,x}$ is a homogeneous polynomial of degree l for all $x \in C$. Using some estimates for germs of homogeneous polynomials [Bor-Sic71, Den-Sta07], it is then not difficult to show that the series $\sum_{l \in \mathbb{N}} a_{lp}(x,y)$ converges locally uniformly and therefore defines an analytic function. It means that f is analytic on a neighborhood of C in $\mathbb{R}^n$, and $k_f(x) \equiv \infty$ on C. $\square$

Remark 2.5.3. If $f : U \to \mathbb{R}$ is a definable function with $U \in \mathcal{S}_{m+n}$ open and if, for each $t \in \mathbb{R}^m$, k_t is the integer satisfying the property $(*)$ of Proposition 2.5.2 (applied to f_t) then one can see (examining the proof of Proposition 2.5.2) that k_t may be bounded away from infinity independently of t.

For $k \in \mathbb{N}^*$, let X^k_{reg} denote the set of points of X at which X is a $\mathscr{C}^k$ manifold.

Theorem 2.5.4. *If $X \in \mathcal{S}_n$ then X_{reg} is definable and dense in X. Indeed, $X^k_{reg} = X_{reg}$ for all k sufficiently large.*

Proof. It is easy to derive from existence of cell decompositions that X_{reg} is dense (cells are analytic manifolds by definition). Let us show that it is definable. The set of points at which a set is locally closed being definable, we can assume X to be locally closed. By Proposition 2.4.2, X is a $\mathscr{C}^\infty$ manifold at x if and only if ρ_X is $\mathscr{C}^\infty$ in the vicinity of x. Let k be an integer for which ρ_X fulfills the property $(*)$ of Proposition 2.5.2. The set of points at which ρ_X is $\mathscr{C}^k$ is definable (see Proposition 2.1.5 (3)) and, in virtue of the property $(*)$, coincides with X_{reg}. $\square$

Remark 2.5.5. It is worthy of notice that the same argument could be used to prove the following parametrized version of the above theorem: if $X \in \mathcal{S}_{m+n}$ then the family $((X_t)_{reg})_{t \in \mathbb{R}^m}$ is a definable family of sets (see Remark 2.5.3).

Corollary 2.5.6. *If $f : X \to \mathbb{R}$ is a definable function then $reg(f)$ is dense in X and definable. Consequently, $sing(f)$ is definable and has lower dimension than X.*

Proof. Let g be the restriction of f to the set of points of X_{reg} at which f is $\mathscr{C}^1$ (this set is dense in X_{reg}, see Remark 1.8.3). Clearly $reg(f) = \pi((\Gamma_g)_{reg})$, where $\pi : X \times \mathbb{R} \to X$ is the canonical projection. The result thus follows from Theorem 2.5.4. □

2.6 Stratifications

Stratifications satisfying regularity conditions constitute very useful tools to perform differential calculus on singular sets. We introduce some of the famous regularity conditions for stratifications, such as Whitney's (b) or Verdier's (w) condition, and show how to construct stratifications satisfying regularity conditions.

Definition 2.6.1. A **stratification of** a definable set X is a finite partition Σ of it into definable $\mathscr{C}^\infty$ submanifolds of $\mathbb{R}^n$, called **strata**. We then say that (X, Σ) is a **stratified set**. A stratification is **compatible** with a set if this set is the union of some strata. A **refinement of a stratification** Σ of X is a stratification of X compatible with every stratum of Σ.

The definition that we have chosen is the most general one. Some authors require in addition that the strata are connected or that the frontier condition holds (see Definition 2.6.15). Connectedness of the strata can always be obtained by refining the stratification, and the issue of the frontier condition will be discussed later on (see Remark 2.6.17).

2.6.1 Whitney and Kuo-Verdier conditions.

The notion of stratification is however generally too weak to perform differential geometry on singular sets, and it is most of the time needed to consider stratifications satisfying extra regularity conditions that describe the way the different pieces glue together. The most famous ones are Whitney's conditions.

We denote by $\mathbb{G}_k^n$ the Grassmannian manifold of k-dimensional linear vector subspaces of $\mathbb{R}^n$. The **angle** between two given vector subspaces E and F of $\mathbb{R}^n$ will be estimated as:

$$\angle(E, F) := \sup_{u \in E, |u|=1} d(u, F).$$

The angle between a vector of $\mathbb{R}^n$ and a vector subspace of $\mathbb{R}^n$ is defined as the angle between the line generated by this vector and this vector subspace.

Definition 2.6.2. Let X and Y be a couple of disjoint submanifolds of $\mathbb{R}^n$ and let $z \in Y \cap cl(X)$. We say that (X, Y) satisfies **Whitney's** (b) **condition at** $z \in Y \cap cl(X)$ if for any sequences $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$, of points of X and Y respectively, converging to z and such that

$$\tau := \lim T_{x_k} X \in \mathbb{G}_p^n \quad \text{and} \quad v := \lim \frac{x_k - y_k}{|x_k - y_k|} \in \mathbf{S}^{n-1}$$

exist (where $p = \dim X$), we have $v \in \tau$.

We will say that (X, Y) satisfies **Whitney's** (a) **condition** at z if $\angle(T_z Y, T_x X)$ tends to zero as $x \in X$ tends to z.

We will say that (X, Y) satisfies the (w) **condition at** z (of Kuo-Verdier) if there exists a constant C such that for $x \in X$ and $y \in Y$ in a neighborhood of z:

$$\angle(T_y Y, T_x X) \le C|y - x|. \tag{2.6.1}$$

Finally, let π be a $\mathscr{C}^\infty$ local retraction onto Y near z (it can easily be checked that the condition below is independent of π). We will say that (X, Y) satisfies the (r) **condition** (of Kuo) at z if:

$$\lim_{x \to z,\, x \in X} \frac{\angle(T_{\pi(x)} Y, T_x X) \cdot |x - z|}{|x - \pi(x)|} = 0. \tag{2.6.2}$$

A stratification Σ is (w)**-regular** (resp. (b)**,** (a)**,** (r)**-regular**) if every couple (X, Y) of strata of Σ satisfies the (w) condition (resp. (b), (a), (r) condition) at every point of Y.

Proposition 2.6.3. *For couples of definable manifolds,* $(w) \Rightarrow (r) \Rightarrow (b) \Rightarrow (a)$.

Proof. $(w) \Rightarrow (r)$ is a straightforward consequence of the definitions. To show that $(r) \Rightarrow (b)$, fix a couple of strata (X, Y) satisfying (r). Up to a coordinate system of Y, we may assume that Y is an open subset of $\mathbb{R}^l \times \{0\}$, $l := \dim Y$, and work near $z = 0$. Denote by π the orthogonal projection onto $\mathbb{R}^l \times \{0\}$.

Thanks to Curve Selection Lemma, we simply have to check Whitney's (b) condition along two definable arcs $x : (0, \varepsilon) \to X$ and $y : (0, \varepsilon) \to Y$ tending to the origin at 0. Since x is a Puiseux arc, we may suppose it to be parametrized by its distance to the origin, i.e., $|x(t)| = t$. We have:

$$|x(t) - \pi(x(t))| \sim t|x'(t) - \pi(x'(t))|. \tag{2.6.3}$$

Denote by P_t the orthogonal projection onto the orthogonal complement of $T_{x(t)} X$. As $P_t(x'(t)) \equiv 0$, we may write using (2.6.3):

$$P_t\left(\frac{x'(t) - \pi(x'(t))}{|x'(t) - \pi(x'(t))|}\right) \sim \frac{t|P_t(\pi(x'(t)))|}{|x(t) - \pi(x(t))|},$$

which tends to zero in virtue of the (r) condition. This shows that the angle between $(x' - \pi(x'))(t)$ and $T_{x(t)}X$ tends to zero as t goes to zero. Since $(x - \pi(x))$ has a Puiseux parametrization we have:

$$\lim_{t \to 0} \frac{x(t) - \pi(x(t))}{|x(t) - \pi(x(t))|} = \lim_{t \to 0} \frac{x'(t) - \pi(x'(t))}{|x'(t) - \pi(x'(t))|}, \tag{2.6.4}$$

which means that the angle between $(x(t) - \pi(x(t)))$ and $T_{x(t)}X$ must tend to zero as well. Moreover, as $(\pi(x(t)) - y(t))$ belongs to $Y = T_{\pi(x(t))}Y$ for every t, the condition (r) entails that the angle between $(\pi(x(t)) - y(t))$ and $T_{x(t)}X$ tends to zero. Together with the preceding sentence, this establishes that the angle between $(x(t) - y(t))$ and $T_{x(t)}X$ tends to zero, yielding Whitney's (b) condition for (X, Y).

It remains to show that (b) implies (a). Again, we will assume $Y = \mathbb{R}^l \times \{0\}$. Suppose that (b) holds and (a) fails at $z \in Y$, i.e., assume that there is a sequence $(x_k)_{k\in\mathbb{N}}$ in X tending to z such that $\tau := \lim T_{x_k}X$ (exists and) does not contain $T_zY = \mathbb{R}^l \times \{0\}$. Let $u \in T_zY \setminus \tau$ and set $y_k := z + \frac{1}{k}u$. Extracting a subsequence if necessary, we may assume that $(x_k)_{k\in\mathbb{N}}$ tends to z as fast as we wish. In particular, we may assume that $\frac{y_k - x_k}{|y_k - x_k|}$ tends to u. By Whitney's (b) condition, this implies that $u \in \tau$, a contradiction. $\square$

Remark 2.6.4. The proof of $(r) \Rightarrow (b)$ relies on Curve Selection Lemma. This implication, unlike the other implications of the above proposition, is no longer true if X and Y are not definable, as shown by the example $X := \{(x, y) : y = \sin \frac{1}{x}, x > 0\}$ and $Y := \{0\} \times (-1, 1)$, which satisfies (w) and not (b).

2.6.2 Existence of regular stratifications of definable sets

To prove existence of Whitney or Kuo-Verdier regular stratifications, we are actually going to establish that we can always construct a stratification satisfying any sufficiently reasonable given regularity condition, which leads us to the following definitions.

Definition 2.6.5. A **regularity condition on stratifications** is the data for every stratified set (A, Σ) of a mathematical formula $\mathbf{G}(x, A, \Sigma)$, where $x \in A$. We say that Σ **satisfies** $(\mathbf{G})$ if $\mathbf{G}(x, A, \Sigma)$ holds true for every $x \in A$.

Such a condition is said to be **local**, if, given any A and Σ, the value of $\mathbf{G}(x, A, \Sigma)$ (true or false) just depends on the germ of (A, Σ) at x (for each $x \in A$), i.e., if for every definable open neighborhood U of x in A, $\mathbf{G}(x, A, \Sigma)$ is equivalent to $\mathbf{G}(x, U, \Sigma \cap U)$, where $\Sigma \cap U$ is the stratification induced by Σ on U.

A regularity condition is **stratifying** if, given A and Σ, as well as $S \in \Sigma$, there is a definable open dense subset W of S such that $\mathbf{G}(x, A, \Sigma)$ holds true $\forall x \in W$.

Whitney and Kuo-Verdier conditions provide examples of local regularity conditions on stratifications. We shall show that they are stratifying. We first show that we can always construct a stratification satisfying a local stratifying condition.

Proposition 2.6.6. *Given a stratifying local condition and a definable set A, there is a stratification of A satisfying this condition. We can require this stratification to be compatible with finitely many given definable subsets of A.*

Proof. Let A be a definable set, $X_1, \dots, X_l$ some definable subsets of A, and let $(\mathbf{G})$ be a local stratifying condition. We construct by decreasing induction on $0 \leq k \leq d+1$, $d := \dim A$, a closed definable subset E_k of A of dimension less than k, such that there is a stratification Σ^k of $A \setminus E_k$ which is compatible with $X_1 \setminus E_k, \dots, X_l \setminus E_k$ and satisfies the condition $(\mathbf{G})$. We can start with $E_{d+1} := A$.

Take then $k \leq d$ for which Σ^{k+1} and E_{k+1} have already been constructed, and let $\mathcal{C}$ be a cell decomposition compatible with $E_{k+1,reg}$ and the X_i's. Denote by $S_1, \dots, S_m$ the cells of $\mathcal{C}$ of dimension k that are included in E_{k+1}. Observe that if we set $Z := E_{k+1} \setminus \cup_{i=1}^m S_i$ then $\dim Z < k$ and

$$\Sigma'^{k+1} := \Sigma^{k+1} \cup \{S_1, \dots, S_m\}$$

is a stratification of $A \setminus Z$. It remains to take off the points of the S_i's at which the condition $(\mathbf{G})$ fails. Since this condition is stratifying, there is for each $i \leq m$ an open dense definable subset $W_i \subset S_i$ such that $\mathbf{G}(x, A \setminus Z, \Sigma'^{k+1})$ holds at every $x \in W_i$. Clearly, if we set $E_k := E_{k+1} \setminus \cup_{i=1}^m W_i$ then $\Sigma^k := \Sigma^{k+1} \cup \{W_1, \dots, W_m\}$ is a stratification of $A \setminus E_k$. Moreover, since the condition $(\mathbf{G})$ is local, and because the germs at the points of $\cup_{i=1}^m W_i$ of the respective strata of Σ'^{k+1} and Σ^k coincide, we see that Σ^k fulfills the condition $(\mathbf{G})$. □

Remark 2.6.7.

(1) The conjunction of several local stratifying conditions being local and stratifying, we have proved that we can construct a stratification satisfying several local stratifying conditions simultaneously.

(2) The fact that we can assume the constructed stratification to be compatible with finitely many definable subsets ensures that we can refine any given stratification into a stratification satisfying a given local stratifying condition.

(3) The algorithm of construction that we gave ensures that, when we wish to refine a stratification Σ that already satisfies $(\mathbf{G})$ on a definable open set U, we do not need to modify Σ on U.

Proposition 2.6.8. *The conditions (a), (b), (r), and (w) are stratifying.*

Proof. Thanks to Proposition 2.6.3, it suffices to show the result for the condition (w). Let Σ be a stratification and let X and Y be two strata. Up to a coordinate system of Y, we may identify Y with a neighborhood of 0 in $\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$ (where $k = \dim Y$). Below, we will sometimes abusively consider Y as a subset of $\mathbb{R}^k$.

Note that, by Theorem 2.1.4, the set of points at which the (w) condition fails is definable. We shall proceed by way of contradiction: assume that there is a definable open subset U of Y such that (w) fails for (X, Y) at every point of U. It means that for any $y \in U$ and any $r > 0$ there is $\omega(y, r) \in X \cap \mathbf{B}(y, r)$ such that:

$$d(\omega(y,r), Y) \leq r \cdot \angle(Y, T_{\omega(y,r)}X). \tag{2.6.5}$$

By Definable Choice (Proposition 2.2.1), $\omega(y, r)$ can be chosen definable. Let

$$W := \omega(U \times (0, \varepsilon)) \subset X.$$

The mapping ω can be seen as a parametrization of the set W. We are going to find another parametrization of this set of type $(y, \xi(y, t))$ in order to make our computations easier. By Lemma 2.3.2, there is a definable open dense subset U' of U such that for any $y \in U' \subset \mathbb{R}^k$, we have $cl(W_y) = cl(W)_y$. Hence, for all $y \in U'$, $0_{\mathbb{R}^{n-k}}$ belongs to $cl(W_y)$. As a matter of fact, for all $y \in U'$ there exists $\xi(y, t)$ in $\mathbf{S}(0_{\mathbb{R}^{n-k}}, t) \cap W_y$, for each $t > 0$ small enough. By Definable Choice again we may assume that ξ is definable.

Write $\xi = (\xi_1, \ldots, \xi_{n-k})$ and apply Proposition 1.8.7 to each of the ξ_i's. This provides a definable partition $\mathcal{P}$ of U' into $\mathscr{C}^\infty$ manifolds such that for every $C \in \mathcal{P}$, $\xi(y, t)$ coincides with a Puiseux series with analytic coefficients on a neighborhood of $C \times \{0\}$ in $C \times \mathbb{R}_+$. Fix $C \in \mathcal{P}$ of dimension k and define a mapping by setting for $y \in C$ and $t > 0$ small

$$g(y, t) := (y, \xi(y, t)).$$

We are ready to show that for every $y \in C$ there is $\varepsilon > 0$ such that for $x \in g(C \times (0, \varepsilon)) \subset W$ close to y we have

$$\angle(Y, T_x X) \lesssim d(x, Y), \tag{2.6.6}$$

which will contradict (2.6.5) and establish the proposition. To prove this, notice that since $g(y, t) \in W$ for all $t > 0$ small and $y \in C$, the vector

$$v_i(y, t) := d_{(y,t)} g(e_i)$$

belongs to $T_{g(y,t)}X$, for any $t > 0$ small, $y \in C$, and $i \leq k$. Furthermore, if the Puiseux expansion of $\xi(y, t)$ starts like $\xi(y, t) = a(y) \cdot t + \ldots$, with $a : C \to \mathbf{S}^{n-k-1}$ analytic function, then for all $i \leq k$:

$$\frac{\partial \xi}{\partial y_i}(y, t) = \frac{\partial a}{\partial y_i}(y) \cdot t + \ldots.$$

As a matter of fact, in the vicinity of any given point of $C \times \{0\}$ we have for $i \le k$:

$$|v_i(y,t) - e_i| \lesssim t = |\xi(y,t)| = d(g(y,t), Y),$$

showing (2.6.6) (since $v_i(y,t) \in T_{g(y,t)}X$). □

2.6.3 Stratifications of mappings

Definition 2.6.9. Let $F : A \to B$ be a definable mapping. We say that $F : (A, \Sigma) \to (B, \Sigma')$ is a **stratified mapping** if Σ and Σ' are stratifications of A and B respectively such that for every stratum $S \in \Sigma$, $F(S)$ is included in an element of $S' \in \Sigma'$ and the restricted mapping $F_{|S} : S \to S'$ is a $\mathscr{C}^\infty$ submersion.

Proposition 2.6.10. *Given a local stratifying regularity condition for stratifications* $(\mathbf{G})$ *and a definable mapping* $F : A \to B$*, we can find two stratifications* Σ *of* A *and* Σ' *of* B *satisfying* $(\mathbf{G})$ *and making of* $F : (A, \Sigma) \to (B, \Sigma')$ *a stratified mapping. Moreover, these stratifications may be required to be compatible with finitely many given definable subsets of* A *and* B*.*

Proof. Let $A_1, \dots, A_k$ (resp. $B_1, \dots, B_\kappa$) be definable subsets of A (resp. B). We are going to prove by decreasing induction on $p \in \{0, \dots, l+1\}$, $l = \dim B$, that for every such p there is a definable closed subset $E_p \subset B$ of dimension less than p such that we can find some stratifications Σ^p and Σ'^p of $F^{-1}(B \setminus E_p)$ and $B \setminus E_p$ respectively, satisfying $(\mathbf{G})$ and such that F maps submersively the strata of Σ_p into the strata of Σ'_p. These stratifications will respectively be compatible with the sets $A_i \setminus F^{-1}(E_p)$ and $B_j \setminus E_p$, $i \le k$, $j \le \kappa$.

In the case $p = l+1$, we set $E_{l+1} := B$ and we are done. Assume thus that E_p, Σ^p, and Σ'^p have been constructed for some $p \le l+1$. If $\dim E_p < p-1$, we are done. Otherwise, let N denote the set of points of E_p at which this set is a smooth manifold of dimension $(p-1)$, and let $\mathscr{S}$ be a stratification of $F^{-1}(N)$ such that F is smooth on each stratum (see Remark 1.8.3).

As $(\mathbf{G})$ is local and stratifying, we can refine $\mathscr{S}$ into a stratification (still denoted $\mathscr{S}$) such that $\mathscr{S} \cup \Sigma^p$ satisfies this condition (see Remark 2.6.7 (3)) and is compatible with the $F^{-1}(N) \cap A_i$'s. Given a point $x \in F^{-1}(N)$, we will denote by S^x the element of $\mathscr{S}$ that contains x. Let D be the closure of the set of points $x \in F^{-1}(N)$ at which the rank of the derivative of the restricted mapping $F_{|S^x} : S^x \to N$ is less than $(p-1)$.

Let now $\mathscr{S}'$ be any stratification of N compatible with $F(D), B_1 \cap N, \dots, B_\kappa \cap N$, as well as with the sets $N \cap F(S)$, $S \in \mathscr{S}$. Let $S_1, \dots, S_m$ be the strata of $\mathscr{S}'$ that have dimension $(p-1)$, and set $\Sigma''^p := \Sigma'^p \cup \{S_1, \dots, S_m\}$. As $(\mathbf{G})$ is stratifying, there is for every i an open definable dense subset Y_i of S_i on which Σ''^p satisfies this condition.

We set $\Sigma'^{p-1} := \Sigma'^p \cup \{Y_1, \dots, Y_m\}$ as well as $\Sigma^{p-1} := \Sigma^p \cup \{F^{-1}(Y_i) \cap S, S \in \mathscr{S}, i \le m\}$. Because $Y_1, \dots, Y_m$ are open in N, which is open in E_p, the set $E_{p-1} := E_p \setminus \cup_{i=1}^m Y_i$ is closed in B. Moreover, by construction, the local condition $(\mathbf{G})$ must hold for the stratifications Σ^{p-1} and Σ'^{p-1}.

To check that F induces a stratified mapping on $A \setminus F^{-1}(E_{p-1})$, take $Z \in \Sigma^{p-1} \setminus \Sigma^p$ (the desired property already holds for the other strata by induction). As $Z \notin \Sigma^p$, $Z = F^{-1}(Y_i) \cap S$, for some $S \in \mathscr{S}$ and some $i \le m$. Since (by Sard's Theorem) $\dim F(D) < p-1$, Y_i must be disjoint from $F(D)$ (recall that $Y_i \subset S_i$, $\dim S_i = p-1$, and $S_i \in \mathscr{S}'$ which is compatible with $F(D)$). Hence, the rank of the restricted mapping $F_{|Z} : Z \to Y_i$ is $(p-1)$, which means that F maps submersively the strata of Σ_p into the strata of Σ'_p. □

Horizontally $\mathscr{C}^1$ maps. We now are going to construct nicer stratifications for definable continuous mappings with bounded first derivative (recall that definable mappings are smooth almost everywhere). In this case, we can have a continuity property of the derivative when passing from one stratum to one another. This will be useful to study the pull-back of a differential form by a definable bi-Lipschitz mapping in Chapter 4 (not necessarily smooth).

Definition 2.6.11. A stratified mapping $h : (X, \Sigma) \to (Y, \Sigma')$ is said to be **horizontally $\mathscr{C}^1$** if for any sequence $(x_l)_{l \in \mathbb{N}}$ in a stratum S of Σ tending to some point x in a stratum $S' \in \Sigma$ and for any sequence $u_l \in T_{x_l} S$ tending to a vector u in $T_x S'$, we have
$$\lim d_{x_l} h_{|S}(u_l) = d_x h_{|S'}(u).$$

If h is horizontally $\mathscr{C}^1$ then the norm of $d_x h_{|S}$ (as a linear map) is bounded away from infinity on every subset of S which is relatively compact in X for every $S \in \Sigma$. The proposition below can be considered as a converse of this observation.

Proposition 2.6.12. *Let $h : X \to Y$ be a definable continuous mapping. If $|d_x h|$ is bounded on every subset of $reg(h)$ which is relatively compact in X then there exist stratifications Σ of X and Σ' of Y making of $h : (X, \Sigma) \to (Y, \Sigma')$ a horizontally $\mathscr{C}^1$ stratified mapping. Moreover, we may require these stratifications to be compatible with finitely many definable subsets of X and Y and to satisfy a given local stratifying condition.*

Proof. Let $\pi_1 : \Gamma_h \to X$ (resp. $\pi_2 : \Gamma_h \to Y$) be the projection onto the source (resp. target) space of h. By Propositions 2.6.8 and 2.6.10, there exist two (w)-regular stratifications Σ_h and Σ' of Γ_h and Y respectively such that $\pi_2 : (\Gamma_h, \Sigma_h) \to (Y, \Sigma')$ is a stratified mapping. Since Σ_h can be required to refine any given stratification, we may assume that h is smooth on the images of the strata of Σ_h under π_1 (see Remark 1.8.3). Let Σ be the stratification of X constituted by the respective images of the strata of Σ_h under the mapping π_1.

Notice that $h : (X, \Sigma) \to (Y, \Sigma')$ is a stratified mapping. In order to show that it is horizontally $\mathscr{C}^1$, fix $S \in \Sigma$, a sequence $x_l \in S$ tending to a point $x \in S' \in \Sigma$, as well as a sequence $u_l \in T_{x_l}S$ tending to some $u \in T_xS'$. Let Z be the stratum of Σ_h that projects onto S via π_1, set $\tau := \lim T_{(x_l,h(x_l))}Z$ (extracting a subsequence, we may assume that this sequence is convergent), and let us show the following

Claim. The restriction of π_1 to τ is one-to-one.

Observe for this purpose that, as $(\Gamma_h)_{reg}$ is dense in Γ_h, for every l we can find $y_l \in (\Gamma_h)_{reg}$ arbitrarily close to $(x_l, h(x_l))$. For each l, let Z^l be the stratum of Σ_h containing y_l. Choosing y_l sufficiently generic, we may assume that Z^l is open in Γ_h, and, extracting a subsequence if necessary, we can assume that $T_{y_l}Z^l$ tends to a limit τ'. Moreover, by the (w) condition, if y_l is sufficiently close to $(x_l, h(x_l))$ then $\tau' \supset \tau$. As $|d_{x_l}h|$ is bounded independently of l, $\pi_{1|\tau'}$ must be one-to-one, which yields the claim.

For every l, there is a unique vector $v_l \in T_{(x_l,h(x_l))}Z$ which projects onto u_l via π_1. As $|d_{x_l}h|$ is bounded independently of l, the norm of v_l must be bounded above, which means that we may assume that v_l is converging to a vector v. We clearly have $\pi_1(v) = u$. Let Z' be the stratum of Σ_h that projects onto S' via π_1 and let v' be the vector tangent to Z' that projects onto u. By the (w) condition $(v - v') \in \tau$, which, since $\pi_1(v) = u = \pi_1(v')$, yields $v = v'$ (by the above claim). Hence, v is tangent to Z', which entails that $\pi_2(v) = d_xh_{|S'}(u)$ and therefore:

$$\lim d_{x_l}h_{|S}(u_l) = \lim \pi_2(v_l) = \pi_2(v) = d_xh_{|S'}(u),$$

which shows that h is horizontally $\mathscr{C}^1$.

Since the stratifications Σ and Σ' that we just constructed are merely provided by a (w)-regular stratification of Γ_h, we see that we can require them to be compatible with any given finite families of definable subsets of X and Y respectively.

By Proposition 2.6.10, there exist refinements of Σ and Σ' satisfying any prescribed stratifying local condition. Since stratified mappings are smooth on strata, the property of being horizontally $\mathscr{C}^1$ is preserved under refinements. $\square$

2.6.4 Some more properties of Whitney and Kuo-Verdier stratifications

Proposition 2.6.13. *Let (X, Y) be a couple of strata and let $y \in Y$. If (X, Y) satisfies Whitney's (b) condition at y then there is a neighborhood U of y such that the restricted mapping $(\pi_Y, \rho_Y)_{|U\cap X} : U \cap X \to Y \times \mathbb{R}$ (see section 2.4) is a submersion.*

Proof. By Whitney's (b) condition, the angle between $\partial_x\rho_Y = 2(x - \pi_Y(x))$ and T_xX tends to zero as $x \in X$ tends to y, which implies that $\rho_{Y|X}$ is a submersion near y. Hence, to complete the proof, it suffices to show that for $x \in X$ sufficiently close to y the restriction of $d_x\pi_Y$ to $\ker d_x\rho_{Y|X}$ is onto. Assume that this fails along a sequence $(x_k)_{k\in\mathbb{N}}$ in X tending to y. Extracting a sequence if necessary, we can assume that $\ker d_{x_k}\rho_Y$ and $T_{x_k}X$ respectively converge to vector spaces L and τ. As Whitney's (b) condition implies (a) (Proposition 2.6.3), we have $T_yY \subset \tau$.

By Proposition 2.4.1, we know that $\partial_{x_k}\rho_Y$ is orthogonal to $T_{\pi_Y(x_k)}Y$, which implies $T_yY \subset L$. As the angle between $\partial_{x_k}\rho_Y$ and $T_{x_k}X$ tends to zero, we see that

$$\lim \ker d_{x_k}\rho_{Y|X} = (\lim \ker d_{x_k}\rho_Y) \cap (\lim T_{x_k}X) = L \cap \tau.$$

As $d_y\pi_Y$ induces the identity map on $T_yY \subset L \cap \tau$, this equality yields that the restriction of $d_{x_k}\pi_Y$ to $\ker d_{x_k}\rho_{Y|X}$ is surjective for every k large, contradicting our assumption on $(x_k)_{k\in\mathbb{N}}$. $\square$

Remark 2.6.14. This proposition yields that if a couple of strata (X, Y) satisfies $cl(X) \cap Y \neq \emptyset$ and is Whitney (b)-regular then $\dim X \geq \dim Y + 1$.

Definition 2.6.15. Let Σ be a stratification of a set A. We say that Σ **satisfies the frontier condition** if for every $S \in \Sigma$ the set $fr(S) \cap A$ is the union of some elements of Σ.

Proposition 2.6.16. *Let Σ be a Whitney (b) regular stratification of a locally closed set. If all the strata of Σ are connected then Σ satisfies the frontier condition.*

Proof. Take a couple of strata (X, Y) that satisfies $fr(X) \cap Y \neq \emptyset$. Arguing by downward induction on the dimension of Y, we can assume that the desired property holds for the strata of dimension bigger than $\dim Y$. Observe that if there is a stratum $Z \subset fr(X)$ such that $cl(Z) \supset Y$ then clearly $cl(X) \supset Y$. As a matter of fact, since we can argue inductively on the dimension of X, we can also assume that $X \cup Y$ is locally closed at every point of Y.

As the strata of Σ are connected, it suffices to show that if $y \in cl(X) \cap Y$ then $\mathbf{B}(y, \alpha) \cap Y \subset cl(X) \cap Y$, for $\alpha > 0$ small. Let (U, π_Y, ρ_Y) be a tubular neighborhood of Y (see Proposition 2.4.1).

Take $y \in cl(X) \cap Y$ and $\alpha > 0$ which is sufficiently small for $\mathbf{B}(y, \alpha) \cap Y$ to be closed in $\mathbf{B}(y, \alpha)$, and set for $\varepsilon > 0$, $A_\varepsilon := \{x \in U \cap X : \rho_Y(x) = \varepsilon\}$. Note that, since π_Y is the identity on Y, it suffices to check that $\mathbf{B}(y, \alpha) \cap Y \subset \pi_Y(A_\varepsilon)$, for all $\varepsilon > 0$ small.

By Proposition 2.6.13, there is a neighborhood W of y such that the restriction of (π_Y, ρ_Y) to $W \cap X$ is a submersion. It means that for $\varepsilon > 0$ small the restriction of π_Y to A_ε is a submersion, which implies that $\pi_Y(A_\varepsilon)$ is an open subset of Y. Moreover, as π_Y is continuous, it is easily checked that for $\alpha > 0$, $\mathbf{B}(y, \alpha) \cap \pi_Y(A_\varepsilon)$ is closed in $\mathbf{B}(y, \alpha) \cap Y$ (since $X \cup Y$ is locally closed at every point of Y), for all $\varepsilon > 0$ small. As the latter subset is connected, this implies that $\mathbf{B}(y, \alpha) \cap Y \subset \pi_Y(A_\varepsilon)$, for all $\varepsilon > 0$ small. $\square$

Remark 2.6.17. This proposition, together with Propositions 2.6.6 and 2.6.8 (see Remark 2.6.7 (1)), implies that we can always construct a stratification of a locally closed definable set satisfying both a given stratifying local condition and the frontier condition.

The following proposition will be needed to study the continuity of the density along the strata of a Whitney stratification in Chapter 4.

Proposition 2.6.18. *Let (X, Y) be a couple of strata with $\dim Y = 1$. If (X, Y) satisfies Whitney's (b) condition at $z \in Y \cap cl(X)$ then it satisfies the condition (r) at this point.*

Proof. Fix such a couple of strata (X, Y) and assume that it satisfies Whitney's (b) condition. We can suppose that Y is an open neighborhood of the origin in $\mathbb{R} \times \{0\}$. Thanks to Curve Selection Lemma (Lemma 2.2.3), it suffices to show that (2.6.2) holds along any definable arc $x : (0, \varepsilon) \to X$ tending to the origin at 0. We may assume that x is parametrized by its distance to the origin, which means that $x'(t)$ does not tend to zero as t goes to zero. Let π be the orthogonal projection onto $\mathbb{R} \times \{0\}$ and P_t the orthogonal projection onto the orthogonal complement of $T_{x(t)}X$. As (X, Y) is (b) regular, we have $\lim_{t\to 0} P_t \left(\frac{x(t)-\pi(x(t))}{|x(t)-\pi(x(t))|}\right) = 0$, which entails (see (2.6.4))

$$\lim_{t\to 0} P_t \left(\frac{x'(t) - \pi(x'(t))}{|x'(t) - \pi(x'(t))|}\right) = 0. \tag{2.6.7}$$

As $P_t(x'(t)) \equiv 0$, we also have:

$$\frac{|P_t(\pi(x'(t)))| \cdot |x(t)|}{|x(t) - \pi(x(t))|} = \frac{|P_t(x'(t) - \pi(x'(t)))| t}{|x(t) - \pi(x(t))|} \overset{(2.6.3)}{\lesssim} \frac{|P_t(x'(t) - \pi(x'(t)))|}{|x'(t) - \pi(x'(t))|}, \tag{2.6.8}$$

which tends to zero by (2.6.7). Now, if $\pi(x'(t))$ does not tend to zero then, as $\angle(Y, T_{x(t)}X) \sim |P_t(\pi(x'(t)))|$ (since Y is one-dimensional), we see that (2.6.8) yields (2.6.2) holds along the arc x. Moreover, in the case where $\pi(x'(t))$ tends to zero, (2.6.2) along x amounts to say that $\angle(Y, T_{x(t)}X)$ tends to zero as t goes to zero (since the ratio $\frac{|x|}{|x-\pi(x)|}$ is bounded), which is true since $(b) \Rightarrow (a)$ (Proposition 2.6.3). $\square$

2.7 Approximations and partitions of unity

Given $Y \in \mathcal{S}_n$, we denote by $\mathcal{S}^+(Y)$ the set of definable positive continuous functions on Y.

Proposition 2.7.1. *Let M be a definable $\mathscr{C}^k$ submanifold of $\mathbb{R}^n$, $k \in \mathbb{N}^*$.*

(i) *(Definable $\mathscr{C}^k$ partitions of unity) Given a finite covering of M by definable open subsets $(U_i)_{i\in I}$ there are finitely many definable $\mathscr{C}^k$ functions $\varphi_j : M \to [0,1]$, $j \in J$, such that $\sum_{j\in J} \varphi_j = 1$ and such that every φ_j is supported in $U_{i(j)}$ for some $i(j) \in I$.*

(ii) *(Definable $\mathscr{C}^k$ approximations) Given a continuous definable function $f : M \to \mathbb{R}$ and $\varepsilon \in \mathcal{S}^+(M)$, there is a $\mathscr{C}^k$ definable function g on M such that $|f-g| < \varepsilon$ on M.*

Proof. We prove these two assertions by induction on $m := \dim M$. Both statement being obvious for $m = 0$, take $m \geq 1$ and denote by $(i)_{<m}$ and $(ii)_{<m}$ the respective induction hypotheses.

We first perform the induction step of (i). Let Σ be a stratification M compatible with all the U_i's and satisfying the frontier condition (see Remark 2.6.17). We denote by $\Sigma_{<m}$ (resp. $\Sigma_{=m}$) the collection of the strata of Σ that have dimension less than (resp. equal to) m, and by Σ_S the set of strata of Σ that are included in $fr(S)$. We fix a tubular neighborhood (U_S, π_S, ρ_S) of every $S \in \Sigma$, and, given $S \in \Sigma_{<m}$ as well as $\delta \in \mathcal{S}^+(S)$, we let

$$U_S^\delta := \{x \in M \cap U_S : \rho_S(x) < \delta(\pi_S(x))\}. \tag{2.7.1}$$

We first point out a useful consequence of $(ii)_{<m}$:

Observation. If $S \in \Sigma_{<m}$ and δ as well as ε belong to $\mathcal{S}^+(S)$ then, by $(ii)_{<m}$, there is a $\mathscr{C}^k$ definable function δ' on S such that $|\delta' - \delta| < \min(\frac{\varepsilon}{2}, \frac{\delta}{4})$. We also can find arbitrarily good $\mathscr{C}^k$ definable approximations of $2|\delta - \delta'|$. Subtracting to δ' such a function if necessary, we see that we can find a $\mathscr{C}^k$ definable function δ' on S satisfying both $|\delta - \delta'| < \varepsilon$ and $\delta' < \delta$. Remark that then $U_S^{\delta'} \subset U_S^\delta$, i.e., we have a fundamental system of tubular neighborhoods defined by $\mathscr{C}^k$ definable functions.

In order to construct our partition of unity let us take a $\mathscr{C}^k$ piecewise polynomial function ψ that satisfies $\psi(x) = 1$ if $x \leq \frac{1}{2}$ and $\psi(x) = 0$ if $|x| \geq \frac{3}{4}$ (one can take for instance $\psi(x) := \frac{1}{a}\int_{\frac{3}{4}}^{x} (t-\frac{1}{2})^k (t-\frac{3}{4})^k dt$ for $x \in [\frac{1}{2}, \frac{3}{4}]$, with $a = \int_{\frac{3}{4}}^{\frac{1}{2}} (t-\frac{1}{2})^k (t-\frac{3}{4})^k dt$). Given $S \in \Sigma_{<m}$ and a $\mathscr{C}^k$ positive definable function δ on S, we then define a $\mathscr{C}^k$ function on U_S (recall that strata are $\mathscr{C}^\infty$ submanifolds of $\mathbb{R}^n$) by setting

$$\psi_S^\delta(x) := \psi\left(\frac{\rho_S(x)}{\delta(\pi_S(x))}\right).$$

By construction, ψ_S^δ is supported in U_S^δ and is equal to 1 on $U_S^{\frac{\delta}{2}}$. Hence, for $\delta < d(x, M \setminus U_S)$, we can extend ψ_S^δ (by 0) to a $\mathscr{C}^k$ function on $M \setminus fr(S)$.

Given $S \in \Sigma_{=m}$ and a $\mathscr{C}^k$ positive definable function δ on S, we set $U_S^\delta := S$ as well as $\psi_S^\delta :\equiv 1$. Given a collection of function $\delta = (\delta_Y)_{Y\in\Sigma}$, where δ_Y is a positive

$\mathscr{C}^k$ definable function for every Y, and a stratum $S \in \Sigma$, we then can define a nonnegative $\mathscr{C}^k$ function on $M \setminus fr(S)$ by setting

$$\widetilde{\psi}_S^\delta(x) := \psi_S^{\delta_S}(x) \cdot \prod_{Y \in \Sigma_S} (1 - \psi_Y^{\frac{\delta_Y}{2}}(x)).$$

As this function vanishes on $U_Y^{\frac{\delta_Y}{4}}$ if $Y \in \Sigma_S$, it can be extended (by 0) to a $\mathscr{C}^k$ function on M. Note that $\widetilde{\psi}_S^\delta(x) = 1$ on $W_S := U_S^{\frac{\delta_S}{2}} \setminus \bigcup_{Y \in \Sigma_S} U_Y^{\frac{\delta_Y}{2}}$ for every $S \in \Sigma$. Since $\cup_{S \in \Sigma} W_S = M$, $\sum_{S \in \Sigma} \widetilde{\psi}_S^\delta(x)$ is bounded below away from zero on M, which makes it possible to set

$$\varphi_S^\delta := \frac{\widetilde{\psi}_S^{\delta_S}}{\sum_{Y \in \Sigma} \widetilde{\psi}_Y^{\delta_Y}}, \tag{2.7.2}$$

so that $\sum_{S \in \Sigma} \varphi_S^\delta \equiv 1$. Since all the U_i's are open and because Σ is compatible with them, every $S \in \Sigma_{<m}$ has a neighborhood U_S^μ, $\mu \in \mathcal{S}^+(S)$, which fits in $U_{i(S)}$ for some $i(S) \in I$. Hence, since we can choose δ_S smaller than μ (and yet smooth, by the above observation), we see that we can assume each φ_S^δ, $S \in \Sigma$, to be supported in some $U_{j(S)}$. The family $(\varphi_S^\delta)_{S \in \Sigma}$ thus constitutes the desired partition of unity.

We now perform the induction step of (ii). Fix $\varepsilon \in \mathcal{S}^+(M)$. Let Σ be a stratification of M such that f is $\mathscr{C}^\infty$ on every stratum (see Remark 1.8.3), and let $\Sigma_{<m}$ as well as $\Sigma_{=m}$ be as in the proof of (i). Let for each $S \in \Sigma$ and each collection of functions $\delta = (\delta_S)_{S \in \Sigma}$ (with δ_S definable $\mathscr{C}^k$ positive function on S, as above), $U_S^{\delta_S}$ and φ_S^δ be as in (2.7.1) and (2.7.2). Given $S \in \Sigma_{<m}$ and $x \in U_S^{\delta_S}$, we set $g_S(x) := f(\pi_S(x))$. We also set $g_S := f$ if $S \in \Sigma_{=m}$. Note that, as f is continuous, it follows from definable choice that if $S \in \Sigma_{<m}$ and δ_S is sufficiently small (and yet positive definable $\mathscr{C}^k$, see the above observation) then $|f - g_S| < \varepsilon$ on $U_S^{\delta_S}$. As a matter of fact, if we set $g := \sum \varphi_S^{\delta_S} g_S$ then $|f - g| = |\sum_{S \in \Sigma} \varphi_S^{\delta_S}(f - g_S)| < \varepsilon$. $\square$

Historical notes. Section 2.1 is constituted by basic model theoretic principles that are applied to our framework. Most of the properties of subanalytic sets were already present in the fundamental work of Łojasiewicz [Loj59, Loj64a, Loj64b]. This material was then rewritten and generalized independently by many people (see in particular [vdD98, Shi97, Cos00] for a complete expository). The presentation which is provided here is fairly close to the introductory book [Cos00], whose content is partially inspired by the book of L. van den Dries' book [vdD98]. The subanalycity of the regular locus is due to M. Tamm [Tam81], although the proof we have presented here was taken from [Kur88]. It is difficult to quote an original reference for tubular neighborhoods provided by closest point retractions; a clear proof, close to our content, can be found in [Pol-Rab84]. Whitney's (b) condition was introduced in [Whi]. Proofs in the semi-analytic category seem to go back to Łojasiewicz. Kuo-Verdier stratifications are generalizations of Whitney's stratifications that appeared later [Kuo69, Ver76]. A proof of existence was given in [Loj-Sta-Wac91]. One of the first results of approximations of continuous definable functions by smooth definable functions with the topology considered in Proposition 2.7.1 (i.e., $|f - g| < \varepsilon$ with $\varepsilon \in \mathcal{S}^+(M)$) is Efroymson's Approximation Theorem [Efr82], which is devoted to the semialgebraic category. This theorem is however much more difficult to prove than the latter proposition since it provides a $\mathscr{C}^\infty$ approximation, which prevents from using partitions of unity. Proposition 2.7.1 is closer to a theorem proved by M. Shiota [Shi97] (see also [Esc02], these works however gives in addition approximation of p derivatives if f is $\mathscr{C}^p$). M. Shiota [Shi86] also proved (in the semialgebraic category) an approximation theorem which provides $\mathscr{C}^\infty$ definable approximations of $\mathscr{C}^p$ definable functions with approximation of the derivatives. Efroymson-Shiota's result was extended to the subanalytic category (and even o-minimal) in [aVa-gVa21] (with the approximation of the first derivative only).

Chapter 3

Lipschitz Geometry

In this chapter, we undertake the description of singularities of globally subanalytic sets from the Lipschitz point of view. We first study the interplay between Lipschitz functions, regular vectors (Definition 3.1.3), and the inner metric, which is the metric provided by the length of the shortest path connecting two given points. We prove in particular that every definable set can be decomposed into Lipschitz cells (Theorem 3.1.18), which enables to compare the inner and outer metrics of a definable set. We then enter the explicit description of the Lipschitz geometry of singularities, introducing and constructing some triangulations, called *metric triangulations*, that completely capture the Lipschitz geometry of definable sets. We derive some consequences about the Lipschitz conic structure of definable singularities. These results recently turned out to be useful to study Sobolev spaces and geometric integration theory on definable sets [Leb16, aVal-gVal21a, aVa-gVa21b, gVa22a, gVa22b, Har-dPa22].

To give an intuitive idea of our concept of metric triangulation on a simple example, let us consider the cusp of equation $y^2 = x^3$ in $\mathbb{R}^2$. It is impossible to find a triangulation of this set which is a bi-Lipschitz homeomorphism. The best that we can do is to construct a homeomorphism which "contracts" the vertical distances by multiplying them by a power of the distance to the origin.

It is getting more complicated as the dimension is increasing. Nevertheless, in higher dimensions, the idea is to find a homeomorphism h, from a simplicial complex onto a given definable set, that contracts distances in a way that we explicitly describe in terms of distances to the faces of the simplices.

Some notations. Given $P \in \mathbb{G}_k^n$, we denote by π_P orthogonal projection onto P. Given $\lambda \in \mathbf{S}^{n-1}$, we denote by N_λ the hyperplane of $\mathbb{R}^n$ normal to the vector λ, and by q_λ the coordinate of $q \in \mathbb{R}^n$ along λ, i.e. the number given by the euclidean inner product of q by λ.

Given $B \in \mathcal{S}_n$ and $\lambda \in \mathbf{S}^{n-1}$, with $B \subset N_\lambda$, as well as a function $\xi : B \to \mathbb{R}$, we set

$$\Gamma_\xi^\lambda := \{q \in \mathbb{R}^n : \pi_{N_\lambda}(q) \in B \quad \text{and} \quad q_\lambda = \xi(\pi_{N_\lambda}(q))\}, \tag{3.0.1}$$

and call this set **the graph of ξ for λ.**

Define finally the m-**support** of a set $A \in \mathcal{S}_{m+n}$ by

$$\text{supp}_m A := \{t \in \mathbb{R}^m : A_t \neq \emptyset\}.$$

3.1 Regular vectors and Lipschitz functions

A mapping $\xi : A \to B$, $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^k$, is said to be **Lipschitz** if there is a constant L such that for all x and x' in A:

$$|\xi(x) - \xi(x')| \le L|x - x'|.$$

We say that ξ is L-**Lipschitz** if we wish to specify the constant. The smallest nonnegative number L having this property is called the **Lipschitz constant of** ξ and is denoted L_ξ. By convention, if A is empty then ξ is Lipschitz and $L_\xi = 0$.

A mapping $\xi : A \to B$ is **bi-Lipschitz** if it is a homeomorphism onto its image such that ξ and ξ^{-1} are Lipschitz.

A definable family $(f_t)_{t \in \mathbb{R}^m}$ is **uniformly Lipschitz** if f_t is L-Lipschitz for all $t \in \mathbb{R}^m$, *with L independent of t*. The **uniformly bi-Lipschitz** families are then defined analogously.

Proposition 3.1.1. *Every definable Lipschitz function $\xi : A \to \mathbb{R}$, $A \in \mathcal{S}_n$, can be extended to an L_ξ-Lipschitz definable function $\overline{\xi} : \mathbb{R}^n \to \mathbb{R}$.*

Proof. Set $\overline{\xi}(q) := \inf\{\xi(p) + L_\xi|q - p| : p \in A\}$ (for $A \neq \emptyset$). By the quantifier elimination principle (Theorem 2.1.4), it is a definable function. An easy computation shows that $\overline{\xi}$ is L_ξ-Lipschitz. $\square$

Remark 3.1.2. Let $A \in \mathcal{S}_{m+n}$ and let a definable function $\xi : A \to \mathbb{R}$ be such that $\xi_t : A_t \to \mathbb{R}$ is a Lipschitz function for every $t \in \mathbb{R}^m$. The respective extensions $\overline{\xi}_t$ of ξ_t, $t \in \mathbb{R}^m$ (with for instance $\overline{\xi}_t \equiv 0$ if $t \notin \text{supp}_m A$), provided by the proof of the above proposition constitute a definable family of functions. We thus can extend definable families of Lipschitz functions to definable families of Lipschitz functions.

3.1.1 Regular vectors

Given a definable set $A \subset \mathbb{R}^n$, let

$$\tau(A) := cl(\{T_x A \in \cup_{k=1}^n \mathbb{G}_k^n : x \in A_{reg}\}).$$

For λ of $\mathbf{S}^{n-1}$ and $Z \subset \cup_{k=1}^{n} \mathbb{G}_k^n$, we set (caution, here Z is not a subset of $\mathbb{R}^n$):

$$d(\lambda, Z) := \inf\{d(\lambda, T) : T \in Z\},$$

with by convention $d(\lambda, \emptyset) := +\infty$.

Definition 3.1.3. Let $A \in \mathcal{S}_n$. An element λ of $\mathbf{S}^{n-1}$ is said to be **regular for the set** A if there is $\alpha > 0$ such that:

$$d(\lambda, \tau(A)) \geq \alpha.$$

More generally, we say that $\lambda \in \mathbf{S}^{n-1}$ is **regular for** $A \in \mathcal{S}_{m+n}$ if there exists $\alpha > 0$ such that for all $t \in \mathbb{R}^m$:

$$d(\lambda, \tau(A_t)) \geq \alpha. \tag{3.1.1}$$

We then also say that λ is **regular for the family** $(A_t)_{t \in \mathbb{R}^m}$.

If $\lambda \in \mathbf{S}^{n-1}$ is regular for $A \in \mathcal{S}_{m+n}$, it is regular for $A_t \in \mathcal{S}_n$ for all $t \in \mathbb{R}^m$. It is however much stronger since in (3.1.1), the angle between λ and the tangent spaces to the fibers is required to be bounded below away from zero by a positive constant *independent of the parameter* t.

The regular vector theorems. The theorems below are two essential ingredients of the construction of metric triangulations (Theorem 3.1.4 will be needed to prove Theorem 3.2.4 and Theorem 3.1.5 will be needed to prove the local version presented in Theorem 3.3.2). The proof of these results being too long to be included in these notes, the reader is referred to [gVa05, gVa23].

Regular vectors do not always exist, even if the considered sets have empty interior, as it is shown by the simple example of a circle. Nevertheless, when the considered sets have empty interior, we can find such a vector, up to a globally subanalytic family of bi-Lipschitz mappings [gVa23, Theorem 1.3.2]:

Theorem 3.1.4. *Let $A \in \mathcal{S}_{m+n}$ be such that A_t has empty interior for every $t \in \mathbb{R}^m$. There exists a uniformly bi-Lipschitz definable family of homeomorphisms $h_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}^m$, such that e_n is regular for the family $(h_t(A_t))_{t \in \mathbb{R}^m}$.*

As we just said, the construction of metric triangulations of germs will require another version of the regular vector theorem (see the introduction of section 3.3 for more details). For $R > 0$ and $n \in \mathbb{N} \setminus \{0, 1\}$, we first set

$$\mathcal{C}_n(R) := \{(t, x) \in [0, +\infty) \times \mathbb{R}^{n-1} : |x| \leq Rt\}. \tag{3.1.2}$$

We also set $\mathcal{C}_1(R) := [0, +\infty)$.

Let $A, B \subset \mathbb{R}^n$. A definable map $h : A \to B$ is **vertical** if it preserves the first coordinate in the canonical basis of $\mathbb{R}^n$, i.e. if for any $t \in \mathbb{R}$, $\pi(h(t,x)) = t$, for all $x \in A_t$, where $\pi : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the first coordinate.

In the case of germs of subsets of $\mathcal{C}_n(R)$, the homeomorphism provided by Theorem 3.1.4 may be required to be a vertical map (note however that in the theorem below, there is no parameter t, unlike in the preceding theorem):

Theorem 3.1.5. *[gVa23, Theorem 1.5.14] Let X be the germ at 0 of a definable subset of $\mathcal{C}_n(R)$ (for some R) of empty interior. There exists a germ of vertical bi-Lipschitz definable homeomorphism (onto its image) $H : (\mathcal{C}_n(R), 0) \to (\mathcal{C}_n(R), 0)$ such that e_n is regular for $H(X)$.*

3.1.2 The inner metric

Any definable arc $\gamma : (\eta, \varepsilon) \to \mathbb{R}^n$, $\eta < \varepsilon$, is piecewise analytic. Its length is therefore well-defined as:

$$lg(\gamma) := \int_\eta^\varepsilon |\gamma'(t)|\, dt.$$

It follows from Puiseux Lemma that this integral is always finite if γ is bounded. We thus may define **the inner metric of a set** $X \in \mathcal{S}_n$ by setting for a and b in the same connected component of X (see Proposition 2.2.4):

$$d_X(a,b) := \inf\{lg(\gamma) : \gamma : [\eta, \varepsilon] \to X,\ \mathscr{C}^0 \text{ definable arc joining } a \text{ and } b\}.$$

We rather have defined a metric on every connected component of X. By convention, $d_X(a,b) = +\infty$, when a and b do not lie in the same connected component of X. Note that for all a and b in X:

$$|a-b| \le d_X(a,b). \tag{3.1.3}$$

A definable mapping $f : X \to Y$ is **Lipschitz with respect to the inner metric** if for some constant L we have for all a and b in X:

$$d_Y(f(a), f(b)) \le L \cdot d_X(a,b).$$

We say **L-Lipschitz with respect to the inner metric** if we wish to specify the constant. By analogy, the restriction of the euclidean metric to X is called the **outer metric of** X. In general, the outer metric is not equivalent to the inner metric as it is shown by the simple example of the cusp $y^2 = x^3$ in $\mathbb{R}^2$.

Proposition 3.1.6. *Let $f : X \to Y$ be a definable continuous mapping and let $L \in \mathbb{R}$. The mapping f is L-Lipschitz with respect to the inner metric of X if and only if $|d_x f| \le L$ for all $x \in reg(f)$.*

Proof. The only if part is obvious. Assume that $|d_x f| \leq L$ on $reg(f)$ and let us show that f is L-Lipschitz with respect to the inner metric. By Proposition 2.6.12, there are Whitney (a)-regular stratifications of X and Y with respect to which f is horizontally $\mathscr{C}^1$. By continuity, we see that $\sup_{x \in S} |d_x f_{|S}| \leq L$ for every stratum S. Let a and b be two points in the same connected component of X and let $\gamma : [0,1] \to X$ be a $\mathscr{C}^0$ definable curve joining a and b. Let $t_0 = 0 \leq t_1 < \dots < t_k = 1$ be such that γ stays in the same stratum on (t_i, t_{i+1}) for all $i < k$. By the Mean Value Theorem, for every $i < k$, we have:

$$d_Y(f(\gamma(t_i)), f(\gamma(t_{i+1}))) \leq lg(f(\gamma_{|[t_i,t_{i+1}]})) \leq L \cdot lg(\gamma_{|[t_i,t_{i+1}]}),$$

which shows that $d_Y(f(a), f(b))$ is not bigger than $L \cdot lg(\gamma)$. □

Remark 3.1.7. In particular, if the derivative of a continuous definable mapping $f : \mathbb{R}^n \to Y$ is bounded on an open dense set by a real number L then f is L-Lipschitz (with respect to the *outer* metric). This fact is of course not always true for functions that are not definable (like the so-called Cantor functions).

3.1.3 Lipschitz cell decompositions

We are going to show that every definable set can be decomposed into Lipschitz cells (Definition 3.1.16 and Theorem 3.1.18), which will entail that any continuous definable function with bounded derivative is piecewise Lipschitz (Corollary 3.1.20). This requires some lemmas that we present first.

Lemma 3.1.8. *Let A and B in $\mathcal{S}_{n+m}$ with $B \subset A$. If $\lambda \in \mathbf{S}^{n-1}$ is regular for A, then it is regular for B.*

Proof. Assume that $\lambda \in \mathbf{S}^{n-1}$ is not regular for B. It means that there is a sequence $((t_i, b_i))_{i \in \mathbb{N}}$, with $b_i \in B_{t_i,reg}$ such that $\tau := \lim T_{b_i} B_{t_i,reg}$ exists and contains λ. Choose for every i a Whitney (a) regular stratification of A_{t_i} compatible with B_{t_i} and $B_{t_i,reg}$ and denote by S_i the stratum containing b_i. Moving slightly b_i if necessary, we may assume that S_i is open in $B_{t_i,reg}$ (since $B_{t_i,reg}$ is open and dense in B_{t_i}), which entails that $T_{b_i} S_i = T_{b_i} B_{t_i,reg}$. As $A_{t_i,reg}$ is dense in A_{t_i}, for every $i \in \mathbb{N}$, we can find a_i in $A_{t_i,reg}$, which is close to b_i. Moreover, possibly extracting a sequence, we may assume that $\tau' := \lim T_{a_i} A_{t_i,reg}$ exists. If a_i is sufficiently close to b_i, by Whitney (a) condition, we deduce that $\tau' \supset \tau$, which contains λ. This yields that λ is not regular for A. □

Remark 3.1.9. It is worthy of notice that the proof of the above lemma shows that the corresponding number α (see (3.1.1)) can remain the same for B.

Lemma 3.1.10. *Given $\nu \in \mathbb{N}$, there exist $\lambda_1, \dots, \lambda_N$ in $\mathbf{S}^{n-1}$ and $\alpha_\nu > 0$ such that for any $P_1, \dots, P_\nu$ in $\bigcup_{l=1}^{n-1} \mathbb{G}_l^n$ we can find $i \leq N$ such that $d(\lambda_i, P_j) > \alpha_\nu$, for all $j \leq \nu$.*

Proof. Given $P_1, \dots, P_\nu$ in $\bigcup_{l=1}^{n-1} \mathbb{G}_l^n$, let $\varphi(P_1, \dots, P_\nu) := \sup_{\lambda \in \mathbf{S}^{n-1}} \min_{j \le \nu} d(\lambda, P_j)$. Since the P_j's have positive codimension, φ is a positive function. As the Grassmannian is compact, φ must be bounded below by some positive number t_ν.

Let $\lambda_1, \dots, \lambda_N$ in $\mathbf{S}^{n-1}$ be such that $\bigcup_{i=1}^N \mathbf{B}(\lambda_i, \frac{t_\nu}{2}) \supset \mathbf{S}^{n-1}$. Suppose that there are $P_1, \dots, P_\nu$ in $\bigcup_{l=1}^{n-1} \mathbb{G}_l^n$ such that for any $i \in \{1, \dots, N\}$ we have $d(\lambda_i, \bigcup_{j=1}^\nu P_j) \le \frac{t_\nu}{2}$. Then any λ in $\mathbf{S}^{n-1}$ satisfies $d(\lambda, \bigcup_{j=1}^\nu P_j) < t_\nu$, in contradiction with our choice of t_ν. It is thus enough to set $\alpha_\nu := \frac{t_\nu}{2}$. □

We recall that we estimate the angle between two vector subspaces P and Q of $\mathbb{R}^n$ in the following way:

$$\angle(P, Q) = \sup\{d(\lambda, Q) : \lambda \text{ is a unit vector of } P\}.$$

Definition 3.1.11. Let $\alpha > 0$ and let $Z \in \mathcal{S}_{m+n}$. We say that Z is **(m, α)-flat** if:

$$\sup\{\angle(P, Q) : P, Q \in \bigcup_{t \in \mathbb{R}^m} \tau(Z_{t,reg})\} \le \alpha.$$

We then also say that $(Z_t)_{t \in \mathbb{R}^m}$ is **α-flat**. When $m = 0$, we say that Z is **α-flat**.

Remark 3.1.12. It follows from Lemma 3.1.10 that if $Z_{1,t}, \dots, Z_{\nu,t}$, $t \in \mathbb{R}^m$, are α_ν-flat definable families (where α_ν is the constant provided by the latter lemma) of subsets of $\mathbb{R}^n$ of empty interiors then one of the λ_i's (that are also provided by the latter lemma) is regular for all these families.

Lemma 3.1.13. *Given $Z \in \mathcal{S}_{m+n}$ and $\alpha > 0$, we can find a finite covering of Z by (m, α)-flat definable subsets of Z.*

Proof. Dividing Z into cells, we may assume that Z is a cell. If $\dim Z = l + m$, we can cover $\mathbb{G}_l^n$ by finitely many balls of radius $\frac{\alpha}{2}$, which gives rise to a covering $U_1, \dots, U_k$ of Z (via the family of mappings $Z_t \ni x \mapsto T_x Z_t$) by (m, α)-flat sets. □

Proposition 3.1.14. *There exist $\lambda_1, \dots, \lambda_N$ in $\mathbf{S}^{n-1}$ such that for any $A_1, \dots, A_p$ in $\mathcal{S}_{m+n}$, there is a cell decomposition $\mathcal{C}$ of $\mathbb{R}^{m+n}$ compatible with all the A_i's and such that for each cell $C \in \mathcal{C}$ satisfying $\dim C_t = n$ (for all $t \in supp_m C$), we may find $\lambda_{j(C)}$, $1 \le j(C) \le N$, regular for the family $(\delta C_t)_{t \in \mathbb{R}^m}$.*

Proof. According to Lemma 3.1.10 (see Remark 3.1.12 and Lemma 3.1.8) it is sufficient to prove by induction on n the following assertions: given $\alpha > 0$ and $A_1, \dots, A_p$ in $\mathcal{S}_{m+n}$, there exists a cell decomposition of $\mathbb{R}^{m+n}$ compatible with $A_1, \dots, A_p$ and such that for every cell $C \subset \mathbb{R}^{m+n}$ of this cell decomposition satisfying $\dim C_t = n$ (for all $t \in \mathrm{supp}_m C$), there are α-flat definable families of sets of empty interior $V_{1,t}, \dots, V_{l,t}$, with $l \le 2n$, such that $\delta C_t \subset \cup_{i=1}^l V_{i,t}$ for all $t \in \mathbb{R}^m$.

For $n = 0$ this is clear. Fix $n \geq 1$, $\alpha > 0$, as well as $A_1, \dots, A_p$ in $\mathcal{S}_{m+n}$. Apply Lemma 3.1.13 to all the cells of a cell decomposition $\mathcal{E}$ compatible with the A_i's, and take a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{m+n}$ compatible with all the elements of the obtained coverings. Applying then the induction hypothesis to the elements of $\pi(\mathcal{D})$ (π being the canonical projection onto $\mathbb{R}^{m+n-1}$), we get a refinement $\mathcal{D}'$ of $\pi(\mathcal{D})$.

Given a cell D of $\mathcal{D}'$, each cell of $\mathcal{E}$ is above D, either the graph of a definable function, say $\zeta_{i,D}$, or a band, say $(\zeta_{i,D}, \zeta_{i+1,D})$, with $\zeta_{i,D} < \zeta_{i+1,D}$ definable functions on D (or $\pm\infty$, see Definition 1.2.1). Let $\mathcal{C}$ be the cell decomposition given by all the graphs $\Gamma_{\zeta_{i,D}}$, $D \in \mathcal{D}'$. To check that it has the required property, fix a cell $C = (\zeta_{i,D}, \zeta_{i+1,D})$, with $\zeta_{i,D} < \zeta_{i+1,D}$ definable functions (or $\pm\infty$) on a cell D of $\mathcal{D}'$ satisfying $\dim D_t = n-1$ for all $t \in \operatorname{supp}_m D$. Since $\mathcal{D}'$ is compatible with the images under π of the (m,α)-flat sets that cover the cells of $\mathcal{E}$, the sets $\Gamma_{\zeta_{i,D}}$ and $\Gamma_{\zeta_{i+1,D}}$ must be (m,α)-flat, and since

$$\delta C_t \subset \left(\Gamma_{\zeta_{i,D}}\right)_t \cup \left(\Gamma_{\zeta_{i+1,D}}\right)_t \cup \pi^{-1}(\delta D_t),$$

we see that the needed fact follows from the induction hypothesis. □

Remark 3.1.15. We have proved a stronger statement: the distance between the regular vector $\lambda_{j(C)}$ and the tangent spaces to δC_t can be bounded below away from zero by a positive number depending only on n, and not on the A_i's. This is due to the fact that in the above proof we apply Lemma 3.1.10 with $\nu = l \leq 2n$.

This proposition will be useful to compare the inner and outer metrics of globally subanalytic sets.

Definition 3.1.16. We define the **Lipschitz cells of** $\mathbb{R}^n$ by induction on n. Let E be a cell of $\mathbb{R}^n$ and denote by D its basis. If $n = 0$ then E is always a Lipschitz cell. If $n > 0$, we say that E is a Lipschitz cell if so is D and if in addition one of the following properties holds:

(i) E is the graph of some Lipschitz definable function $\xi : D \to \mathbb{R}$.

(ii) E is a band (ξ_1, ξ_2), $\xi_1 < \xi_2$, where ξ_1 is either $-\infty$ or a Lipschitz definable function on D, and ξ_2 is either $+\infty$ or a Lipschitz definable function on D.

Lemma 3.1.17. *If E is a Lipschitz cell of $\mathbb{R}^n$ then the outer and inner metrics of E are equivalent, i.e., there is a constant C such that for all x and y in E:*

$$|x - y| \leq d_E(x, y) \leq C|x - y|.$$

Moreover, the constant C just depends on n and on the Lipschitz constants of the functions defining E.

Proof. The definition of a Lipschitz cell being inductive, this lemma can be proved by induction on n. It is an easy exercise to construct a path joining two given points using the functions defining the cell, and to estimate its length in terms of the Lipschitz constants of these functions. □

This leads us to the following "Lipschitz cell decomposition Theorem".

Theorem 3.1.18. *Given $A_1, \dots, A_l$ in $\mathcal{S}_{m+n}$, there is a definable partition $\mathcal{P}$ of $\mathbb{R}^{m+n}$ compatible with $A_1, \dots, A_l$ and such that for each $V \in \mathcal{P}$ there is a linear isometry Λ of $\mathbb{R}^n$ for which $\bigcup_{t\in\mathbb{R}^m}\{t\} \times \Lambda(V_t)$ is a cell and $\Lambda(V_t)$ is a Lipschitz cell for every $t \in \mathbb{R}^m$. Moreover:*

(i) The Lipschitz constants of the functions defining the cells $\Lambda(V_t)$ can be bounded by a function of n (thus independent of t and of the A_i's).

(ii) The corresponding linear isometry Λ can be chosen among a finite family that only depends on n (and not on the A_i's).

Proof. We prove this proposition by induction on n (for $n = 0$ the result is trivial). Taking a cell decomposition of $\mathbb{R}^{m+n}$ if necessary, we may assume that the family $A_1, \dots, A_l$ is reduced to one single set A which is a cell.

We start with the case where A_t has empty interior for all $t \in \mathrm{supp}_m A$. Fix $\alpha > 0$ sufficiently small for Lemma 3.1.10 to hold with $\nu = 1$. By Lemma 3.1.13, A can be covered by finitely many (m, α)-flat sets. Take a definable partition of A compatible with the elements of this covering, and let E be an element of this partition. By Lemmas 3.1.10 (with $\nu = 1$) and 3.1.8, the family $(E_t)_{t\in\mathbb{R}^m}$ has a regular vector (which, up to a linear isometry, can be assumed to be e_n). Take a cell decomposition compatible with E and let C be a cell included in E. The set C is thus the graph (for e_n) of some function, say $\xi : B \to \mathbb{R}$, and the derivative of ξ_t must be bounded independently of $t \in \mathbb{R}^m$. By induction, we know that we can cover B by some sets $W_1, \dots, W_l$ such that for all $t \in \mathbb{R}^m$ and each i, $W_{i,t}$ is a Lipschitz cell after a possible linear isometry. By Proposition 3.1.6 and Lemma 3.1.17, ξ_t induces a Lipschitz function on each $W_{i,t}$.

We now carry out the induction step in the case where $\dim A_t = n$ for all $t \in \mathrm{supp}_m A$. By Proposition 3.1.14, splitting our set A into several sets if necessary, we may assume that $(\delta A_t)_{t\in\mathbb{R}^m}$, has a regular vector (which again, up to a linear isometry, can be supposed to be e_n). Take a cell decomposition $\mathcal{C}$ of $\mathbb{R}^{m+n}$ compatible with $\bigcup_{t\in\mathbb{R}^m}\{t\} \times \delta A_t$. If $C \in \mathcal{C}$ is such that $C_t \subset \delta A_t$ for all $t \in \mathbb{R}^m$ then C_t is for each $t \in \mathrm{supp}_m C$ the graph of some function ξ_t, $t \in \mathrm{supp}_m C$, that has bounded gradient (independently of t). By the same argument as in the case $\dim A_t < n$, the family ξ_t induces a family of Lipschitz functions on the elements of a partition of the basis of C (into sets which are cells up to a linear isometry) obtained from the induction hypothesis. As cells are connected, a cell decomposition of $\mathbb{R}^n$ which is compatible with δA_t is compatible with A_t.

That the Lipschitz constants of the functions defining these Lipschitz cells can be bounded by a function of n comes down from Remark 3.1.15 (and Lemma 3.1.10 in the case $\dim A_t < n$) together with Lemma 3.1.17. Assertion (ii) follows from the fact that in Proposition 3.1.14, the family $\lambda_1, \dots, \lambda_N$ is independent of the A_i's. $\square$

Thanks to Lemma 3.1.17, the above theorem has the following immediate consequence.

Corollary 3.1.19. *Every set $A \in \mathcal{S}_{m+n}$ admits a definable partition $\mathcal{P}$ such that for every $V \in \mathcal{P}$ and $t \in \mathbb{R}^m$ the inner and outer metrics of V_t are equivalent. The constants of these equivalences can be bounded by a function of n.*

Since every definable function is piecewise continuous, thanks to Proposition 3.1.6, we then derive:

Corollary 3.1.20. *Let $\xi_t : A_t \to \mathbb{R}$, $t \in \mathbb{R}^m$, $A \in \mathcal{S}_{m+n}$, be a definable family of functions, and set $K_{\xi_t} := \sup_{x \in reg(\xi_t)} |\partial \xi_t(x)|$, for each $t \in \mathbb{R}^m$. There are a constant C (depending only on n) and a definable partition $\mathcal{P}$ of A such that ξ_t is $C \cdot K_{\xi_t}$-Lipschitz on V_t, if V is an element of $\mathcal{P}$ and $t \in \mathbb{R}^m$ is such that $K_{\xi_t} < +\infty$.*

The following corollary unravels the close interplay between Lipschitz functions and regular vectors.

Corollary 3.1.21. *The vector $\lambda \in \mathbf{S}^{n-1}$ is regular for the set $A \in \mathcal{S}_{m+n}$ if and only if there are finitely many uniformly Lipschitz definable families of functions $\xi_{i,t} : B_{i,t} \to \mathbb{R}$, $t \in \mathbb{R}^m$, $i = 1, \dots, p$, with $B_i \subset \mathbb{R}^m \times N_\lambda$, such that for all $t \in \mathbb{R}^m$:*

$$A_t = \cup_{i=1}^p \Gamma^\lambda_{\xi_{i,t}}.$$

Proof. As the "if" part is clear, we will focus on the converse. Up to a linear isometry we can assume that $\lambda = e_n$. Take a cell decomposition compatible with A and let C be a cell included in A. This cell cannot be a band since e_n is regular for A (see Lemma 3.1.8). It is thus the graph of a function $\xi : D \to \mathbb{R}$, with $D \in \mathcal{S}_{m+n-1}$. For every $t \in \mathbb{R}^m$, the function $\xi_t : D_t \to \mathbb{R}$ has bounded first derivative (with a bound independent of t). By Corollary 3.1.20, there is a definable partition $\mathcal{P}$ of D, such that the family $\xi_{t|V_t}$ is uniformly Lipschitz for every $V \in \mathcal{P}$. $\square$

Remark 3.1.22. By Proposition 3.1.1 (see Remark 3.1.2), we may extend the $\xi_{i,t}$'s to N_λ. Using the min operator, it is then not difficult to show (see [gVa23, Proposition 1.4.7]) that we can assume that these extensions satisfy $\xi_{1,t} \le \dots \le \xi_{p,t}$ on N_λ (and $A_t \subset \cup_{i=1}^p \Gamma^\lambda_{\xi_{i,t}}$).

3.1.4 Comparing the inner and outer metrics

We define the **diameter** of a set $X \subset \mathbb{R}^n$ by

$$\operatorname{diam}(X) := \sup\{|x-y| : x \in X, y \in X\}. \tag{3.1.4}$$

Proposition 3.1.23. *Given $A \in \mathcal{S}_{m+n}$, there is a constant C such that for all $t \in \mathbb{R}^m$, we have for each x and y in the same connected component of A_t:*

$$d_{A_t}(x,y) \le C\, diam(A_t). \tag{3.1.5}$$

Proof. Let $U_1, \dots, U_k$ be the elements of the partition of A provided by Corollary 3.1.19, and set $V_{i,t} := cl(U_{i,t}) \cap A_t$ for $i \le k$ and $t \in \mathbb{R}^m$. It follows from Curve Selection Lemma (Lemma 2.2.3) that the inner and outer metrics of the $V_{i,t}$ are equivalent, and that the constants of these equivalences are bounded independently of t. We proceed by induction on k. If $k=1$ then the result immediately follows from the fact that the inner and outer metrics of $V_{1,t}$ are equivalent. Set $B_t^j := \cup_{i=1}^j V_{i,t}$, for $j \le k$, and assume that the desired fact holds for B_t^{k-1}.

For $t \in \mathbb{R}^m$, let E_t denote the set of all the couples $(x,y) \in B_t^{k-1} \times V_{k,t}$ of elements that are in the same connected component of B_t^k. Thanks to our induction hypothesis, it suffices to show that

$$\sup\{d_{B_t^k}(x,y) : (x,y) \in E_t, t \in \mathbb{R}^m\} < +\infty.$$

Let $t \in \operatorname{supp}_m A$ and $(x,y) \in E_t$, with $x \neq y$, and $\varepsilon > 0$. Take a continuous definable arc $\gamma : [0,1] \to B_t^k$ joining x and y such that $lg(\gamma) \le d_{B_t^k}(x,y) + \varepsilon$, and set $s_0 := \sup\{s : \gamma([0,s]) \subset B_t^{k-1}\}$ as well as $z := \gamma(s_0)$. The arc $\gamma_1 := \gamma_{|[0,s_0]}$ joins x and z, while the arc $\gamma_2 := \gamma_{|[s_0,1]}$ joins z and y. But since we obviously have

$$lg(\gamma_1) \le d_{B_t^{k-1}}(x, \gamma(s_0)) + \varepsilon \quad \text{and} \quad lg(\gamma_2) \le d_{V_{k,t}}(\gamma(s_0), y) + \varepsilon,$$

the required fact follows from the induction hypothesis together with the fact that the inner and outer metrics of $V_{k,t}$ are equivalent (since ε can be taken arbitrarily small). ☐

The inner metric $A \times A \ni (x,y) \mapsto d_A(x,y)$ is not necessarily a definable function. We however have:

Lemma 3.1.24. *Given a definable connected set A, there is a continuous definable function $\rho : A \times A \to [0,+\infty)$ such that $d_A(x,y) \sim \rho(x,y)$ on $A \times A$.*

Proof. By Corollary 3.1.19, we can cover A with finitely many definable connected sets, say $V_1, \dots, V_k$, such that the inner and outer metrics of each V_j are equivalent. Possibly replacing these sets with their closure in A, we can assume that they are closed in A. It is then easy to check that for x and y in A we have $d_A(x,y) \sim \rho(x,y) := \inf \sum_{i=1}^{l-1} |x_i - x_{i+1}|$, where the infimum is taken over all the families $x_1, \dots, x_l$, $l \leq k$, of points of A such that $x_1 = x$, $x_l = y$, and such that for each $i < l$, the points x_i and x_{i+1} both lie in V_{j_i}, for some $j_i \leq k$. $\square$

Definition 3.1.25. We say that $A \in \mathcal{S}_n$ is **connected at** $x \in cl(A)$ if $\mathbf{B}(x,\varepsilon) \cap A$ is connected for all $\varepsilon > 0$ small enough. We will say that it is **connected along** $Z \subset cl(A)$ if it is connected at each point of Z. We say that A is **normal** if it is connected at each $x \in cl(A)$.

Proposition 3.1.26. *If $A \in \mathcal{S}_n$ is normal, connected, and bounded then there are a constant C and a positive rational number θ such that*

$$|x-y| \leq d_A(x,y) \leq C|x-y|^{\theta}.$$

Proof. Let ρ be provided by applying Lemma 3.1.24 to A. By (3.1.5), $d_A(x,y)$ is bounded on $A \times A$, and hence, so is ρ. The desired fact follows by applying Łojasiewicz's inequality (Theorem 2.2.5) to $\rho(x,y)$ and $|x-y|$, which nevertheless requires to check that $\rho(x(t),y(t))$ tends to zero, for each couple of definable arcs $x : (0,\varepsilon) \to A$ and $y : (0,\varepsilon) \to A$ tending to the same point $z \in cl(A)$ (as A is bounded, definable arcs in A must have endpoints in $cl(A)$). We can assume $x(t)$ and $y(t)$ to be parameterized by their distance to z. If $B_t := \mathbf{S}(z,t) \cap A$ is not connected (for $t > 0$ small) then, as A is connected at z (since it is normal), $z \in A$ (this fact can be deduced from the local conic structure theorem which is proved in section 3.4 independently), in which case the needed fact is clear. Otherwise, if B_t is connected, we can write

$$\rho(x(t),y(t)) \lesssim d_A(x(t),y(t)) \leq d_{B_t}(x(t),y(t)) \overset{(3.1.5)}{\leq} Ct,$$

which yields that $\rho(x(t),y(t))$ tends to zero as t goes to zero. $\square$

Remark 3.1.27. The assumption "A normal" is clearly necessary and it is easy to see that the proposition is no longer true if one drops the boundedness assumption.

In [Bir-Mos00], a related theorem ensures that every semialgebraic set can be *normally embedded* (their proof goes over the globally subanalytic category), which means that, given a semialgebraic set A, we can construct a definable homeomorphism (onto its image) $h : A \to \mathbb{R}^k$, bi-Lipschitz with respect to the inner metric and such that the inner and outer metrics of $h(A)$ are equivalent.

3.2 Metric triangulations

Some preliminary definitions. A **simplex** of $\mathbb{R}^n$ of dimension k is the convex hull of $(k+1)$ affine independent (i.e., not contained in an affine space of dimension $(k-1)$) elements $u_0, \dots, u_k$ of $\mathbb{R}^n$. We say that $\{u_0, \dots, u_k\}$ **generates** σ.

A **face** of σ is then a simplex σ' generated by a subset of $\{u_0, \dots, u_k\}$. A face τ of σ satisfying $\dim \tau < \dim \sigma$ is called a **proper face**. An **open simplex** is a simplex from which the proper faces have been deleted.

A **simplicial complex** K of $\mathbb{R}^n$ is a *finite* collection of open simplices of $\mathbb{R}^n$ such that for any σ_1 and σ_2 in K, the set $cl(\sigma_1) \cap cl(\sigma_2)$ is a common face of $cl(\sigma_1)$ and $cl(\sigma_2)$[1]. We denote by $|K|$ the **polyhedron** of K which is the set constituted by the union of all the elements of K.

A **triangulation of a set** $X \in \mathcal{S}_n$ is a globally subanalytic homeomorphism $\Psi : |K| \to X$, with K simplicial complex of $\mathbb{R}^n$.

Basic idea. We are going to define a notion of triangulation adapted to the study of Lipschitz geometry. A *metric triangulation* will be a homeomorphism from a simplicial complex onto the given set, which means that it will be a triangulation in the usual meaning. Of course, the distances are modified by such a homeomorphism. We shall require that over each simplex the distances are preserved up to "some contractions" which will be characterized by finitely many iterations of sums, products, and powers of distance functions to the faces. Such functions will be called *standard simplicial functions*. Indeed, Definition 3.2.2 will involve standard simplicial functions on $\sigma \times \sigma$ depending on two variables q and q' in σ. These functions are such combinations of distance functions $d(q, \tau)$ and $d(q', \tau)$, τ proper face of σ.

What will matter is that two sets having the same metric triangulation (with the same coordinate systems and equivalent contraction functions) will be definably bi-Lipschitz homeomorphic (Proposition 3.2.3).

Definition of metric triangulations and the main theorem. It will not be possible to express the contractions along the directions of the canonical basis. The reason is that this kind of mappings alter not only distances, but also angles. Hence, we first introduce the concept of tame system of coordinates along which the contractions will apply.

Definition 3.2.1. Let σ be an open simplex of $\mathbb{R}^n$ and denote by $\pi_i : \mathbb{R}^n \to \mathbb{R}^i$ the projection onto the i first coordinates. A **tame system of coordinates of** σ is a mapping $\mu : \sigma \to [0,1]^n$, $\mu = (\mu_1, \dots, \mu_n)$, which is a homeomorphism onto its image of the following form for $q = (q_1, \dots, q_n) \in \sigma$ and $i \in \{1, \dots, n\}$:

$$\mu_i(q) = \begin{cases} \frac{q_i - \zeta_i(\pi_{i-1}(q))}{\zeta_i'(\pi_{i-1}(q)) - \zeta_i(\pi_{i-1}(q))} & \text{if } \zeta_i < \zeta_i' \\ 0 & \text{whenever } \zeta_i = \zeta_i' \text{ on } \pi_{i-1}(\sigma), \end{cases} \tag{3.2.1}$$

[1]It should be emphasized that our notion of simplicial complex is slightly different from the usual one since we take *open* simplices.

where ζ_i and ζ_i' are linear functions on $\pi_{i-1}(\sigma)$ satisfying for all $q \in \sigma$

$$\zeta_i(\pi_{i-1}(q)) \leq q_i \leq \zeta_i'(\pi_{i-1}(q)).$$

A **standard simplicial function on** σ is a function $\varphi(q)$ given by the quotient of finite sums of monomials of type

$$d(q, \tau_1)^{\alpha_1} \cdots d(q, \tau_k)^{\alpha_k}, \tag{3.2.2}$$

with $\alpha_1, \dots, \alpha_k$ positive rational numbers and $\tau_1, \dots, \tau_k$ faces of σ. A **standard simplicial function on** $\sigma \times \sigma$ is a function $\varphi(q, q')$ given by the quotient of finite sums of monomials as in (3.2.2) but involving both functions of type $\sigma \ni q \mapsto d(q, \tau)$ and $\sigma \ni q' \mapsto d(q', \tau)$, τ proper face of σ (standard simplicial functions on $\sigma \times \sigma$ are actually just needed if we triangulate unbounded sets, see Remark 3.2.9).

Definition 3.2.2. Let $X \in \mathcal{S}_n$. A **metric triangulation of** X is a triangulation $\Psi : |K| \to X$ of X such that for every $\sigma \in K$ there exist $\varphi_{\sigma,1}, \dots, \varphi_{\sigma,n}$, standard simplicial functions on $\sigma \times \sigma$ satisfying for any q and q' in σ:

$$|\Psi(q) - \Psi(q')| \sim \sum_{i=1}^{n} \varphi_{\sigma,i}(q, q') \cdot |\mu_{\sigma,i}(q) - \mu_{\sigma,i}(q')|, \tag{3.2.3}$$

with $\mu_\sigma := (\mu_{\sigma,1}, \dots, \mu_{\sigma,n})$ tame system of coordinates of σ. The functions $\varphi_{i,\sigma}$ are then called **the contraction functions** of the metric triangulation Ψ.

Given some subsets $A_1, \dots, A_\kappa$ of X, we say that a metric triangulation $h : |K| \to X$ is **compatible with** $A_1, \dots, A_\kappa$ if each $\Psi^{-1}(A_i)$ is the union of some elements of K.

Let us make more precise the extent to which the metric structure of the set is captured by the triangulation by stating the following immediate proposition which, roughly speaking, yields that two subsets having the same metric triangulation are definably bi-Lipschitz homeomorphic:

Proposition 3.2.3. *Let X_1 and X_2 be elements of $\mathcal{S}_n$ and let $\Psi_1 : |K| \to X_1$ and $\Psi_2 : |K| \to X_2$ respectively be metric triangulations of X_1 and X_2. Assume that these metric triangulations involve the same contraction functions $\varphi_{i,\sigma}$ and the same tame systems of coordinates $(\mu_{\sigma,1}, \dots, \mu_{\sigma,n})$ on every simplex $\sigma \in K$. Then $\Psi_2\Psi_1^{-1}$ is a definable bi-Lipschitz homeomorphism from X_1 onto X_2.*

Proof. This follows from (3.2.3) for both Ψ_1 and Ψ_2. □

We are going to prove that every globally subanalytic set admits a metric triangulation. We shall establish it not only for a definable set, but for a definable family, up to a partition of the parameter space. This will yield that definable bi-Lipschitz triviality holds almost everywhere (Corollary 3.2.12).

Theorem 3.2.4. *Given $A_1, \dots, A_\kappa$ in $\mathcal{S}_{m+n}$, there is a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for each $B \in \mathcal{P}$ there are a simplicial complex K of $\mathbb{R}^n$ and a definable family of metric triangulations $\Psi_t : |K| \to \mathbb{R}^n$ (of $\mathbb{R}^n$) compatible with $A_{1,t}, \dots, A_{\kappa,t}$ for each $t \in B$. The contraction functions and the tame systems of coordinates of Ψ_t are independent of $t \in B$.*

The proof of this theorem requires two preliminary lemmas that will also be used in the proof of the local version displayed in Theorem 3.3.2.

Lemma 3.2.5. *Let $\xi : A \to \mathbb{R}$ be a definable nonnegative function, $A \in \mathcal{S}_{m+n}$. There exist some definable subsets $W_1, \dots, W_k$ of $cl(A)$ and a definable partition $\mathcal{P}$ of A such that for any $V \in \mathcal{P}$ there are some rational numbers $\alpha_1, \dots, \alpha_k$ such that for each $t \in \mathbb{R}^m$ we have on $V_t \subset \mathbb{R}^n$:*

$$\xi_t(x) \sim d(x, W_{1,t})^{\alpha_1} \cdots d(x, W_{k,t})^{\alpha_k}. \tag{3.2.4}$$

Proof. We prove it by induction on n. For $n = 1$, the result follows from the Preparation Theorem (Theorem 1.8.2). Take $n \geq 2$ and assume that the proposition is true for $(n-1)$. Denote by $\Lambda_1, \dots, \Lambda_\kappa$ the linear isometries involved in Theorem 3.1.18 (see (ii) of this theorem). We will sometimes regard them as changes of coordinates of $\mathbb{R}^{n+m}$, preserving the m first variables.

For each $i \leq \kappa$, applying the Preparation Theorem to $\xi \circ \Lambda_i^{-1} : \Lambda_i(A) \to \mathbb{R}$, we obtain a definable partition of $\Lambda_i(A)$. The images of all the elements of this partition under the map Λ_i^{-1} constitute (for each $i \leq \kappa$) a partition of A, denoted by $\mathcal{P}_i$. Apply Theorem 3.1.18 to the elements of a common refinement of all the $\mathcal{P}_i$'s and denote by $\mathcal{P}$ the resulting partition of A.

Since it suffices to show the result for the restriction of ξ to each $C \in \mathcal{P}$, let us fix such C. By Theorem 3.1.18, there is $i \leq \kappa$ such that $\Lambda_i(C)$ is a cell and $\Lambda_i(C)_t$ is a Lipschitz cell for each $t \in \mathbb{R}^m$. If $\Lambda_i(C)_t$ is the graph of some uniformly Lipschitz family of functions, the needed estimate easily follows by induction on n. Otherwise, $\Lambda_i(C) = (\eta_1, \eta_2)$ with $\eta_1 < \eta_2$ definable functions on the basis of $\Lambda_i(C)$ such that $(\eta_{1,t})_{t \in \mathbb{R}^m}$ and $(\eta_{2,t})_{t \in \mathbb{R}^m}$ are uniformly Lipschitz (or $\pm\infty$).

As $\mathcal{P}$ is compatible with the $\mathcal{P}_j$'s, the function $\xi \circ \Lambda_i^{-1}$ is reduced on C. Since we can argue up to a linear isometry, we will from now identify Λ_i with the identity. Hence, there are $r \in \mathbb{Q}$ as well as some functions a and θ on the basis B of C such that for $x = (\tilde{x}, x_n) \in C_t$, $t \in \mathbb{R}^m$, we have:

$$\xi_t(x) \sim a_t(\tilde{x}) |x_n - \theta_t(\tilde{x})|^r.$$

Thanks to the induction hypothesis, we thus only have to check the result for the function $|x_n - \theta_t(\tilde{x})|$. As $\Gamma_\theta \cap C = \emptyset$, we can assume for every $(t, \tilde{x}) \in B$, either $\theta_t(\tilde{x}) \leq \eta_{1,t}(\tilde{x})$ or $\theta_t(\tilde{x}) \geq \eta_{2,t}(\tilde{x})$. Suppose for instance that $\theta_t(\tilde{x}) \leq \eta_{1,t}(\tilde{x})$, and write for $t \in \mathrm{supp}_m C$ and $x = (\tilde{x}, x_n) \in C_t$:

$$x_n - \theta_t(\tilde{x}) = (x_n - \eta_{1,t}(\tilde{x})) + (\eta_{1,t}(\tilde{x}) - \theta_t(\tilde{x})).$$

There is a definable partition of C such that the two terms of the sum appearing in the right-hand-side of this equality are comparable on every element E of this partition. As they are both nonnegative, we then see that either $|x_n - \theta_t(\tilde{x})| \sim (x_n - \eta_{1,t}(\tilde{x}))$ or $|x_n - \theta_t(\tilde{x})| \sim (\eta_{1,t}(\tilde{x}) - \theta_t(\tilde{x}))$ on E_t for each $t \in \mathbb{R}^m$. In the second case, the desired result comes down from the induction hypothesis (since $(\eta_{1,t}(\tilde{x}) - \theta_t(\tilde{x}))$ is an $(n-1)$-variable function). Moreover, since $\eta_{1,t}$ is Lipschitz, $|x_n - \eta_{1,t}(\tilde{x})| \sim d(x, \Gamma_{\eta_{1,t}})$ on E_t. $\square$

Remark 3.2.6. The constants of the equivalence in the above lemma depend on t. The exponents $\alpha_1, \dots, \alpha_k$ however just depend on $V \in \mathcal{P}$.

The second result that we shall need in the proof of Theorem 3.2.4 is an elementary fact about families of functions that will help us to refine partitions.

Lemma 3.2.7. *Given some definable families of Lipschitz functions $c_{1,t}, \dots, c_{k,t}$ on $\mathbb{R}^{n-1}$, $t \in \mathbb{R}^m$ we can find some definable families of Lipschitz functions $\xi_{1,t} \le \dots \le \xi_{l,t}$ on $\mathbb{R}^{n-1}$ and a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{m+n-1}$ such that for every $D \in \mathcal{D}$, the collection of functions*

$$c_{1,t}(\tilde{x}), \dots, c_{k,t}(\tilde{x}), |x_n - c_{1,t}(\tilde{x})|, \dots, |x_n - c_{k,t}(\tilde{x})|$$

is totally ordered on $(\xi_{i,t|D_t}, \xi_{i+1,t|D_t})$ for each $t \in \mathbb{R}^m$ and $i \in \{0, \dots, l\}$ (with $\xi_{0,t} \equiv -\infty$ and $\xi_{l+1,t} \equiv \infty$).

Proof. Take a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{m+n-1}$ compatible with the sets $Z_{ij} := \{(t,z) \in \mathbb{R}^{m+n-1} : c_{i,t}(z) \le c_{j,t}(z)\}$. Let us complete the finite collection constituted by the $c_{i,t}$'s with the functions $(c_{i,t} + c_{j,t})$, $(c_{i,t} - c_{j,t})$, and $\frac{c_{i,t}+c_{j,t}}{2}$, $i, j \in \{1, \dots, k\}$ (which are also Lipschitz), and denote by $\xi_{1,t}, \dots, \xi_{l,t}$ the completed family. Using the min operator if necessary, it is then not difficult to see that we can assume $\xi_{1,t} \le \dots \le \xi_{l,t}$ (see [gVa23, Proposition 1.4.7]).

To check that it has the required property fix a cell D of $\mathcal{D}$ and observe that the compatibility of $\mathcal{D}$ with the Z_{ij}'s entails that the $c_{i,t}$'s are comparable with each other on $E_{p,t} := (\xi_{p,t|D_t}, \xi_{p+1,t|D_t})$, for all p. Moreover, since the graphs of the $c_{i,t}$'s are included in the union of the graphs of the $\xi_{i,t}$'s, the function $(x_n - c_{i,t}(\tilde{x}))$ has constant sign on $E_{p,t}$, for each i and each p. If, for instance, $(x_n - c_{i,t}(\tilde{x})) > 0$ and $(x_n - c_{j,t}(\tilde{x})) < 0$ on $E_{p,t}$, then since

$$(x_n - c_{i,t}(\tilde{x})) - (c_{j,t}(\tilde{x}) - x_n) = 2\left(x_n - \frac{c_{i,t} + c_{j,t}}{2}\right),$$

we see that the inclusion $\Gamma_{\frac{c_{i,t}+c_{j,t}}{2}} \subset \bigcup_{\iota=1}^{l} \Gamma_{\xi_{\iota,t}}$ entails that $|x_n - c_{i,t}(\tilde{x})|$ and $|x_n - c_{j,t}(\tilde{x})|$ are comparable with each other on $E_{p,t}$, for all p. The inclusion of the graphs of the functions $(c_{i,t} + c_{j,t})$ and $(c_{i,t} - c_{j,t})$ in $\bigcup_{\iota=1}^{l} \Gamma_{\xi_{\iota,t}}$ can be used analogously to prove that $c_{i,t}$ is comparable with $(x_n - c_{j,t}(\tilde{x}))$ on $E_{p,t}$ for all p, i, and j. $\square$

Remark 3.2.8. The proof has established that if the $c_{i,t}$'s are L-Lipschitz for some $t \in \mathbb{R}^m$ then $\xi_{p,t}$ is $2L$-Lipschitz (for this value of t) for all $p \leq l$ (since we only complete the collection with the families $(c_{i,t} + c_{j,t})$, $(c_{i,t} - c_{j,t})$, and $\frac{c_{i,t}+c_{j,t}}{2}$).

proof of Theorem 3.2.4. We shall actually prove by induction on n the following stronger statements:

$(\mathbf{H_n})$. Let $A_1, \dots, A_\kappa$ in $\mathcal{S}_{m+n}$ and let $\eta_1, \dots, \eta_l$ be some definable nonnegative functions on $\mathbb{R}^{m+n}$. There is a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for each $B \in \mathcal{P}$, there is a definable family of metric triangulations $\Psi_t : |K| \to \mathbb{R}^n$, $t \in B$, of $\mathbb{R}^n$ compatible with $A_{1,t}, \dots, A_{\kappa,t}$ and such that for each $\sigma \in K$, $t \in B$, and $i \leq l$, the function $\eta_{i,t} \circ \Psi_{t|\sigma}$ is $\sim$ to a standard simplicial function $v_{i,\sigma} : \sigma \to \mathbb{R}$. Moreover, the contraction functions and the tame systems of coordinates of Ψ_t, as well as the functions $v_{i,\sigma}$, are independent of $t \in B$.

For $n = 0$ the result is clear. Take $n \in \mathbb{N}^*$, assume $(\mathbf{H_{n-1}})$, and fix $A_1, \dots, A_\kappa$ in $\mathcal{S}_{m+n}$ as well as some definable nonnegative functions $\eta_1, \dots, \eta_l$ on $\mathbb{R}^{m+n}$.

We denote by $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection omitting the last coordinate. Below, we sometimes regard π as a projection $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n-1}$ (still omitting the last coordinate). We also sometimes take for granted that a family of functions $\xi_t : \mathbb{R}^{n-1} \to \mathbb{R}$, $t \in \mathbb{R}^m$, gives rise to a map $\xi : \mathbb{R}^{m+n-1} \to \mathbb{R}^m \times \mathbb{R}$, $(t, x) \mapsto (t, \xi_t(x))$.

By Lemma 3.2.5, there is a definable partition $\mathcal{P}$ of $\mathbb{R}^{m+n}$ and some definable subsets $W_1, \dots, W_k$ of $\mathbb{R}^{m+n}$ such that for every $C \in \mathcal{P}$ and each $i \leq l$, we can find some rational numbers $r_1, \dots, r_k$ such that for $x \in C_t$ (for each $t \in \mathbb{R}^m$):

$$\eta_{i,t}(x) \sim d(x, W_{1,t})^{r_1} \cdots d(x, W_{k,t})^{r_k}. \tag{3.2.5}$$

By Theorem 3.1.4, there is a uniformly bi-Lipschitz definable family of mappings (that we will identify with the identity) such that for each $t \in \mathbb{R}^m$ the respective images under this family of homeomorphisms of the $\delta A_{i,t}$'s and the $\delta W_{i,t}$'s, as well as the images of the sets δC_t, $C \in \mathcal{P}$, are included in the union of a finite number of graphs of definable (families of) Lipschitz functions $\theta_{1,t}, \dots, \theta_{\lambda,t}$ defined on $\mathbb{R}^{n-1}$ (see Corollary 3.1.21 and Remark 3.1.2).

By Lemma 3.2.7 (applied to the $\theta_{j,t}$'s and to the families of $(n-1)$-variable functions $\mathbb{R}^{n-1} \ni \tilde{x} \mapsto d(\tilde{x}, \pi(\delta W_{i,t} \cap \Gamma_{\theta_{j,t}})))$, there exist a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{m+n-1}$ and finitely many Lipschitz functions $\xi_{1,t} \leq \cdots \leq \xi_{N,t}$ such that for every $D \in \mathcal{D}$ and $t \in \mathbb{R}^m$, all the functions $|x_n - \theta_{j,t}(\tilde{x})|$, $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, are comparable with each other and comparable with the functions $\mathbb{R}^{n-1} \ni \tilde{x} \mapsto d(\tilde{x}, \pi(\delta W_{i,t} \cap \Gamma_{\theta_{j,t}}))$, $i \leq k$, $j \leq \lambda$, on the set $(\xi_{\iota,t|D_t}, \xi_{\iota+1,t|D_t})$ for all $\iota \leq N$. Adding some graphs if necessary, we can assume that $\bigcup_{i=1}^N \Gamma_{\xi_i} \supset \bigcup_{i=1}^\lambda \Gamma_{\theta_i}$.

Consider a cell decomposition $\mathcal{C}$ of $\mathbb{R}^{m+n}$ compatible with the Γ_{ξ_i}'s, the $A_{i,t}$'s, and the $W_{i,t}$'s. Take then a common refinement $\mathcal{E}$ of $\pi(\mathcal{C})$ and $\mathcal{D}$ such that all the functions $(\xi_{i+1} - \xi_i)$ and $(\xi_i - \theta_j)$ have constant sign (in $\{-1, 0, 1\}$, see Definition

1.6.1) on every cell and apply ($\mathbf{H_{n-1}}$) to the cells of $\mathcal{E} \subset \mathcal{S}_{m+(n-1)}$ to get a partition $\mathcal{P}$. Fix B in $\mathcal{P}$ and $t \in B$ (we argue for one single fixed t, the proof of the definability of the family of triangulations is easy and will be left to the reader); there is (for this fixed value ot t) a metric triangulation (K, Ψ_t) of $\mathbb{R}^{n-1}$ compatible with all the sets E_t, $E \in \mathcal{E}$.

Let $\zeta_1 \leq \cdots \leq \zeta_N$ be piecewise linear functions over $|K|$ such that for every i, $\zeta_i \equiv \zeta_{i+1}$ on the set $\{\xi_{i,t} \circ \Psi_t = \xi_{i+1,t} \circ \Psi_t\}$ (this set is a subcomplex of K since $(\xi_{i+1} - \xi_i)$ has constant sign on every cell of $\mathcal{E}$). Let also $\zeta_0 := \zeta_1 - 1$ and $\zeta_{N+1} := \zeta_N + 1$, as well as

$$Z := \{(p, y) \in |K| \times \mathbb{R} \;:\; \zeta_0(p) < y < \zeta_{N+1}(p)\}.$$

We obtain a polyhedral decomposition of Z by taking the respective inverse images by $\pi_{|Z}$ of the simplices of K as well as all the images of the simplices of $|K|$ by the mappings $p \mapsto (p, \zeta_i(p))$, $0 \leq i \leq N+1$. After a barycentric subdivision of this polyhedra, we get a simplicial complex $\widehat{K}$ satisfying $|\widehat{K}| = Z$.

Define now the desired homeomorphism $\widehat{\Psi}_t$ in the following way:

$$\widehat{\Psi}_t(p, \nu\,\zeta_i(p) + (1-\nu)\zeta_{i+1}(p)) := (\Psi_t(p), \nu\,\xi_{i,t} \circ \Psi_t(p) + (1-\nu)\xi_{i+1,t} \circ \Psi_t(p))$$

for $1 \leq i \leq N-1$, $p \in |K|$ and $\nu \in [0,1]$. For $p \in |K|$ and $\nu \in [0,1)$, set:

$$\widehat{\Psi}_t(p, \nu\,\zeta_0(p) + (1-\nu)\,\zeta_1(p)) := (\Psi_t(p), \xi_{1,t}(\Psi_t(p)) - \frac{\nu}{1-\nu})$$

as well as

$$\widehat{\Psi}_t(p, \nu\,\zeta_{N+1}(p) + (1-\nu)\,\zeta_N(p)) := (\Psi_t(p), \xi_{N,t}(\Psi_t(p)) + \frac{\nu}{1-\nu}).$$

This clearly defines a homeomorphism $\widehat{\Psi}_t : |\widehat{K}| \to \mathbb{R}^n$. By construction, the $A_{j,t}$'s and the $W_{i,t}$'s are images of open simplices.

We shall check that, over each simplex $\sigma \in \widehat{K}$, the mapping $\widehat{\Psi}_t$ fulfills (3.2.3) (for some functions $\varphi_{\sigma,i}$ and some tame systems of coordinates that we shall introduce). Let $\sigma \in \widehat{K}$ and let τ be the simplex of K containing $\pi(\sigma)$.

Thanks to the induction hypothesis, we may find some functions $\varphi_{\tau,1}, \dots, \varphi_{\tau,n-1}$ and a tame system of coordinates $(\mu_{\tau,1}, \dots, \mu_{\tau,n-1})$ such that for any p and p' in τ:

$$|\Psi_t(p) - \Psi_t(p')| \sim \sum_{j=1}^{n-1} \varphi_{\tau,j}(p,p')|\mu_{\tau,j}(p) - \mu_{\tau,j}(p')|. \tag{3.2.6}$$

There is $0 \leq i \leq N$ such that $\sigma \subset [\zeta_i, \zeta_{i+1}]$. If $\sigma \subset \Gamma_{\zeta_i}$ or $\sigma \subset \Gamma_{\zeta_{i+1}}$ then the result follows from (3.2.6) and the Lipschitzness of the ξ_i's. Otherwise, let q and q' be two points of σ. These two points may be expressed

$$q = (p, \nu\zeta_i(p) + (1-\nu)\zeta_{i+1}(p)) \quad \text{and} \quad q' = (p', \nu'\zeta_i(p') + (1-\nu')\zeta_{i+1}(p')),$$

for some (p, p') in $\tau \times \tau$ and (ν, ν') in $(0,1)^2$. Define then

$$q'' := (p, \nu' \zeta_i(p) + (1-\nu')\zeta_{i+1}(p)).$$

We begin with the case where $1 \le i \le N-1$. We may consider ν, ν', p, p', and q'' as functions of q and q'. Due to the definition of $\widehat{\Psi}_t$, since $\xi_{i,t}$ and $\xi_{i+1,t}$ are Lipschitz functions, we must have over $\sigma \times \sigma$:

$$|\hat{\Psi}_t(q) - \hat{\Psi}_t(q')| \sim |\hat{\Psi}_t(q) - \hat{\Psi}_t(q'')| + |\Psi_t(p) - \Psi_t(p')|. \tag{3.2.7}$$

As $\pi(q) = \pi(q'')$, by definition of $\widehat{\Psi}_t$, we have:

$$|\hat{\Psi}_t(q) - \hat{\Psi}_t(q'')| \sim (\xi_{i+1,t}(\Psi_t(p)) - \xi_{i,t}(\Psi_t(p))) \cdot |\nu - \nu'|.$$

Thanks to the induction hypothesis, we can assume that the triangulation (K, Ψ_t) is such that $(\xi_{i+1,t} - \xi_{i,t}) \circ \Psi_t$ is $\sim$ to a standard simplicial function on τ. The composite $(\xi_{i+1,t} - \xi_{i,t}) \circ \Psi_t \circ \pi$ is thus $\sim$ to a standard simplicial function on σ that we will denote by $\varphi_{n,\sigma}$. The functions ζ_i and ζ_{i+1} define a tame coordinate of $\mathbb{R}^n$ (as in (3.2.1)) that we will denote by $\mu_{\sigma,n}$. Observe that $\nu = \mu_{\sigma,n}(q)$ and $\nu' = \mu_{\sigma,n}(q')$. The preceding estimate may thus be rewritten as:

$$|\hat{\Psi}_t(q) - \hat{\Psi}_t(q'')| \sim \varphi_{\sigma,n}(q) \,\cdot\, |\mu_{\sigma,n}(q) - \mu_{\sigma,n}(q')\,|. \tag{3.2.8}$$

Define for $j < n$:

$$\varphi_{\sigma,j}(q, q') = \varphi_{\tau,j}(\pi(q), \pi(q')) \quad \text{and} \quad \mu_{\sigma,j}(q) := \mu_{\tau,j}(\pi(q)).$$

Then by (3.2.6), (3.2.7), and (3.2.8), we get the desired equivalence (in the case $1 \le i \le N-1$). Observe that the introduced function $\varphi_{\sigma,n}$ just depends on q. In the case $i = 0$ or N, it will depend on both q and q'.

Let us now focus on the case $i = 0$ (the case $i = N$ is completely analogous and will be left to the reader). Note that we have by construction over $\sigma \times \sigma$:

$$|\hat{\Psi}_t(q) - \hat{\Psi}_t(q'')| \sim \frac{1}{(q_n - \zeta_0(p))(q'_n - \zeta_0(p'))} \cdot |\nu - \nu'|, \tag{3.2.9}$$

where q_n and q'_n respectively denote the last coordinate of q and q'. Remark also that $|\nu - \nu'|$ is $\sim$ to the difference of the tame coordinates of q and q'' defined by ζ_0 and ζ_1 (as in (3.2.1)). As $|q_n - \zeta_0(p)| \sim d(q, \Gamma_{\zeta_0})$, which is $\sim$ to a standard simplicial function on σ, we may apply the same argument as in the case $1 \le i \le N-1$, which completes the case $i = 0$. Since we can do the same job in the case $i = N$, this yields (3.2.3) for $\widehat{\Psi}_t$.

It remains to check that we can assume that the functions $\eta_{j,t} \circ \widehat{\Psi}_t$ are $\sim$ to standard simplicial functions over any $\sigma \in \widehat{K}$. For this purpose, let us fix $\sigma \in \widehat{K}$. If

$\widehat{\Psi}_t(\sigma) \subset \Gamma_{\xi_{i,t}}$ for some i, the result follows by induction. So, assume that $\widehat{\Psi}_t(\sigma) \subset (\xi_{i,t}, \xi_{i+1,t})$, for some $0 \leq i \leq N$ (setting $\xi_{0,t} \equiv -\infty$ and $\xi_{N+1,t} \equiv +\infty$).

The set $\widehat{\Psi}_t(\sigma)$ thus must fit in some C_t, with $C \in \mathcal{P}$ (we recall that for each $C \in \mathcal{P}$ we have $\delta C_t \subset \cup_{\iota=1}^{\lambda} \Gamma_{\theta_{\iota,t}} \subset \cup_{\iota=1}^{N} \Gamma_{\xi_{\iota,t}}$, which is therefore disjoint from $\widehat{\Psi}_t(\sigma) \subset (\xi_{i,t}, \xi_{i+1,t})$), and hence the $\eta_{j,t}$'s are $\sim$ on $\widehat{\Psi}_t(\sigma)$ to products of powers of distances to the $W_{j,t}$'s (see (3.2.5)). It thus suffices to show the result for the functions $q \mapsto d(\widehat{\Psi}_t(q), W_{j,t})$, $j = 1, \dots, k$. Fix $j \leq k$.

As $(\widehat{\Psi}_t, \widehat{K})$ is compatible with $W_{j,t}$, either $\widehat{\Psi}_t(\sigma) \subset W_{j,t}$ or $d(x, W_{j,t}) = d(x, \delta W_{j,t})$ for all $x \in \widehat{\Psi}_t(\sigma)$. In the former case, the result is obvious since $q \mapsto d(\widehat{\Psi}_t(q), W_{j,t})$ is zero over σ. We thus can suppose that $d(x, W_{j,t}) = d(x, \delta W_{j,t})$ on $\widehat{\Psi}_t(\sigma)$.

By definition of the $\theta_{\iota,t}$'s, $\delta W_{j,t} \subset \cup_{\iota=1}^{\lambda} \Gamma_{\theta_{\iota,t}}$. Moreover, for each $\iota \in \{1, \dots, \lambda\}$, since $\theta_{\iota,t}$ is Lipschitz, we have for $x = (\tilde{x}, x_n) \in \widehat{\Psi}_t(\sigma) \subset \mathbb{R}^{n-1} \times \mathbb{R}$:

$$d(x, \delta W_{j,t} \cap \Gamma_{\theta_{\iota,t}}) \sim |x_n - \theta_{\iota,t}(\tilde{x})| + d(\tilde{x}, \pi(\delta W_{j,t} \cap \Gamma_{\theta_{\iota,t}})). \tag{3.2.10}$$

As both terms of the right-hand-side are positive, the sum is $\sim$ to the max of these two terms, that is to say, is $\sim$ to one of them since they are comparable over $\widehat{\Psi}_t(\sigma)$ (thanks to the definition of the ξ_i's). Note that clearly $d(x, \delta W_{j,t}) = \min\limits_{1 \leq \iota \leq \lambda} d(x, \delta W_{j,t} \cap \Gamma_{\theta_{\iota,t}})$. But as by construction the functions

$$g_{\iota,t}(\tilde{x}) := d(\tilde{x}, \pi(\delta W_{j,t} \cap \Gamma_{\theta_{\iota,t}}))$$

are comparable with each other and comparable with all the functions $x = (\tilde{x}, x_n) \mapsto |x_n - \theta_{\iota,t}(\tilde{x})|$ on $\widehat{\Psi}_t(\sigma)$, by (3.2.10), the function $d(x, \delta W_{j,t})$ is equivalent over $\widehat{\Psi}_t(\sigma)$ to one of the functions $g_{\iota,t}(\tilde{x})$ or to some function $|x_n - \theta_{\iota,t}(\tilde{x})|$.

Thanks to the induction hypothesis, we can assume that the triangulation (Ψ_t, K) is such that for each $\iota \leq \lambda$ the function $g_{\iota,t} \circ \Psi_t$ is $\sim$ to a standard simplicial function on τ. Hence, by the preceding paragraph, it only remains to prove that the functions $q = (p, q_n) \mapsto |\widehat{\Psi}_{t,n}(q) - \theta_{\iota,t}(\Psi_t(p))|$ (setting $\widehat{\Psi}_t := (\Psi_t, \widehat{\Psi}_{t,n})$) are $\sim$ over σ to a standard simplicial function (independent of t).

For this purpose, fix a positive integer $\iota \leq \lambda$. As $\Gamma_{\theta_{\iota,t}} \subset \bigcup_{j=1}^{N} \Gamma_{\xi_{j,t}}$, we have on $\pi(\widehat{\Psi}_t(\sigma))$ either $\theta_{\iota,t} \geq \xi_{i+1,t}$ or $\theta_{\iota,t} \leq \xi_{i,t}$. For simplicity, we will assume that the latter inequality holds. On σ we have:

$$|\widehat{\Psi}_{t,n} - \theta_{\iota,t} \circ \Psi_t| = (\widehat{\Psi}_{t,n} - \xi_{i,t} \circ \Psi_t) + (\xi_{i,t} \circ \Psi_t - \theta_{\iota,t} \circ \Psi_t). \tag{3.2.11}$$

In the case $0 < i < N$, by (3.2.8), we have over σ for $q = (p, q_n)$:

$$\hat{\Psi}_{t,n}(q) - \xi_{i,t}(\Psi_t(p)) \sim \mu_{\sigma,n}(q) \cdot \varphi_{\sigma,n}(q). \tag{3.2.12}$$

Moreover, by (3.2.9) (with $\nu' = 0$), a similar estimate holds in the case $i = 0$ (or N). Observe also that $\mu_{\sigma,n}$ is obviously $\sim$ to a standard simplicial function. Equivalence

(3.2.12) therefore yields that the first summand of the right-hand-side of (3.2.11) is $\sim$ to a standard simplicial function. Since by induction we can assume that so is $|\xi_{i,t} \circ \Psi_t - \theta_{\iota,t} \circ \Psi_t|$, the conclusion thus comes down from (3.2.11). □

Remark 3.2.9. It is worthy of notice that the above proof has explicitly shown that if σ is an open simplex such that $\Psi(\sigma)$ is bounded then it is not necessary to involve in the definition of metric triangulations standard simplicial functions depending on both q and q'. More precisely, instead of (3.2.3), it is enough to require on such σ:

$$|\Psi(q) - \Psi(q')| \sim \sum_{i=1}^{n} \varphi_{\sigma,i}(q) \cdot |\mu_{\sigma,i}(q) - \mu_{\sigma,i}(q')|, \tag{3.2.13}$$

with $\varphi_{\sigma,i}(q)$ standard simplicial functions on σ for each i. Moreover, for such a simplex σ, the simplices involved in the expression of $\varphi_{\sigma,i}$ are by construction of dimension $(i-2)$ for all $i \geq 2$, and $\varphi_{\sigma,1} \equiv 1$. In addition to this, for such a simplex σ, the contraction functions $\varphi_{\sigma,i}$ can be required to be bounded away from infinity.

Remark 3.2.10. By definition, each tame coordinate is characterized by two linear functions. In the proof of the above theorem, for every simplex σ, the respective graphs of the linear functions defining $(\mu_{\sigma,1}, \dots, \mu_{\sigma,n})$ are images under canonical projections of simplices of the constructed simplicial complex.

A consequence of Theorem 3.2.4 is the following corollary that tells us how many classes subanalytic bi-Lipschitz equivalence admits.

Corollary 3.2.11. *Up to globally subanalytic bi-Lipschitz mappings, globally subanalytic sets are countable.*

Proof. Up to globally subanalytic bi-Lipschitz mappings, globally subanalytic sets are clearly at least countable. Let us show that they are at most countable.

By Proposition 3.2.3, two sets that can be triangulated by the same simplicial complex K, the same contraction functions $\varphi_{\sigma,i}$, and the same tame systems of coordinates are definably bi-Lipschitz homeomorphic.

The vertices of the simplicial complex provided by Theorem 3.2.4 can be chosen in $\mathbb{Q}^n$. The class of such finite simplicial complexes is countable. Moreover, given such a simplicial complex, the tame systems of coordinates of the constructed triangulation in the proof of the latter theorem can be chosen among a finite family (see Remark 3.2.10). The contractions are given by finitely many combinations of sums, products, and powers of distances to the faces. As these powers belong to $\mathbb{Q}$, the standard simplicial functions of such simplicial complexes are countable. □

We are going to derive other consequences of Theorem 3.2.4. We will say that $A \in \mathcal{S}_{m+n}$ is **definably bi-Lipschitz trivial along** $U \subset \mathbb{R}^m$ if there exist $t_0 \in U$ and a definable family of bi-Lipschitz homeomorphisms $h_t : A_{t_0} \to A_t$, $t \in U$. As a byproduct of Theorem 3.2.4 and Proposition 3.2.3, we have:

Corollary 3.2.12. *Given $A \in \mathcal{S}_{m+n}$, there exists a definable partition of $\mathbb{R}^m$ such that A is definably bi-Lipschitz trivial along each element of this partition.*

Remark 3.2.13. Since Theorem 3.2.4 ensures that we can triangulate several families simultaneously, we can trivialize several definable families simultaneously. Namely, given some definable subsets $B_1, \dots, B_k$ of A, possibly refining the partition provided by the above corollary, we can require the trivialization of A that we have along every element of this partition to be also a trivialization of the B_i's.

We are going to see that, refining the partition given by this corollary if necessary, we can require the trivialization h_t to be bi-Lipschitz with respect to t on compact sets. This will yield that definable local bi-Lipschitz triviality is a stratifying condition for stratifications (Corollary 3.2.16). This requires the following proposition which can be regarded as a Lipschitz version of Proposition 2.3.6.

Proposition 3.2.14. *Let $A \in \mathcal{S}_{m+n}$ and let $f_t : A_t \to \mathbb{R}$ be a definable family of functions. If f_t is Lipschitz for all $t \in \mathbb{R}^m$ then there exists a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathcal{P}$, $f : A \to \mathbb{R}$, $(t, x) \mapsto f_t(x)$ induces a Lipschitz function on $A \cap K$, for every compact subset K of $B \times \mathbb{R}^n$.*

Proof. We prove the result by induction on m. The case $m = 0$ being vacuous, assume the result to be true for $(m-1)$, $m \geq 1$. By Proposition 3.1.1 (see Remark 3.1.2), we may assume that $A = \mathbb{R}^{m+n}$. By Proposition 2.3.6, there is a definable partition $\mathcal{P}$ of the parameter space, such that f is continuous on every $B \times \mathbb{R}^n$, $B \in \mathcal{P}$. Fix an element $B \in \mathcal{P}$ (we shall refine several times the partition $\mathcal{P}$).

We start with the (easier) case where $\dim B < m$. In this case, by Lemmas 3.1.13 and 3.1.10, there is a partition of B such that every element of this partition has a regular vector, that, without loss of generality, we can assume to be $e_m \in \mathbf{S}^{m-1}$. Thanks to Corollary 3.1.21, it is therefore enough to deal with the case where B is the graph of a Lipschitz function, say $\xi : D \to \mathbb{R}$, $D \in \mathcal{S}_{m-1}$. The result in this case now follows from the induction hypothesis applied to the function $D \times \mathbb{R}^n \ni (t, x) \mapsto f(t, \xi(t), x)$.

We now address the case $\dim B = m$. The function $B \ni t \mapsto L_{f_t}$ being definable, partitioning B if necessary, we can assume this function to be continuous on this set. In particular, it is bounded on compact subsets of B. Let Z be the set of points $q \in \Gamma_f$ for which there exists a sequence $q_k \in (\Gamma_f)_{reg}$ tending to q such that

$$(0_{\mathbb{R}^m}, e_{n+1}) \in \lim T_{q_k}(\Gamma_f)_{reg},$$

where e_{n+1} is the last vector of the canonical basis of $\mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^m \times \mathbb{R}^{n+1} \to \mathbb{R}^m$ denote the projection omitting the last $(n+1)$ coordinates. We claim that $\pi(Z)$ has dimension less than m.

Assume otherwise. Take a Whitney (a)-regular definable stratification of Γ_f compatible with Z and let $S \subset Z$ be a stratum such that $\pi(S)$ has dimension m. Let

S' be the set of points of S at which $\pi_{|S}$ is a submersion. Since $\pi(S)$ is of dimension m, by Sard's Theorem, the set S' cannot be empty. Moreover, by definition of S', $T_q S'$ is transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ at any point q of S'. Let $q \in S' \subset Z$. By definition of Z, there is a sequence q_k tending to q such that $(0_{\mathbb{R}^m}, e_{n+1}) \in \tau_q := \lim T_{q_k}(\Gamma_f)_{reg}$, and Whitney (a) condition ensures that $\tau_q \supset T_q S'$. Consequently, τ_q is transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ as well. But since L_{f_t} is locally bounded (it was assumed to be continuous), the vector e_{n+1} does not belong to $\lim T_{x_k} \Gamma_{f_{t_k}}$, if $q_k = (t_k, x_k)$ in $\mathbb{R}^m \times \mathbb{R}^{n+1}$ (extracting a sequence if necessary, we can assume that this limit exits), which means that

$$(\lim T_{q_k} \Gamma_f) \cap \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1} \neq \lim \left(T_{q_k} \Gamma_f \cap \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}\right)$$

(since the latter does not contain the vector $(0_{\mathbb{R}^m}, e_{n+1})$ while the former does), and hence, that τ_q cannot be transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ (since otherwise the intersection with the limit would be the limit of the intersection). A contradiction.

This establishes that $\dim \pi(Z) < m$. Since we can refine $\mathcal{P}$ into a partition which is compatible with $\pi(Z)$, we thus see that we can suppose $B \subset \mathbb{R}^m \setminus \pi(Z)$ (we are dealing with the case $\dim B = m$). For $(t, R) \in (B \setminus \pi(Z)) \times [0, +\infty)$ set:

$$\varphi(t, R) := \sup\{|\frac{\partial f}{\partial t}(t, x)| : x \in \mathbf{B}(0_{\mathbb{R}^n}, R),\ f \text{ is } \mathscr{C}^1 \text{ at } (t, x)\}$$

(which is finite, by definition of Z). As φ is definable, up to a partition of B, this function may be assumed to be continuous (and thus bounded on compact sets) for $R \geq \zeta(t)$, with $\zeta : B \to \mathbb{R}$ definable function. The function f therefore induces a function which is Lipschitz with respect to the inner metric on every compact set of $B \times \mathbb{R}^n$. By Corollary 3.1.19, up to an extra refinement of the partition, we can suppose the inner and outer metrics of B to be equivalent, which means that so are the inner and outer metrics of $B \times \mathbb{R}^n$, establishing that f is Lipschitz on every compact set of $B \times \mathbb{R}^n$. $\square$

Definition 3.2.15. A stratification Σ of a set X is **locally definably bi-Lipschitz trivial at** $x_0 \in S \in \Sigma$ if there are a tubular neighborhood (V_S, π_S) of S (see Proposition 2.4.1) and an open neighborhood W of x_0 in S for which there is a definable bi-Lipschitz homeomorphism $\Lambda : \pi_S^{-1}(W) \to \pi_S^{-1}(x_0) \times W$ satisfying:

(i) $\pi_S(\Lambda^{-1}(x, y)) = y$, for all $(x, y) \in \pi_S^{-1}(x_0) \times W$.

(ii) $\Sigma_{x_0} := \{\pi_S^{-1}(x_0) \cap Y : Y \in \Sigma\}$ is a stratification of $\pi_S^{-1}(x_0)$, and $\Lambda(\pi_S^{-1}(W) \cap Y) = (\pi_S^{-1}(x_0) \cap Y) \times W$, for all $Y \in \Sigma$.

We now can derive the following consequence of Theorem 3.2.4:

Corollary 3.2.16. *Being locally definably bi-Lipschitz trivial is a local stratifying condition. Consequently, every stratification can be refined into a stratification having this property.*

Proof. This condition is obviously local. Note that if S is a stratum of a stratification Σ of a set X and (V_S, π_S, ρ_S) is a tubular neighborhood of S then the family $\pi_S^{-1}(t) \cap X$, $t \in S$, is definable. By Corollary 3.2.12, it must be definably bi-Lipschitz trivial along each element of a definable partition $\mathcal{P}$ of S. The trivialization can be assumed to be a trivialization of every stratum (see Remark 3.2.13). By Proposition 3.2.14, the trivialization is locally bi-Lipschitz with respect to $t \in S$, after a possible refinement of the partition $\mathcal{P}$. By Sard's Theorem, there is a nowhere dense definable subset Z of S such that $S \setminus Z$ contains no singular value of the restriction of π_S to the strata (i.e. no point x_0 of type $x_0 = \pi_S(z)$ with $z \in Y \in \Sigma$ and $\pi_{S|Y} : Y \to S$ non submersive at z), which means that Σ_{x_0} (see Definition 3.2.15 (ii)) is a stratification of $\pi^{-1}(x_0)$ for every $x_0 \in S \setminus Z$. The elements of $\mathcal{P}$ that are open in S constitute together an open dense subset U of S, and the stratification Σ is locally definably bi-Lipschitz trivial at every point of $U \setminus Z$, which yields the first sentence. The last sentence of the corollary is due to Proposition 2.6.6. $\square$

3.3 Metric triangulations: a local version

We are going to prove a stronger version of our metric triangulation theorem for germs of definable sets (Theorem 3.3.2). In section 3.4, we will rely on this so as to describe the "Lipschitz conic structure" of definable germs (Theorem 3.4.1), which will yield their "Lipschitz contractibility" (Corollary 3.4.5) as well as the definable bi-Lipschitz invariance of their link (Corollary 3.4.18).

The idea is that, since $X \in \mathcal{S}_n$ is definably bi-Lipschitz homeomorphic to

$$\check{X} := \{(t, x) \in \mathbb{R} \times X : t = |x|\}, \tag{3.3.1}$$

which is a subset of $\mathcal{C}_{n+1}(1)$ (see (3.1.2) for $\mathcal{C}_n(R)$), the study of germs of subsets of $\mathbb{R}^n$ reduces to that of germs of subsets of $\mathcal{C}_{n+1}(1)$ (see the proof of Theorems 3.4.1 and 3.4.14). We thus will focus in this section on germs at 0 of subsets of $\mathcal{C}_n(R)$, $R > 0$. This amounts to replace the function $x \mapsto |x|$ with the function $\rho(x_1, \dots, x_n) = x_1$, which fits better to triangulation problems for it is a linear function.

In the case of germs of definable subsets of $\mathcal{C}_n(R)$, $R > 0$, we are going to construct some definable metric triangulations having specific properties. It was already pointed out in Remark 3.2.9 that when the triangulated set is bounded, it is not necessary to involve contraction functions $\varphi_{\sigma,i}$ that depend on both q and q'. It is therefore not surprising that in the case of triangulations of germs of definable subsets of $\mathcal{C}_n(R)$, we will just need contractions that are standard simplicial functions on σ, for every simplex σ. But the main improvement provided by this local version is that these contractions will be decreasing faster than the distance to the origin as we are drawing near this point (up to some constant, see Definition 3.3.1 and Theorem 3.3.2). These facts will be essential in section 3.4.

Definition 3.3.1. Let σ be an open simplex of $\mathbb{R}^n$ with $0 \in cl(\sigma)$. A nonnegative function φ on σ is **subhomogeneous** if there is a constant C such that for all $s \in (0,1]$ and $q \in \sigma$ we have $\varphi(sq) \leq Cs\varphi(q)$.

If $\mu = (\mu_1, \dots, \mu_n)$ is a tame system of coordinates on σ and $i \in \{1, \dots, n\}$, we say that μ_i is **radially constant** if $\mu_i(sq) = \mu_i(q)$, for all $s \in (0,1]$ and $q \in \sigma$.

In the theorem below, $X_{[0,\varepsilon]}$ stands for the restriction of X to $[0,\varepsilon]$ (see (2.3.1)).

Theorem 3.3.2. *Let X be a definable subset of $\mathcal{C}_n(R)$, $R > 0$. For $\varepsilon > 0$ small enough, there is a metric triangulation $\Psi : (|K|, 0) \to (X_{[0,\varepsilon]}, 0)$ satisfying (3.2.13) and such that:*

(i) Ψ is a vertical Lipschitz mapping.

(ii) For each $\sigma \in K$ and each $2 \leq i \leq n$, the contraction function $\varphi_{\sigma,i}$ (see (3.2.13)) is a bounded subhomogeneous standard simplicial function on σ and the tame coordinate $\mu_{\sigma,i}$ is radially constant. Moreover, $\varphi_{\sigma,1} \equiv 1$.

Furthermore, we may require this triangulation to be compatible with finitely many given germs of definable subsets of X.

The proof of this theorem occupies the remaining part of this section.

A preliminary lemma. We denote by $\mathcal{S}_{n,0}$ the set constituted by all the germs at the origin of definable subsets of $\mathbb{R}^n$. In the lemma below, all the considered germs are germs at the origin.

Lemma 3.3.3. *Let $A_1, \dots, A_\kappa$ be germs of definable subsets of $\mathcal{C}_n(R)$, $R > 0$, and $\eta_1, \dots, \eta_l$ be germs of nonnegative definable functions on $\mathcal{C}_n(R)$. There exist a germ of definable vertical bi-Lipschitz homeomorphism (onto its image) $H : (\mathcal{C}_n(R), 0) \to (\mathcal{C}_n(R), 0)$ and a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n$ such that:*

(i) $\mathcal{D}$ is compatible with (some representatives of the germs of) $H(A_1), \dots, H(A_\kappa)$.

(ii) Every cell of $\mathcal{D}$ is either a band or the graph of a Lipschitz function.

(iii) On each cell D of $\mathcal{D}$, every germ $\eta_i \circ H^{-1}$ is $\sim$ to a function of the form:

$$|x_n - \theta(\tilde{x})|^r a(\tilde{x}), \qquad (\tilde{x}, x_n) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}, \tag{3.3.2}$$

where a is a definable function of constant sign, θ is a Lipschitz definable function (on the basis of D), and $r \in \mathbb{Q}$.

Proof. Apply Lemma 3.2.5 (with $m = 0$ and $A = \mathcal{C}_n(R)$) to each of the functions η_j, $j = 1, \dots, l$ and take a common refinement of the obtained partitions. This provides a finite partition $V_1, \dots, V_b$ of $\mathcal{C}_n(R)$ together with some definable subsets $W_1, \dots, W_c$ of $\mathcal{C}_n(R)$, such that on each V_i each function η_j is equivalent to a product of powers of functions of type $x \mapsto d(x, W_k)$, $k \le c$.

Possibly refining the partition $V_1, \dots, V_b$, we may assume that the W_k's are unions of elements of this partition. The function $d(x, W_k)$ is then on V_i for each k and i either identically 0 or equal to $d(x, \delta W_k)$. We therefore can suppose that the W_k's have empty interior, possibly replacing them with the δW_k's (if a function η_j is identically zero on some V_i then (iii) is trivial on this set).

Apply now Theorem 3.1.5 to the union of the δA_i's, the δV_i's, and the W_i's. This provides a germ of vertical bi-Lipschitz homeomorphism onto its image $H : (\mathcal{C}_n(R), 0) \to (\mathcal{C}_n(R), 0)$ such that e_n is regular for the respective images of these sets under H. It means that these sets are sent by H into the union of the graphs of some definable Lipschitz functions $\theta_1 \le \cdots \le \theta_d$ defined on $\mathbb{R}^{n-1}$ (see Remark 3.1.22).

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the canonical projection. Thanks to Lemma 3.2.7 (applied to the θ_i's and to all the $(n-1)$-variable functions $\tilde{x} \mapsto d(\tilde{x}, \pi(W_k \cap \Gamma_{\theta_i}))$ with $m = 0$), we know that there exist some definable functions $\xi_1 \le \cdots \le \xi_p$ and a cell decomposition $\mathbb{R}^{n-1}$, say $\mathcal{E}$, such that for every $E \in \mathcal{E}$ and over each $[\xi_{i,|E}, \xi_{i+1,|E}]$, $i < p$, the family of functions

$$|x_n - \theta_\nu(\tilde{x})|, \; d(\tilde{x}, \pi(W_k \cap \Gamma_{\theta_\nu})), \; (\theta_\nu - \theta_{\nu'})(\tilde{x}), \quad \nu' < \nu \le d, \; k \le c,$$

where $(\tilde{x}, x_n) \in [\xi_{i,|E}, \xi_{i+1,|E}] \subset \mathbb{R}^{n-1} \times \mathbb{R}$, is totally ordered. Adding some graphs if necessary, we can assume that $\bigcup_{i=1}^{p} \Gamma_{\xi_i} \supset \bigcup_{i=1}^{d} \Gamma_{\theta_i}$. Moreover, refining $\mathcal{E}$ if necessary, we can also assume $(\xi_i - \xi_{i+1})$ to have constant sign on every cell (see Definition 1.6.1) for all i, and $\mathcal{E}$ to be compatible with the cells of $\pi(\mathcal{F})$, where $\mathcal{F}$ is some cell decomposition of $\mathbb{R}^n$ compatible with the $H(A_i)$'s and the Γ_{ξ_i}'s.

The respective graphs of the restrictions of the ξ_i's to the cells of $\mathcal{E}$ now induce a cell decomposition of $\mathbb{R}^n$ compatible with the $H(A_i)$'s that we will denote by $\mathcal{D}$. Since the ξ_i's are Lipschitz functions, we already see that (i) and (ii) hold.

To prove (iii), fix a cell D of $\mathcal{D}$. If this cell is a graph, (3.3.2) is obvious. Otherwise, since $H^{-1}(D)$ is included in V_j for some $j \le b$ (for the $H(\delta V_j)$'s are included in the graphs of the $\theta_i' s$ which are cells of $\mathcal{D}$), we know that for each k the function η_k is $\sim$ on $H^{-1}(D)$ to a product of powers of functions of the finite family $x \mapsto d(x, W_k)$, $k \le c$. As H is bi-Lipschitz, this entails that for each k the function $\eta_k \circ H^{-1}$ is $\sim$ on D to a product of powers of functions of the family $x \mapsto d(x, H(W_k))$, $k \le c$. As a matter of fact, it is enough to check that each function $x \mapsto d(x, H(W_k))$, $k \in \{1, \dots, c\}$, admits an estimate like displayed in (3.3.7), for some function θ independent of k.

Fix $k \leq c$. As the θ_i's are Lipschitz functions, we have for any $\nu \in \{1, \dots, d\}$:

$$d(x, H(W_k) \cap \Gamma_{\theta_\nu}) \sim |x_n - \theta_\nu(\tilde{x})| + d(\tilde{x}, \pi(H(W_k) \cap \Gamma_{\theta_\nu})), \tag{3.3.3}$$

where $x = (\tilde{x}, x_n)$ in $\mathbb{R}^{n-1} \times \mathbb{R}$. The terms of the right-hand-side are nonnegative and comparable with each other (for partial order relation $\leq$) over the cell D (by choice of the ξ_i's). The left-hand-side is therefore $\sim$ to one of them on D.

Note that, as the $H(W_k)$'s are included in the graphs of the θ_ν's we have: $d(x, H(W_k)) = \min\limits_{1 \leq \nu \leq d} d(x, H(W_k) \cap \Gamma_{\theta_\nu})$. Hence, by (3.3.3), each function $d(x, H(W_k))$ is equivalent over D either to one of the functions $\tilde{x} \mapsto d(\tilde{x}, \pi(H(W_k) \cap \Gamma_{\theta_\nu}))$ (which is an $(n-1)$-variable function) or to $x = (\tilde{x}, x_n) \mapsto |x_n - \theta_\nu(\tilde{x})|$, for some $\nu \in \{1, \dots, d\}$. It thus only remains to check that on each cell, the same ν can be chosen for all k. But, since the finite family constituted by the functions $|x_n - \theta_\nu(\tilde{x})|$, $\nu \leq d$, together with the functions $(\theta_\nu - \theta_{\nu'})$, $\nu' < \nu \leq d$, is totally ordered on each cell, there is $\nu_0 \leq d$ (for each cell) such that each of the functions $|x_n - \theta_\nu(\tilde{x})|$, $\nu \in \{1, \dots, d\}$, is either equivalent to $|x_n - \theta_{\nu_0}(\tilde{x})|$ or to an $(n-1)$-variable function (see (1.6.2) and (1.6.3)).

That a has constant sign (in (3.3.2)) may always be obtained up to a refinement of the cell decomposition. $\square$

Proof of Theorem 3.3.2. We shall use an argument which is similar to the one we used in the proof of Theorem 3.2.4, proving inductively the following statements:

($\mathbf{H_n}$) Let X be a definable subset of $\mathcal{C}_n(R)$, $R > 0$, and $A_1, \dots, A_\kappa$ some definable subsets of X. Given finitely many bounded nonnegative definable function-germs (at the origin) $\eta_1, \dots, \eta_l$ on $\mathcal{C}_n(R)$, there exist $\varepsilon > 0$ and a metric triangulation

$$\Psi : (|K|, 0) \to (X_{[0,\varepsilon]}, 0)$$

of $X_{[0,\varepsilon]}$ compatible with $A_{1_{[0,\varepsilon]}}, \dots, A_{\kappa_{[0,\varepsilon]}}$, satisfying properties (i) and (ii) of the theorem, and such that for each $j \leq l$ the function $\eta_j \circ \Psi$ is $\sim$ to the germ of a standard simplicial function on each simplex. Moreover, if $\eta_j(x) \lesssim x_1$ for $x = (x_1, \dots, x_n) \in X_{[0,\varepsilon]}$, then we can require $\eta_j \circ \Psi$ to be subhomogeneous.

The assertion ($\mathbf{H_1}$) being trivial (Ψ is then the identity map), let us prove ($\mathbf{H_n}$), assuming ($\mathbf{H_{n-1}}$), $n > 1$. Fix $X, A_1 \dots, A_\kappa$ and $\eta_1, \dots, \eta_l$ as in ($\mathbf{H_n}$). We denote by $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the $(n-1)$ first coordinates.

Step 1. We define a $\mathscr{C}^0$ triangulation $\widehat{\Psi} : |\widehat{K}| \to X$.

Apply Lemma 3.3.3 to the family constituted by X, the A_i's, and the set $\mathcal{C}_n(R)$ itself, together with the functions $\eta_1, \dots, \eta_l$. We get a (germ of) vertical bi-Lipschitz map $H : \mathcal{C}_n(R) \to \mathcal{C}_n(R)$ and a cell decomposition $\mathcal{D}$ such that (i), (ii), and (iii) of the latter lemma hold. As we may work up to a vertical bi-Lipschitz map, we will identify H with the identity map.

By (ii) of Lemma 3.3.3, every cell of $\mathcal{D}$ included in $\mathcal{C}_n(R)$ which is not a band is the graph of a Lipschitz function. We thus can include all such cells in the union of the respective graphs of some definable Lipschitz functions $\xi_1 \leq \cdots \leq \xi_N$ defined on $\mathcal{C}_{n-1}(R)$ (see Remark 3.1.22).

Refine the cell decomposition $\pi(\mathcal{D})$ into a cell decomposition $\mathcal{F}$ compatible with the zero loci of the functions $(\xi_j - \xi_{j+1})$, $j < N$, and apply the induction hypothesis to the family constituted by the cells of $\mathcal{F}$ to get a homeomorphism $\Psi : (|K|, 0) \to (\mathcal{C}_{n-1}(R)_{[0,\varepsilon]}, 0)$, with $\varepsilon > 0$ and K simplicial complex of $\mathbb{R}^{n-1}$.

We are going to lift Ψ to a homeomorphism $\widehat{\Psi} : |\widehat{K}| \to \mathcal{C}_n(R)$. We first define the simplicial complex $\widehat{K}$ in a similar way as in the proof of Theorem 3.2.4.

Let $\zeta_1 \leq \cdots \leq \zeta_N$ be piecewise linear functions over $|K|$ such that for each i $\zeta_i \equiv \zeta_{i+1}$ on the set $\{\xi_i \circ \Psi = \xi_{i+1} \circ \Psi\}$ (this set is a subcomplex of K). Since the graphs of the ξ_i's are subsets of $\mathcal{C}_n(R)$, all these functions vanish at the origin, and we can assume (up to a translation) that so do the ζ_i's. Let also

$$Z := \{(p, y) \in |K| \times \mathbb{R} \ : \ \zeta_1(p) \leq y \leq \zeta_N(p)\}.$$

We obtain a polyhedral decomposition of Z by taking the respective inverse images by $\pi_{|Z}$ of the simplices of K as well as all the images of the simplices of K by the mappings $p \to (p, \zeta_i(p))$, $1 \leq i \leq N$. After a barycentric subdivision of this polyhedra, we get a simplicial complex $\widehat{K}$.

Define now over $\widehat{K}$ a mapping $\widehat{\Psi} : |\widehat{K}| \to \mathbb{R}^n$ in the following way:

$$\widehat{\Psi}(p, s\,\zeta_i(p) + (1-s)\zeta_{i+1}(p)) := (\Psi(p), s\,\xi_i \circ \Psi(p) + (1-s)\,\xi_{i+1} \circ \Psi(p)),$$

for $1 \leq i < N$, $p \in |K|$, and $s \in [0,1]$. By construction, this mapping is a homeomorphism onto its image. The cells of $\mathcal{D}$ that lie in $\mathcal{C}_n(R)$ are unions of images under $\widehat{\Psi}$ of open simplices. Since $\mathcal{D}$ is compatible with X and the A_i's, the restriction of $\widehat{\Psi}$ to $\widehat{\Psi}^{-1}(X)$ is a triangulation of X compatible with $A_{1_{[0,\varepsilon]}}, \dots, A_{\kappa_{[0,\varepsilon]}}$.

Let us now fix an open simplex $\sigma \in \widehat{K}$ and denote by τ the simplex of K that contains $\pi(\sigma)$. Let $i < N$ be such that $\sigma \subset [\zeta_i, \zeta_{i+1}]$.

Step 2. We check that over σ the mapping $\widehat{\Psi}$ satisfies an inequality of type (3.2.3), for some subhomogeneous standard simplicial functions $\varphi_{\sigma,1}, \dots, \varphi_{\sigma,n}$ and some tame system of coordinates μ_σ that we shall introduce.

Thanks to the induction hypothesis, we may find standard simplicial functions $\varphi_{\tau,1}, \dots, \varphi_{\tau,n-1}$ and a tame system of coordinates $(\mu_{\tau,1}, \dots, \mu_{\tau,n-1})$ such that for p and p' in τ:

$$|\Psi(p) - \Psi(p')| \sim \sum_{j=1}^{n-1} \varphi_{\tau,j}(p)|\mu_{\tau,j}(p) - \mu_{\tau,j}(p')|. \tag{3.3.4}$$

If $\sigma \subset \Gamma_{\zeta_i}$ or $\sigma \subset \Gamma_{\zeta_{i+1}}$ then the result follows from (3.3.4) and the Lipschitzness of the ξ_i's. Otherwise, let q and q' be two points of σ. Such points may be expressed

$$q = (p, s\zeta_i(p) + (1-s)\zeta_{i+1}(p)) \quad \text{and} \quad q' = (p', s'\zeta_i(p') + (1-s')\zeta_{i+1}(p'))$$

for some (p, p') in $\tau \times \tau$ and (s, s') in $(0,1)^2$, and we then can define

$$q'' := (p, s'\zeta_i(p) + (1-s')\zeta_{i+1}(p)).$$

We will consider s, s', p, p', and p'' as functions of q and q'. By definition of $\widehat{\Psi}$, since ξ_i and ξ_{i+1} are Lipschitz functions, we have over $\sigma \times \sigma$:

$$|\widehat{\Psi}(q) - \widehat{\Psi}(q')| \sim |\widehat{\Psi}(q) - \widehat{\Psi}(q'')| + |\Psi(p) - \Psi(p')|. \tag{3.3.5}$$

As $\pi(q) = \pi(q'')$, by definition of $\widehat{\Psi}$, we have:

$$|\widehat{\Psi}(q) - \widehat{\Psi}(q'')| \sim (\xi_{i+1}(\Psi(p)) - \xi_i(\Psi(p))) \cdot |s - s'|.$$

Thanks to the induction hypothesis, we can assume that the triangulation (K, Ψ) is such that $(\xi_{i+1} - \xi_i) \circ \Psi$ is $\sim$ to a subhomogeneous standard simplicial function on τ. The composite $(\xi_{i+1} - \xi_i) \circ \Psi \circ \pi$ is thus $\sim$ to a subhomogeneous standard simplicial function on σ that we will denote by $\varphi_{\sigma,n}$. The functions ζ_i and ζ_{i+1} define a tame coordinate of $\mathbb{R}^n$ (as in (3.2.1)) that we will denote by $\mu_{\sigma,n}$. Observe that $s = \mu_{\sigma,n}(q)$ and $s' = \mu_{\sigma,n}(q')$. The preceding estimate can therefore be rewritten as:

$$|\widehat{\Psi}(q) - \widehat{\Psi}(q'')| \sim \varphi_{\sigma,n}(q) \cdot |\mu_{\sigma,n}(q) - \mu_{\sigma,n}(q')\,|. \tag{3.3.6}$$

Define then for $j < n$ and $q \in \sigma$:

$$\varphi_{\sigma,j}(q) := \varphi_{\tau,j}(\pi(q)) \quad \text{and} \quad \mu_{\sigma,j}(q) := \mu_{\tau,j}(\pi(q)).$$

By (3.3.4), (3.3.5), and (3.3.6), we get the desired equivalence. Since $\zeta_i(0) = \zeta_{i+1}(0) = 0$, it is clear that $\mu_{\sigma,n}$ is radially constant (see (3.2.1)).

Step 3. We check that for each j the function $\eta_j \circ \widehat{\Psi}$ is $\sim$ to a standard simplicial function on σ (we recall that σ is a fixed element of $\widehat{K}$).

For this purpose, fix a positive integer $j \leq l$. If $\widehat{\Psi}(\sigma) \subset \Gamma_{\xi_\iota}$, with $\iota = i$ or $i+1$ (we recall that $\sigma \subset [\zeta_i, \zeta_{i+1}]$, which entails $\widehat{\Psi}(\sigma) \subset [\xi_i, \xi_{i+1}]$), then we are done since, thanks to the induction hypothesis, we can assume that $\tau \ni \tilde{q} \mapsto \eta_j\big(\Psi(\tilde{q}), \xi_\iota \circ \Psi(\tilde{q})\big)$ is $\sim$ to a standard simplicial function. Otherwise, by definition of $\widehat{\Psi}$, we must have $\widehat{\Psi}(\sigma) \subset (\xi_i, \xi_{i+1})$.

By construction, $\widehat{\Psi}(\sigma)$ is included in a cell $D \in \mathcal{D}$. By (3.3.2) we know that there are functions a and θ on the basis E of D, as well as $\alpha \in \mathbb{Q}$, such that for $x = (\tilde{x}, x_n) \in \widehat{\Psi}(\sigma)$

$$\eta_j(x) \sim |x_n - \theta(\tilde{x})|^\alpha a(\tilde{x}). \tag{3.3.7}$$

Thanks to $(\mathbf{H_{n-1}})$, we can assume that $a \circ \Psi$ is $\sim$ to a standard simplicial function. We thus merely have to check that the function $(\tilde{q}, q_n) \mapsto |\widehat{\Psi}_n(q) - \theta(\Psi(\tilde{q}))|$ is equivalent to a standard simplicial function (here we set $\widehat{\Psi} = (\Psi, \widehat{\Psi}_n)$).

As $\Gamma_\theta \subset \bigcup_{k=1}^N \Gamma_{\xi_k}$, we have on $\pi(\widehat{\Psi}(\sigma))$ either $\theta \geq \xi_{i+1}$ or $\theta \leq \xi_i$. For simplicity, we will assume that the latter inequality holds. On σ we have:

$$|\widehat{\Psi}_n - \theta \circ \Psi| = (\widehat{\Psi}_n - \xi_i \circ \Psi) + (\xi_i \circ \Psi - \theta \circ \Psi). \tag{3.3.8}$$

By (3.3.6), we have over σ for $q = (\tilde{q}, q_n)$:

$$\widehat{\Psi}_n(q) - \xi_i(\Psi(\tilde{q})) \sim \mu_{\sigma,n}(q) \cdot \varphi_{\sigma,n}(q). \tag{3.3.9}$$

The function $\mu_{\sigma,n}(q)$ is obviously $\sim$ to a standard simplicial function. Thus, as by induction $|\xi_i \circ \Psi - \theta \circ \Psi|$ can also be assumed to be equivalent to a standard simplicial function, the required fact follows from (3.3.8) and (3.3.9).

Step 4. We check that if $\eta_j(x) \lesssim x_1$ for $x = (x_1, \dots, x_n) \in \widehat{\Psi}(\sigma)$, $j \leq l$, then we can require in addition the function $\eta_j \circ \widehat{\Psi}$ to be subhomogeneous on σ.

For simplicity, we will say that a function f on $\widehat{\Psi}(\sigma)$ (resp. $\Psi(\tau)$) is **$\widehat{\Psi}$-subhomogeneous** (resp. **Ψ-subhomogeneous**) if $f \circ \widehat{\Psi}$ (resp. $f \circ \Psi$) is subhomogeneous. The induction hypothesis thus allows us to assume that finitely many given definable $(n-1)$-variable functions that are $\lesssim x_1$ near the origin are Ψ-subhomogeneous.

If $\widehat{\Psi}(\sigma) \subset \Gamma_{\xi_i}$ or $\widehat{\Psi}(\sigma) \subset \Gamma_{\xi_{i+1}}$ then the result can easily be deduced from the induction hypothesis. Otherwise, recall that we have assumed that $\theta \leq \xi_i$ (right before (3.3.8), see (3.3.7) for the definition of θ) on the basis of D, which entails that for $(\tilde{x}, x_n) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$:

$$\eta_j(\tilde{x}, x_n) \sim \min\left(|x_n - \xi_i(\tilde{x})|^\alpha a(\tilde{x}), |\xi_i(\tilde{x}) - \theta(\tilde{x})|^\alpha a(\tilde{x})\right), \tag{3.3.10}$$

if α is negative, and

$$\eta_j(\tilde{x}, x_n) \sim \max\left(|x_n - \xi_i(\tilde{x})|^\alpha a(\tilde{x}), |\xi_i(\tilde{x}) - \theta(\tilde{x})|^\alpha a(\tilde{x})\right), \tag{3.3.11}$$

in the case where α is nonnegative.

Note that $\eta_j(x) \lesssim x_1$ entails $\eta_j(x) \sim \min(\eta_j(x), x_1)$, which means that it is enough to check that $\min(\eta_j(x), x_1)$ is $\widehat{\Psi}$-subhomogeneous. Thanks to the induction hypothesis, we can assume that $\tau \ni \tilde{x} \mapsto \min(|\xi_i - \theta|(\tilde{x})^\alpha a(\tilde{x}), \tilde{x}_1)$, $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{n-1})$, is Ψ-subhomogeneous. Hence, in virtue of (3.3.10) and (3.3.11), it is enough to show that the function $\min(|x_n - \xi_i(\tilde{x})|^\alpha a(\tilde{x}), x_1)$ is $\widehat{\Psi}$-subhomogeneous (the min and max of $\widehat{\Psi}$-subhomogeneous functions are $\widehat{\Psi}$-subhomogeneous - note also that $\min(\max(u,v),w) = \max(\min(u,w), \min(v,w))$).

For simplicity, we define a function on D by setting for $x = (\tilde{x}, x_n) \in D$:

$$F(x) := |x_n - \xi_i(\tilde{x})|^\alpha \cdot a(\tilde{x}),$$

and a function on the basis E of D by setting for $\tilde{x} \in E$

$$G(\tilde{x}) := |\xi_{i+1}(\tilde{x}) - \xi_i(\tilde{x})|^\alpha \cdot a(\tilde{x}).$$

Observe that if we set for $x = (\tilde{x}, x_n) \in D$

$$\nu(x) := \frac{x_n - \xi_i(\tilde{x})}{\xi_{i+1}(\tilde{x}) - \xi_i(\tilde{x})}$$

then we have:

$$F(x) = \nu(x)^\alpha \cdot G(\tilde{x}).$$

Remark that $\nu(\widehat{\Psi}(sq))$ is constant with respect to s, which entails that for $s \in [0,1]$ and $q = (\tilde{q}, q_n) \in \sigma$:

$$F(\widehat{\Psi}(sq)) = \nu(\widehat{\Psi}(q))^\alpha \cdot G(\Psi(s\tilde{q})). \tag{3.3.12}$$

We first suppose that α is negative. Thanks to the induction hypothesis, we can assume that $\tilde{x} \mapsto \min(G(\tilde{x}), \tilde{x}_1)$, $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{n-1})$ is Ψ-subhomogeneous. This implies (multiplying by $\nu^\alpha(x)$ and applying (3.3.12)) that $x = (x_1, \dots, x_n) \mapsto \min(F(x), \nu^\alpha(x)x_1)$ is $\widehat{\Psi}$-subhomogeneous, which entails that so is the function $\min(F(x), \nu^\alpha(x)x_1, x_1)$. But, as α is negative,

$$\min(F(x), \nu^\alpha(x)x_1, x_1) = \min(F(x), x_1),$$

so that we can conclude that $\min(F(x), x_1)$ is $\widehat{\Psi}$-subhomogeneous, as required.

We now suppose that α is nonnegative. As $\eta_j(x) \lesssim x_1$, (3.3.11) then implies that $F(x) \lesssim x_1$ on σ, which entails that $G(\tilde{x}) \lesssim \tilde{x}_1$, and therefore $G(\tilde{x}) \sim \min(G(\tilde{x}), \tilde{x}_1)$ on $\pi(\sigma)$, which, thanks to the induction hypothesis, can be assumed to be Ψ-subhomogeneous. By (3.3.12), this entails that F is $\widehat{\Psi}$-subhomogeneous. □

Remark 3.3.4. It is worthy of notice that the proof has established (see the induction assumption) that, given some definable functions $\eta_1, \dots, \eta_l$ on $\mathcal{C}_n(R)$ satisfying $\eta_j(x) \lesssim x_1$ for all j (if $x = (x_1, \dots, x_n) \in \mathcal{C}_n(R)$), we can construct metric triangulation Ψ in such a way that $\eta_j \circ \Psi$ is subhomogeneous for all j.

3.4 Local conic structure

Since definable sets can be triangulated [Loj64a], their germs are homeomorphic to cones over their links. Although this homeomorphism cannot be chosen bi-Lipschitz, we are going to see that metric triangulations enclose information on the way this homeomorphism affects the Lipschitz geometry (Theorem 3.4.1 just below). This result recently turned out to be helpful to compute the cohomology of L^p forms of subanalytic varieties [gVa12, gVa21] as well as to investigate the theory of Sobolev spaces of these varieties [aVa-gVa21b, gVa22a, gVa22b].

Given a point $x_0 \in \mathbb{R}^n$ and $A \in \mathcal{S}_n$, we denote by $x_0 * A$ the cone over A with vertex at x_0, i.e., we set:

$$x_0 * A := \{tx + (1-t)x_0 : x \in A \text{ and } t \in [0,1]\}.$$

A **retraction by deformation of** $X \subset \mathbb{R}^n$ **onto** $x_0 \in X$ is a continuous map $r : [0,1] \times X \to X, (s,x) \mapsto r_s(x)$, such that $r_1 : X \to X$ is the identity map, $r_s(x_0) = x_0$ for all s, and $r_0 \equiv x_0$.

Theorem 3.4.1. *Let $X \in \mathcal{S}_n$ and $x_0 \in X$. For $\varepsilon > 0$ small enough, there exists a definable homeomorphism*

$$H : x_0 * (\mathbf{S}(x_0,\varepsilon) \cap X) \to \overline{\mathbf{B}}(x_0,\varepsilon) \cap X,$$

satisfying $H_{|\mathbf{S}(x_0,\varepsilon)\cap X} = Id$, preserving the distance to x_0, and having the following Lipschitzness properties:

(i) H is Lipschitz and the natural retraction by deformation onto x_0

$$r : [0,1] \times \overline{\mathbf{B}}(x_0,\varepsilon) \cap X \to \overline{\mathbf{B}}(x_0,\varepsilon) \cap X,$$

defined by

$$r(s,x) := H(sH^{-1}(x) + (1-s)x_0),$$

is also Lipschitz. Moreover, there is a constant C such that for every fixed $s \in [0,1]$, the retraction r_s defined by $x \mapsto r_s(x) := r(s,x)$, is Cs-Lipschitz.

(ii) For each $\eta > 0$, the restriction of H^{-1} to $\{x \in X : \eta \le |x - x_0| \le \varepsilon\}$ is Lipschitz and, for each $s \in (0,1]$, the map $r_s^{-1} : \overline{\mathbf{B}}(x_0, s\varepsilon) \cap X \to \overline{\mathbf{B}}(x_0,\varepsilon) \cap X$ is Lipschitz.

Proof. We may assume $x_0 = 0$. We recall that $\check{X} = \{(t,x) \in \mathbb{R} \times X : t = |x|\}$. Applying Theorem 3.3.2 to this set (which is a subset of $\mathcal{C}_{n+1}(1)$) provides a vertical Lipschitz definable homeomorphism $\Psi : |K| \to \check{X}_{[0,\varepsilon]}$, with ε positive real number and K simplicial complex of $\mathbb{R}^{n+1}$.

Every point of $0*(\mathbf{S}(0,\varepsilon)\cap X)$ can be written tq with $t \in [0,1]$ and $q \in \mathbf{S}(0,\varepsilon)\cap X$. For such t and q we set:

$$H(tq) := \pi \circ \Psi(t\Psi^{-1}(\varepsilon,q)),$$

where $\pi : \check{X} \to X$ is induced by the projection omitting the first coordinate. This defines a homeomorphism on $0 * \mathbf{S}(0,\varepsilon) \cap X$ which, since Ψ is vertical, satisfies $|H(tq)| = t|q|$, showing that H preserves the distance to the origin. The statements about the Lipschitzness properties of H and H^{-1} directly follow from (3.2.13) together with (i) and (ii) of Theorem 3.3.2.

Moreover, if for $x \in \overline{\mathbf{B}}(0,\varepsilon) \cap X$ we set $r(s,x) = H(sH^{-1}(x))$ then by definition of H we have

$$r(s,x) = \pi \circ \Psi(s\,\Psi^{-1} \circ \pi^{-1}(x))$$

(to see this, observe that if we denote by $\tilde{r}_s$ the retraction that sits on the right-hand-side of this equality then $\tilde{r}_s = r_s$ on $\mathbf{S}(0,\varepsilon) \cap X$, for all s, and since $r_s \circ r_t = r_{st}$ as well as $\tilde{r}_s \circ \tilde{r}_t = \tilde{r}_{st}$, this equality must continue to hold on $\mathbf{B}(0,\varepsilon) \cap X$). As π is a bi-Lipschitz homeomorphism, the Lipschitzness properties of r thus follow from (3.2.13) together with (i) and (ii) of Theorem 3.3.2. □

Remark 3.4.2. It follows from the last sentence of Theorem 3.3.2 that, given finitely many definable set-germs $X_1, \dots, X_k$ at $x_0 \in \cap_{i=1}^k X_i$, the respective homeomorphisms of the Lipschitz conic structure of the X_i's provided by Theorem 3.4.1 can be required to be induced by the same homeomorphism $H : x_0 * \mathbf{S}(x_0,\varepsilon) \to \overline{\mathbf{B}}(x_0,\varepsilon)$.

Remark 3.4.3. The Lipschitz constant of r_s^{-1} (see (ii)) is bounded away from infinity if s stays bounded away from 0. Indeed, if $s \geq \delta > 0$ then $r_\delta = r_{\frac{\delta}{s}} \circ r_s$ entails $r_s^{-1} = r_{\frac{\delta}{s}} \circ r_\delta^{-1}$, and the Lipschitz constants of both $r_{\frac{\delta}{s}}$ and r_δ^{-1} are bounded independently of $s \geq \delta$. This also can be deduced from the proof of the above theorem and (3.2.13).

Remark 3.4.4. By Remark 3.3.4, if $\eta_1, \dots, \eta_l$ are definable function-germs on X satisfying $\eta_j(x) \lesssim |x|$ then we can require that $\eta_j(r_s(x)) \leq Cs\eta_j(x)$, for all $s \in [0,1]$, $x \in X$, $j \leq l$, for some constant C independent of x and s.

In particular, we have established the following

Corollary 3.4.5. *Let $X \in \mathcal{S}_n$ and $x_0 \in X$. For every $\varepsilon > 0$ small enough there exists a definable Lipschitz retraction by deformation $r : [0,1] \times X \cap \mathbf{B}(x_0,\varepsilon) \to X \cap \mathbf{B}(x_0,\varepsilon), (s,x) \mapsto r_s(x)$, onto x_0. Moreover, r_s is Cs-Lipschitz for some $C > 0$ independent of s, and can be required to preserve finitely many given definable subsets of X for all $s \in (0,1]$.*

The second consequence that we would like to point out will lead us to the notion of link that will be studied in section 3.4.2.

Corollary 3.4.6. *Let $X \in \mathcal{S}_n$ and let $x_0 \in \mathbb{R}^n$. Up to a definable bi-Lipschitz homeomorphism, the set $\mathbf{S}(x_0,\varepsilon) \cap X$, $\varepsilon > 0$, is independent of $\varepsilon > 0$ small enough.*

The **link** of $X \in \mathcal{S}_n$ at $x_0 \in \mathbb{R}^n$, denoted $lk(X,x_0)$, will be the subset $\mathbf{S}(x_0,\varepsilon) \cap X$, $\varepsilon > 0$ small (by the just above corollary, it is well defined *up to a definable bi-Lipschitz homeomorphism*).

3.4.1 Some lemmas about definable set and map germs

Theorem 3.3.2 makes it possible to establish that the link of a globally subanalytic set is invariant under globally subanalytic bi-Lipschitz mappings (Corollary 3.4.18), which requires some preliminaries that we carry out in this section.

Given a definable Lipschitz map-germ F at $0_{\mathbb{R}^n}$ the limit $\lim_{t\to 0}\frac{1}{t}(F(tx)-F(0))$ exists for all x and clearly defines an L_F-Lipschitz mapping, which means that definable Lipschitz maps are Gateau differentiable. Indeed:

Lemma 3.4.7. *Let A be a definable subset of $\mathbf{S}(0_{\mathbb{R}^n},\varepsilon)$, $\varepsilon>0$, and let $F: 0_{\mathbb{R}^n} * A \to B$ be a Lipschitz definable map. If we set $G(x)=\lim_{t\to 0}\frac{1}{t}(F(tx)-F(0))$ then for all $(t,x)\in[0,1]\times A$*

$$F(tx)=F(0)+tG(x)+t\mu(tx),$$

for some definable continuous mapping μ tending to zero at the origin.

Proof. Possibly replacing A with $cl(A)$, we may assume that this set is closed. Notice that $\tilde{F}_t(x):=\frac{1}{t}(F(tx)-F(0))$, $x\in A$, is L_F-Lipschitz for each $t\in(0,1]$, and hence, so is G. It suffices to show that $(t,x)\mapsto\tilde{F}_t(x)$ extends continuously on $[0,1]\times A$. Let (t_i,x_i) be a sequence in $(0,1]\times A$ tending to some point $(0,a)$ in this set. By Ascoli's Theorem, extracting a sequence if necessary, we can assume that $\tilde{F}_{t_i}$ uniformly converges to G. Hence, as $(\tilde{F}_{t_i}(x_i)-G(x_i))$ and $(G(x_i)-G(a))$ both tend to zero, $\tilde{F}_{t_i}(x_i)$ must tend to $G(a)$. □

Given a and b in $\mathbb{R}^n$, we denote by ab the line segment joining a and b.

Lemma 3.4.8. *Let K be a simplicial complex of $\mathbb{R}^n$ and let a germ of Lipschitz definable map $H:(|K|,0)\to(\mathbb{R}^k,0)$ satisfy $|H(q)|\sim|q|$. The angle at $H(q)$ between the ray $0H(q)$ and the tangent half-line to the arc $H(0q)$ (the image of the ray $0q$ under H) at $H(q)$ tends to zero as $q\neq 0$ goes to the origin.*

Proof. Let $\pi_1:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$ (resp. $\pi_2:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^k$) denote the projection onto the n first (resp. k last) coordinates. Take a definable arc $\alpha:(0,\varepsilon)\to|K|$ tending to the origin at 0, let

$$l:=\lim_{s\to 0}\frac{H(\alpha(s))}{|H(\alpha(s))|},$$

and let l' denote the limit of the unit tangent vector at $H(\alpha(s))$ to the image of the ray that stems from $\alpha(s)$, i.e.:

$$l':=\lim_{s\to 0}\lim_{t\to 1_-}\frac{\frac{d}{dt}\beta(s,t)}{|\frac{d}{dt}\beta(s,t)|},\qquad\text{where}\quad\beta(s,t)=H(t\cdot\alpha(s)).$$

Thanks to Curve Selection Lemma (Lemma 2.2.3), it suffices to show $l=l'$.

Let then $\alpha(s,t) := t \cdot \alpha(s)$ and define $\gamma(s,t) := (\alpha(s,t), \beta(s,t)) \in \Gamma_H$. Set $u := \lim_{s\to 0} \frac{\gamma(s,1)}{|\gamma(s,1)|}$ and $u' := \lim_{s\to 0} \lim_{t\to 1_-} \frac{\frac{d}{dt}\gamma(s,t)}{|\frac{d}{dt}\gamma(s,t)|}$. We claim that $u = u'$.

Since for every s, $\{\alpha(s,t) : t \in \mathbb{R}\}$ is a line, we clearly have $\frac{\pi_1(u)}{|\pi_1(u)|} = \frac{\pi_1(u')}{|\pi_1(u')|}$. Take a Whitney (b) regular stratification of Γ_H (see Propositions 2.6.6 and 2.6.8) compatible with $\{0_{\mathbb{R}^{n+k}}\}$. Let S be the stratum that contains $\gamma(s,t)$ for $t < 1$ close to 1 (for $s > 0$ small enough it is independent of s). As H is Lipschitz, π_1 must induce a one-to-one map on

$$\tau := \lim_{s\to 0} \lim_{t\to 1_-} T_{\gamma(s,t)} S,$$

which contains u (by Whitney (b) condition) and u' (by definition). Hence, $\frac{\pi_1(u)}{|\pi_1(u)|} = \frac{\pi_1(u')}{|\pi_1(u')|}$ entails $u = u'$, yielding our claim.

As $|H(q)| \sim |q|$, the vector u cannot be included in $\ker \pi_2$, so that by definition of u and l we must have $l = \frac{\pi_2(u)}{|\pi_2(u)|}$ (since the curve $\gamma(s,1)$ lies on Γ_H). But $u' = u$ is not included in the kernel of π_2 either, so that by definition of u' and l', we also have $l' = \frac{\pi_2(u')}{|\pi_2(u')|}$. We thus get $l = \frac{\pi_2(u)}{|\pi_2(u)|} = \frac{\pi_2(u')}{|\pi_2(u')|} = l'$, as required. □

Lemma 3.4.9. *Let $A \in \mathcal{S}_{n,0}$ and let $f : (A,0) \to (\mathbb{R},0)$ be a germ of definable function satisfying*

$$|f(x)| \lesssim |x|, \tag{3.4.1}$$

for $x \in A$ close to the origin. If $f_{|\mathbf{S}(0,r)\cap A}$ is L-Lipschitz for every $r > 0$ small, with $L \in \mathbb{R}$ independent of r, then f is the germ of a Lipschitz function.

Proof. By Curve Selection Lemma, it suffices to check the Lipschitz condition along two definable arcs $x : (0,\varepsilon) \to A$ and $y : (0,\varepsilon) \to A$ tending to the origin at 0. We may assume that x is parametrized by its distance to the origin. Let for simplicity $t_r := |y(r)|$. As x is a Puiseux arc satisfying $|x(r)| = r$, its expansion starts like $x(r) = ar + \dots$, with $a \in \mathbf{S}^{n-1}$, and therefore:

$$|x(r) - x(t_r)| \lesssim |r - t_r| \le |x(r) - y(r)|. \tag{3.4.2}$$

By Proposition 1.8.4, $f(x(r))$ is a Puiseux arc. Since $|f(x(r))| \lesssim r$ for r positive small, this entails that f is Lipschitz along $x(r)$. Hence,

$$|f(x(r)) - f(x(t_r))| \lesssim |x(r) - x(t_r)| \overset{(3.4.2)}{\lesssim} |x(r) - y(r)|. \tag{3.4.3}$$

As f is L-Lipschitz on the spheres

$$|f(x(t_r)) - f(y(r))| \le L|x(t_r) - y(r)| \le L|x(t_r) - x(r)| + L|x(r) - y(r)|,$$

which, together with (3.4.2) and (3.4.3), implies the desired inequality. □

Definition 3.4.10. Let $A \in \mathcal{S}_{n,0}$. A nonnegative Lipschitz definable function-germ $\rho : (A, 0) \to (\mathbb{R}, 0)$ satisfying

$$\rho(x) \sim |x| \tag{3.4.4}$$

is called **a radius function**.

Lemma 3.4.11. *Let $A \in \mathcal{S}_{n,0}$ and let $\rho : (A, 0) \to (\mathbb{R}, 0)$ be a radius function. Let in addition $x : [0, \varepsilon) \to A$ and $y : [0, \varepsilon) \to A$ be two $\mathscr{C}^0$ definable paths such that*

$$x(0) = y(0) = 0_{\mathbb{R}^n} \quad \text{and} \quad \rho(x(r)) = \rho(y(r)), \quad \text{for all } r \in (0, \varepsilon).$$

If x and y have the same tangent half-line at the origin then $\frac{x-y}{|x-y|}$ and $\frac{y}{|y|}$ have different limits as r goes to zero.

Proof. We claim that we can assume $A = \mathbb{R}^n$. Indeed, for every r, the restriction $f_r := \rho_{|\mathbf{S}(0,r)\cap A}$ may then be extended to a definable L_ρ-Lipschitz function by setting

$$\tilde{f}_r(q) := \inf\{\rho(p) + L_\rho|p - q| : p \in \mathbf{S}(0, r) \cap A\}. \tag{3.4.5}$$

Clearly, $\tilde{\rho}(q) := \tilde{f}_{|q|}(q)$ extends ρ and satisfies (3.4.4). Moreover, as it is L_ρ-Lipschitz on every sphere $\mathbf{S}(0, r)$, by Lemma 3.4.9, it is Lipschitz on a neighborhood of the origin in $\mathbb{R}^n$, yielding that we can assume $A = \mathbb{R}^n$.

We now shall show that there is $\varepsilon > 0$ such that for almost every q in a neighborhood of the origin (as ρ is definable, its derivative exists almost everywhere)

$$\partial_q \rho \cdot \frac{q}{|q|} \geq \varepsilon. \tag{3.4.6}$$

Thanks to Curve Selection Lemma, it suffices to show that for every definable arc $\gamma(r)$ the limit $l := \lim_{r\to 0} \partial_{\gamma(r)}\rho \cdot \frac{\gamma(r)}{|\gamma(r)|}$ is positive. As γ is a Puiseux arc (see Proposition 1.8.4), we may assume that it is parametrized by its distance to the origin. If $\gamma(r) = ar + \dots$, with $a \in \mathbf{S}^{n-1}$, then $\gamma'(r) = a + \dots$, so that

$$\lim_{r\to 0} \frac{d\rho(\gamma(r))}{dr} = \lim_{r\to 0} \partial_{\gamma(r)}\rho \cdot \gamma'(r) \tag{3.4.7}$$

has the same sign as l (positive, negative, or zero). Since ρ is positive and vanishes at the origin, $\frac{d\rho(\gamma(r))}{dr}$ must be positive for r positive, which means that l must be nonnegative. If $l = 0$ then, by (3.4.7), we get $\rho(\gamma(r)) \ll r$, in contradiction with (3.4.4), yielding (3.4.6).

Now, we assume that the conclusion of the lemma fails, i.e., that the limit of the segment xy (in the projective space) is the same as the limit of the segment $0y$. Let us move slightly y to some close point z such that ρ is almost everywhere differentiable on the segment xz (almost every $z \in \mathbb{R}^n$ has this property). By Definable Choice (Proposition 2.2.1), we can assume that z is a definable path.

By assumption, x and y share the same half-tangent at the origin. If z is sufficiently close to y then the paths $\frac{x}{|x|}$ and $\frac{z}{|z|}$ have the same limit in $\mathbf{S}^{n-1}$, say v. Notice that $v = \lim_{r\to 0} \frac{q(r)}{|q(r)|}$, for every path $q : (0, \varepsilon) \to \mathbb{R}^n$ such that $q(r)$ belongs to the segment $x(r)z(r)$ for all r. By (3.4.6), we thus have $\partial_q \rho \cdot v \geq \frac{\varepsilon}{2}$ for any point q in the segment $x(r)z(r)$, $r > 0$ small enough (since v is close to $\frac{q}{|q|}$).

As we have assumed that $\frac{x-y}{|x-y|}$ tends in the projective space to the same limit as y, if z is sufficiently close to y then $\frac{x-z}{|x-z|}$ must converge to $\pm v$, say v for simplicity. By the above paragraph, this implies that $\partial_q \rho . \frac{x-z}{|x-z|} \geq \frac{\varepsilon}{4}$, for any q in the segment $x(r)z(r)$, $r > 0$ small enough. Hence, ρ is strictly monotonic on the segment xz, with a derivative bounded away from zero by $\frac{\varepsilon}{4}$. This contradicts the fact that the points $x(r)$ and $y(r)$ belong to the same level surface of ρ (and that z is very close to y). $\square$

We now can extend Lemma 3.4.9 to all the radius functions:

Lemma 3.4.12. *Let $A \in \mathcal{S}_{n,0}$ and let $\rho : A \to \mathbb{R}$ be a radius function. Let $f : (A, 0) \to (\mathbb{R}, 0)$ be a definable function-germ satisfying (3.4.1) near the origin. If $f_{|\{\rho=r\}}$ is L-Lipschitz for every $r > 0$ small, with $L \in \mathbb{R}$ independent of r, then f is the germ of a Lipschitz function.*

Proof. One more time, it is enough to check the Lipschitz condition along two definable curves x and y ending at the origin, and we may assume that x is parametrized in such a way that $\rho(x(r)) = r$.

Set for simplicity $t_r := \rho(y(r))$. If x and y are not tangent to each other at the origin then $|x(r)| \lesssim |x(r) - y(r)|$ and $|y(r)| \lesssim |x(r) - y(r)|$, so that the result follows from assumption (3.4.1). If they have the same half tangent at the origin, then, by Lemma 3.4.11, the angle between the vectors $x(t_r)$ and $\pm(x(t_r) - y(r))$ does not go to zero. This implies (since x is definable arc) that the angle between $(x(t_r) - x(r))$ and $(x(t_r) - y(r))$ is bounded below away from zero (for $r \neq t_r$; if $r \equiv t_r$ the needed fact is obvious). We thus have:

$$|x(r) - x(t_r)| \lesssim |x(r) - y(r)| \tag{3.4.8}$$

(since $|x(r) - y(r)| \ll |x(r) - x(t_r)|$ would entail that this angle goes to zero). Since $f(x(r))$ is a Puiseux arc satisfying $|f(x(r))| \lesssim r$, the function f is Lipschitz along the arc $x(r)$. We thus can finish the proof with exactly the same computation as in the proof of Lemma 3.4.9, writing (3.4.3) and replacing (3.4.2) with (3.4.8). $\square$

We also can derive a bi-Lipschitz version of this lemma.

Lemma 3.4.13. *Let $A \in \mathcal{S}_{n,0}$ and let $f : (A, 0) \to (B, 0)$ be a germ of definable map. Let $\alpha : (B, 0) \to \mathbb{R}$ be a radius function such that $\rho(x) := \alpha(f(x))$ defines a* *radius function on A. If the function $f_{|\{\rho=r\}}$ is L-bi-Lipschitz for every $r>0$ small, with $L\in\mathbb{R}$ independent of r, then f is the germ of a bi-Lipschitz mapping.*

Proof. As α and ρ are both radius functions, we have near the origin $|x|\sim\rho(x)=\alpha(f(x))\sim|f(x)|$. Hence, by Lemma 3.4.12, as $f_{|\{\rho=r\}}$ is L-Lipschitz for every $r>0$ small, with $L\in\mathbb{R}$ independent of r, f is the germ of a Lipschitz mapping.

We now check that f is one-to-one. If p and q are two points of A such that $f(p)=f(q)$ then $\rho(p)=\alpha(f(p))=\alpha(f(q))=\rho(q)$. But, as f is one-to-one on the level surfaces of ρ, this implies $p=q$.

It remains to show that f^{-1} is Lipschitz. The restriction of f^{-1} to the set

$$\{x\in f(A):\alpha(x)=r\}=f(\{x\in A:\rho(x)=r\})$$

is L-Lipschitz by assumption, for all $r>0$ small enough. Therefore, again due to Lemma 3.4.12, f^{-1} must be a Lipschitz mapping. $\square$

3.4.2 Uniqueness of the link

Theorem 3.4.14. *Let $X\in\mathcal{S}_{n,0}$ and let $\rho:(X,0)\to(\mathbb{R},0)$ be a radius function. There exists a germ of definable bi-Lipschitz homeomorphism $g:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ preserving X and such that $|g(x)|=\rho(x)$ for all $x\in X$.*

Proof. We may extend ρ to a radius function to a definable open neighborhood of the origin U in $\mathbb{R}^n$ (see (3.4.5)). We recall that $\check{U}:=\{(t,x)\in\mathbb{R}\times U:t=|x|\}$ (see (3.3.1)), which is a subset of $\mathcal{C}_{n+1}(1)$. As the mapping $P:\check{U}\to U$, induced by the restriction of the projection omitting the first coordinate, is a bi-Lipschitz homeomorphism, it is enough to construct a germ of definable bi-Lipschitz homeomorphism $g:(\check{U},0)\to(\check{U},0)$ preserving $\check{X}$ and satisfying $g_1(x)=\check{\rho}(x)$, for all $x\in\check{U}$, where $g_1(x)$ is the first coordinate of $g(x)$ in the canonical basis and $\check{\rho}(x):=\rho(P(x))$.

Let $\Psi:(|K|,0)\to(\check{U}_{[0,\varepsilon]},0)$ be the vertical map given by Theorem 3.3.2 applied to $\check{X}$ and $\check{U}$. Define a mapping $\Lambda:|K|\to|K|$ by setting

$$\Lambda(q):=\frac{\check{\rho}(\Psi(q))}{q_1}\cdot q,$$

if $q=(q_1,\dots,q_{n+1})\in|K|$ is nonzero, and $\Lambda(0):=0$. Note that Λ preserves the germs of the open simplices that have the origin in their closure. Define then a mapping g from $\check{U}$ to itself by:

$$g(x):=\Psi\circ\Lambda\circ\Psi^{-1}(x),\quad x\in\check{U}.$$

We first check g gives rise to an onto map-germ, i.e., that $g(\check{U})=\check{U}$ as germs, which amounts to check that $\Lambda(\Psi^{-1}(\check{U}))=\Psi^{-1}(\check{U})$, as germs. Take $y=(y_1,\dots,y_{n+1})\in$

$\Psi^{-1}(\check{U})$. As our problem is local at 0, we will assume that $y_1 < \varepsilon$, where $\varepsilon > 0$ is sufficiently small for $\{q \in \Psi^{-1}(\check{U}) : \check{\rho} \circ \Psi(q) \le \varepsilon\}$ to be compact (note that, since Ψ is vertical and ρ is a radius function, $\check{\rho} \circ \Psi(q) \sim q_1 \sim |q|$ on $\Psi^{-1}(\check{U})$). A preimage of y under Λ is given by a point $q \in \mathscr{L}_y := \{ty : t \in (0,\infty)\}$, satisfying $\check{\rho} \circ \Psi(q) = y_1$. If $y_1 \in [0, \check{\rho} \circ \Psi(y)]$ then, as ρ is continuous and vanishes at 0, it is clear that there is q on $\mathscr{L}_y$ between 0 and y satisfying $\check{\rho} \circ \Psi(q) = y_1$. If $y_1 \in [\check{\rho} \circ \Psi(y), \varepsilon)$ then, as $\{\check{\rho} \circ \Psi \le \varepsilon\}$ is compact, the half-line $\mathscr{L}_y$ cannot meet the frontier of $\Psi^{-1}(\check{U})$ before $\check{\rho} \circ \Psi$ reaches ε, which means that there is $q \in \Psi^{-1}(\check{U})$ on $\mathscr{L}_y$ such that $\check{\rho} \circ \Psi(q) = y_1$, as required.

Observe also that, as Ψ is vertical, we have for $x \in \check{U}$:

$$g_1(x_1, \dots, x_{n+1}) = \frac{\check{\rho}(x)}{x_1} \cdot x_1 = \check{\rho}(x),$$

as required. To finish the proof, we have to check the bi-Lipschitzness of g.

By Lemma 3.4.13, it is enough to prove that for every t the restriction of g to $\{\check{\rho} = t\}$ is L-bi-Lipschitz with $L \in \mathbb{R}$ independent of t. To show this, take two definable curves $x : (0,\varepsilon) \to \check{U}$ and $y : (0,\varepsilon) \to \check{U}$ such that $\check{\rho}(x(t)) = \check{\rho}(y(t)) = t$ for any $t > 0$ small enough, and let $p(t) = (p_1(t), \dots, p_{n+1}(t))$ and $q(t) = (q_1(t), \dots, q_{n+1}(t))$ be the respective preimages of these two arcs under Ψ. Set $\nu(t) := \frac{p_1(t)}{q_1(t)}$, and note that, since ρ is a radius function and Ψ is vertical, we have $\nu \sim 1$ (for simplicity, and because we can interchange x and y, we will assume that $\nu(t) \le 1$).

Claim. We have:

$$|\Psi(p) - \Psi(q)| \sim |\Psi(\nu q) - \Psi(p)|. \tag{3.4.9}$$

Let $\alpha(s,t) = \Psi(sq(t))$, $s \in [\nu(t), 1]$ (if $\nu \equiv 1$ then the above claim is trivial, we will thus suppose $\nu < 1$). By Lemma 3.4.7, as Ψ is Lipschitz and satisfies $|\Psi(z)| \sim |z|$ for z close to zero, the angle between $\alpha(s,t)$ and $\Psi(q(t))$ tends to zero as $t \to 0$ (uniformly in $s \in [\nu(t), 1]$). Furthermore, by Lemma 3.4.8, we know that the angle between $\frac{\partial \alpha}{\partial s}(s,t)$ and $\alpha(s,t)$ tends to zero as $t \to 0$ (uniformly in $s \in [\nu(t), 1]$). Hence, since $\Psi(q) - \Psi(\nu q) = \int_{\nu(t)}^1 \frac{\partial \alpha}{\partial s}(s,t) ds$, we conclude that

$$\lim_{t \to 0} \frac{\Psi(q) - \Psi(\nu q)}{|\Psi(q) - \Psi(\nu q)|} = \lim_{t \to 0} \frac{\Psi(q(t))}{|\Psi(q(t))|}. \tag{3.4.10}$$

Observe also that, in virtue of (ii) of Theorem 3.3.2 and (3.2.13), we must have:

$$|\Psi(\nu q) - \Psi(p)| \lesssim |\Psi(p) - \Psi(q)|.$$

Therefore, if (3.4.9) fails then the angle between $(\Psi(q) - \Psi(\nu q))$ and $(\Psi(q) - \Psi(p))$ tends to 0, which, by (3.4.10), means that so does the angle between $\Psi(q) - \Psi(p) = y - x$ and $\Psi(q) = y$, in contradiction with Lemma 3.4.11, yielding (3.4.9).

Observe now that since $\check{\rho}(x(t)) = \check{\rho}(y(t)) = t$, by definition of Λ we must have

$$\Lambda(q(t)) = \frac{t}{p_1(t)}\nu(t)q(t) \quad \text{and} \quad \Lambda(p(t)) = \frac{t}{p_1(t)}p(t),$$

so that, since the homothetic transformation $|K| \ni z \mapsto \frac{t}{p_1(t)}z$ preserves the tame coordinates $\mu_{\sigma,i}$ for $i \geq 2$ (see (ii) of Theorem 3.3.2) and $\frac{t}{p_1(t)} \sim 1$, we have:

$$|\Psi(\Lambda(q)) - \Psi(\Lambda(p))| \overset{(3.2.13)}{\sim} |\Psi(\nu q) - \Psi(p)| \overset{(3.4.9)}{\sim} |\Psi(q) - \Psi(p)|,$$

establishing the bi-Lipschitz character of g. □

Remark 3.4.15. The just constructed mapping g only depends on the triangulation Ψ. As a matter of fact, if we have several definable set-germs $X_1, \dots, X_l$ at $0_{\mathbb{R}^n}$, we may demand g to be the same for all the X_i's (taking a triangulation Ψ compatible with all the X_i).

Given two definable sets X and Y (resp. germs of definable sets), we write $X \approx Y$ if there is a definable bi-Lipschitz homeomorphism (resp. a germ of definable bi-Lipschitz homeomorphism) sending X onto Y. An immediate consequence of the preceding theorem is:

Corollary 3.4.16. *Let $X \in \mathcal{S}_{n,0}$ and let $\rho : (X, 0) \to (\mathbb{R}, 0)$ be a radius function. For any $r > 0$ small, $\{x \in X : \rho(x) = r\} \approx lk(X, 0)$.*

Another consequence of Theorem 3.4.14 is the following:

Theorem 3.4.17. *Let X and Y be two germs of definable sets at the origin. If $X \approx Y$ then there exists a germ of definable bi-Lipschitz homeomorphism Φ sending X onto Y and preserving the distance to the origin.*

Proof. Let $h : (X, 0) \to (Y, 0)$ be a germ of definable bi-Lipschitz homeomorphism and define a radius function on X by setting for $x \in X$, $\rho(x) := |h(x)|$. By Theorem 3.4.14, there is a germ of definable homeomorphism $g : (X, 0) \to (X, 0)$ such that $|g(x)| = \rho(x)$ for all $x \in X$, which implies that the mapping $\Phi := h \circ g^{-1}$ is bi-Lipschitz and preserves the distance to the origin. □

Observe that Lemma 3.4.13 and Theorem 3.4.17 establish that the metric type of a definable set-germ X is characterized by the metric types of all the sections $\mathbf{S}(0, r) \cap X$, $r > 0$ small, and vice-versa. In particular, the above theorem yields the definable bi-Lipschitz invariance of the link:

Corollary 3.4.18. *Let X and Y be two definable set-germs at the origin. If $X \approx Y$ then $lk(X, 0) \approx lk(Y, 0)$.*

Remark 3.4.19. In the just above corollary and in Theorem 3.4.17, X and Y do not necessarily lie in the same euclidean space. Moreover, if we define the link as the generic fiber of the distance to the origin (in the field of convergent Puiseux series), a converse is possible [gVa07]. This fact can indeed be derived from Lemma 3.4.13 and standard arguments of algebraic geometry. We also stress the fact that, in Theorem 3.4.17 and Corollary 3.4.18, if the homeomorphisms between X and Y are given by homeomorphisms of the ambient space, then the provided can also be required to be induced by such homeomorphisms (since Theorem 3.4.14 itself provides such a homeomorphism).

Historical notes. The Lipschitz cell decomposition theorem (Theorem 3.1.18) is due to K. Kurdyka and A. Parusiński [Kur92, Kur-Par06]. It is called in [Bir-Mos00] the "Pancake Decomposition Theorem". Statement (ii) of this theorem is however an addendum to the latter works that was obtained in [gVa05] in order to show Lemma 3.2.5. Let us mention that this fact was upgraded by W. Pawłucki [Paw09] who showed that the linear changes of coordinates may always be expressed as a permutation of the vectors of the canonical basis. Proposition 3.1.26 seems to be new and has been added because it is useful to the study of Sobolev spaces of subanalytic manifolds (via Morrey's embedding) recently developped by the author of these notes [aVal-gVal21a, aVa-gVa21b, gVa22a, gVa22b].

The study of bi-Lipschitz equivalence of analytic singularities was initiated by T. Mostowski [Mos85] who focused on complex analytic sets and continued by A. Parusiński [Par94a] who explored the real case (see also [Ngu-gVa16, Hal-Yin18]). The material of sections 3.2, 3.3, and 3.4 is however due to the author of these notes. Metric triangulations were introduced in [gVa05] to establish Corollary 3.2.12 (and where they are called Lipschitz triangulations), which is the Lipschitz counterpart of a result about topological stability sometimes referred as Hardt's Theorem [Har80]. Theorem 3.3.2 and uniqueness of the link (Corollary 3.4.18) were established in [gVa07] ((ii) of Theorem 3.3.2 was added in [gVa21]). In the case of curves, a first insight had been made earlier by L. Birbrair and his Hölder complex decomposition [Bir99]. Corollary 3.2.11 was established in [gVa08] while Corollary 3.2.12 was already present in [gVa05]. Corollary 3.4.5 was proved in [gVa12] and the description of the Lipschitz conic structure of globally subanalytic sets may be found in [gVa21]. Existence of $\mathscr{C}^0$ triangulations and local $\mathscr{C}^0$ retracts however goes back to as far as the original work of S. Łojasiewicz [Loj64a, Loj64b].

Chapter 4

Geometric measure theory

We study the Hausdorff measure of globally subanalytic sets as well as integrals of globally subanalytic functions. It is easy to see that the antiderivatives of a globally subanalytic function are not necessarily globally subanalytic. After recalling some basic formulas, we will show that functions defined by integrals of globally subanalytic families of functions may be described by polynomials in some globally subanalytic functions and their logarithms (Corollary 4.2.7), that we call $\mathcal{L}_{og}$*-functions* (Definition 4.2.2). This will yield that if A is subanalytic then $\mathcal{H}^l(A \cap \mathbf{B}(0,r))$ ($\mathcal{H}^l$ being the Hausdorff measure) has an analytic expansion in r and $\ln r$, and will lead us to the notion of density that we will study on stratified sets (section 4.5). On the way, we give several results of measure theory of globally subanalytic sets and families that are of their own interest (sections 4.3 and 4.4). We end this chapter (section 4.6) by establishing Stokes' formula on globally subanalytic sets (possibly singular).

4.1 Cauchy-Crofton's formula

As this will be the central tool of our study, we enclose a proof of this formula. For simplicity, we will just focus on proving a formula that relates the Hausdorff measure of a definable set to the cardinal of its sections by affine spaces of complement dimension (Theorem 4.1.2). There exist much more general versions [Fed69] but this will enough for our purpose. We first recall some basic techniques of integration theory.

Given $A \in \mathcal{S}_n$, we denote by $\mathcal{H}^k(A)$ the k-dimensional Hausdorff measure of A. We recall that $\pi_P : \mathbb{R}^n \to \mathbb{R}^n$ stands for the orthogonal projection onto $P \in \mathbb{G}^n_k$.

The generalized Jacobian. We will often apply the so-called *co-area formula* (see below). This requires the following notion.

Definition 4.1.1. Let $f : A \to \mathbb{R}^m$ be a definable mapping with $\dim A \geq m$. Take a point $x \in reg(f)$ at which A is a manifold of dimension $\dim A$ and denote by $M_x(f)$ the Jacobian matrix of f with respect to some orthonormal bases of T_xA and $\mathbb{R}^m$. The **generalized Jacobian of f at x** is defined as

$$J_x(f) := \sqrt{\det\left(M_x(f)^t \cdot M_x(f)\right)}.$$

Here $M_x(f)^t$ stands for the transpose of the matrix $M_x(f)$. The generalized Jacobian coincides with the square root of the sum of the squares of the minors of order m of $M_x(f)$. It is of course independent of the choice of the orthonormal bases.

It is worthy of notice that if $\dim A = m$ then the generalized Jacobian equals the usual Jacobian, the absolute value of the determinant of the Jacobian matrix of f. Note also that if $m = 1$, i.e., if f is a function, then $J_x(f)$ is merely the norm of the gradient of f at x.

The co-area formula. This formula will be useful in section 4.5 to estimate the variation of the measure of globally subanalytic sets. A proof can be found in [Fed69, Kra-Par08]. These two books actually provide it in a much more general context than globally subanalytic sets.

Let $f : A \to \mathbb{R}^m$ be a definable mapping, $E \subset A$ be a definable subset, and set $l := \dim A$. If $l \geq m$ then we have:

$$\int_{y\in\mathbb{R}^m} \mathcal{H}^{l-m}(f^{-1}(y) \cap E)\, d\mathcal{H}^m(y) = \int_{x\in E} J_x(f)\, d\mathcal{H}^l(x). \tag{4.1.1}$$

Proof. Generally, a smoothness assumption is put on f. As f is here just assumed to be definable, let us explain how to derive this formula from the classical one. There are some stratifications Σ of E and Σ' of $\mathbb{R}^m$ such that f induces a smooth submersion on every $S \in \Sigma$ to a stratum $S' \in \Sigma'$ (that depends on S). Fix such S and S', and let us emphasize that for $x \in S$, $J_x(f)$ is not the Jacobian of the mapping induced by f from S to S' that, to avoid any confusion, we will denote by $g : S \to S'$. As strata are disjoint, is suffices to show:

$$\int_{y\in S'} \mathcal{H}^{l-m}(f^{-1}(y) \cap S)\, d\mathcal{H}^m(y) = \int_{x\in S} J_x(f)\, d\mathcal{H}^l(x).$$

Note first that if $\dim S < l$ and $\dim S' < m$ then $\mathcal{H}^l(S) = \mathcal{H}^m(S') = 0$, which means that both sides vanish. If $\dim S < l$ and $\dim S' = m$ then $\mathcal{H}^l(S) = 0$ and (since g is a submersion) $\dim(f^{-1}(y) \cap S) < l - m$ which means that $\mathcal{H}^{l-m}(f^{-1}(y) \cap S) = 0$, and hence that both sides are also zero. If $\dim S = l$ and $\dim S' < m$ then $J_x(f) = 0$ on S (since the derivative of f has rank less than m) and $\mathcal{H}^m(S') = 0$, in which case both sides are also zero. Finally, in the case $\dim S = l$ and $\dim S' = m$, as $J_x(f) = J_x(g)$ on S, this follows from the classical coarea formula [Kra-Par08]. $\square$

The measure $\gamma_{l,n}$. Cauchy-Crofton's formula involves integration on the Grassmannian manifold $\mathbb{G}_l^n$, which requires to introduce a measure on this set. We sketch the classical construction of what is usually called the *Haar measure*.

Denote by $\mathcal{O}_n$ the group of linear isometries of $\mathbb{R}^n$. This set may be identified with a definable compact subset of $\mathbb{R}^{n^2}$. Denote by d_n its dimension. We first define a measure θ_n on $\mathcal{O}_n$ satisfying $\theta_n(\mathcal{O}_n) = 1$ by setting, for $W \subset \mathcal{O}_n$, $\theta_n(W) := \frac{\mathcal{H}^{d_n}(W)}{\mathcal{H}^{d_n}(\mathcal{O}_n)}$.

Fix now any $V \in \mathbb{G}_l^n$ and set for $U \subset \mathbb{G}_l^n$:

$$\gamma_{l,n}(U) := \theta_n(\{L \in \mathcal{O}_n : LV \in U\}).$$

It is not difficult to derive from the definitions that, since the elements of $\mathcal{O}_n$ are isometries, the number $\gamma_{l,n}(U)$ is independent of $V \in \mathbb{G}_l^n$, and that $\gamma_{l,n}$ is a measure on $\mathbb{G}_l^n$. As integration over $\mathbb{G}_l^n$ will always be considered with respect to this measure, we will not specify the considered measure when integrating on $\mathbb{G}_l^n$ (simply writing dP if $P \in \mathbb{G}_l^n$ is the variable of integration).

Cauchy-Crofton's formula. Given a set E, let $\operatorname{card} E$ denote its **cardinal**, i.e., the number of elements of E, with the convention that $\operatorname{card} E := \infty$ if E is infinite.

Given $l \in \{1, \dots, n\}$, $P \in \mathbb{G}_l^n$, and $y \in P$, we denote by N_P^y the $(n-l)$-dimensional affine space passing through y and directed by the orthogonal complement of P in $\mathbb{R}^n$.

Theorem 4.1.2. *(Cauchy-Crofton's formula) There exists a positive constant $\beta_{l,n}$ such that for any $A \in \mathcal{S}_n$ we have:*

$$\mathcal{H}^l(A) = \beta_{l,n} \int_{P \in \mathbb{G}_l^n} \int_{y \in P} card\,(N_P^y \cap A)\, d\mathcal{H}^l(y)\, dP. \tag{4.1.2}$$

Proof. We may assume that $\dim A = l$ since otherwise both sides of the equality are 0 or ∞. Thanks to Corollary 1.8.9, we know that for each $A \in \mathcal{S}_n$ there is $m_A \in \mathbb{N}$ such that $\operatorname{card}(N_P^y \cap A) \neq j$ for all integers $j > m_A$, all $P \in \mathbb{G}_l^n$, and all $y \in P$.

Step 1. We establish the desired formula in the case where A is a definable subset of a vector space $Q \in \mathbb{G}_l^n$.

Let $\lambda_P : Q \to P$ be the restriction of π_P. Take its matrix with respect to orthonormal bases of Q and P, denote by $\alpha_{l,n}(P)$ the absolute value of its determinant, and define the desired constant $\beta_{l,n}$ by setting:

$$\beta_{l,n} := \Big(\int_{P \in \mathbb{G}_l^n} \alpha_{l,n}(P) dP \Big)^{-1}.$$

For almost every P, λ_P is a linear automorphism, which entails that we have $\operatorname{card}\,(\pi_P^{-1}(y) \cap A) = 1$ for all $y \in \lambda_P(A)$, and since $\mathcal{H}^l(\lambda_P(A)) = \alpha_{l,n}(P)\mathcal{H}^l(A)$, we can evaluate the right-hand-side of (4.1.2) as follows:

$$\beta_{l,n}\int_{\mathbb{G}_l^n}\int_{\lambda_P(A)} d\mathcal{H}^l(y)dP = \beta_{l,n}\left(\int_{\mathbb{G}_l^n}\alpha_{l,n}(P)\mathcal{H}^l(A)dP\right) = \mathcal{H}^l(A).$$

Step 2. We prove that there exists a positive constant C such that for any $\alpha \in (0, \frac{1}{2}]$ and any α-flat set $A \in \mathcal{S}_n$, we have:

$$|\mathcal{H}^l(A) - \mu(A)| \leq C\alpha\mathcal{H}^l(A),$$

where $\mu(A)$ stands for the right-hand-side of (4.1.2).

Let $\alpha \in (0, \frac{1}{2}]$ and let $A \in \mathcal{S}_n$ be α-flat, which implies that we can find $Q \in \mathbb{G}_l^n$ such that $\angle(P, Q) < \alpha$, for all $P \in \tau(A)$ (see the beginning of section 3.1.1 for $\tau(A)$). We will assume that $Q = \mathbb{R}^l \times \{0_{\mathbb{R}^{n-l}}\}$ for simplicity (this is possible thanks to the definition of the measure on the Grassmannian). This means that the set A coincides with the disjoint union of the graphs of some definable mappings, $\xi_i : A_i \to \mathbb{R}^{n-l}$, $i = 1, \dots, s$, $A_i \subset \mathbb{R}^l$, with $|d_x\xi_i| \leq \alpha$ almost everywhere. Fix $i \leq s$ and observe that, by the coarea formula (applied to $\pi_{Q|\Gamma_{\xi_i}}$), we have

$$|\mathcal{H}^l(A_i) - \mathcal{H}^l(\Gamma_{\xi_i})| \leq \alpha\mathcal{H}^l(A_i). \tag{4.1.3}$$

For simplicity, set for $E \in \mathcal{S}_n$ and $P \in \mathbb{G}_l^n$:

$$\nu_P(E) := \int_{y\in P} \operatorname{card}\left(N_P^y \cap E\right) d\mathcal{H}^l(y).$$

Since $\angle(T_x\Gamma_{\xi_i}, Q) < \alpha$ at every $x \in \Gamma_{\xi_{i,reg}}$, it is an easy exercise of linear algebra to establish that for $x \in Q$ we have $|J_x(\pi_{P|Q}) - J_{(x,\xi_i(x))}(\pi_{P|\Gamma_{\xi_i}})| \leq \alpha$. By the coarea formula (4.1.1), this implies that we have (using again $|d_x\xi_i| \leq \alpha$):

$$|\nu_P(\Gamma_{\xi_i}) - \nu_P(A_i)| \leq 2\alpha\mathcal{H}^l(\Gamma_{\xi_i}). \tag{4.1.4}$$

Integrating with respect to P we get:

$$|\mu(\Gamma_{\xi_i}) - \mu(A_i)| \leq C\alpha\mathcal{H}^l(\Gamma_{\xi_i}), \tag{4.1.5}$$

for some constant C. As the A_i's are subsets of an l-dimensional vector subspace of $\mathbb{R}^n$, thanks to step 1, we know that Cauchy-Crofton's formula must hold for each of them. Making use of this formula, we immediately derive from (4.1.5):

$$|\mu(\Gamma_{\xi_i}) - \mathcal{H}^l(A_i)| \leq C\alpha\mathcal{H}^l(\Gamma_{\xi_i}). \tag{4.1.6}$$

Note that as $\alpha \leq \frac{1}{2}$, (4.1.3) entails that we have $\mathcal{H}^l(A_i) \leq 2\mathcal{H}^l(\Gamma_{\xi_i})$, and consequently

$$|\mu(\Gamma_{\xi_i}) - \mathcal{H}^l(\Gamma_{\xi_i})| \leq |\mu(\Gamma_{\xi_i}) - \mathcal{H}^l(A_i)| + |\mathcal{H}^l(A_i) - \mathcal{H}^l(\Gamma_{\xi_i})| \leq (C+2)\alpha\mathcal{H}^l(\Gamma_{\xi_i}),$$

by (4.1.3) and (4.1.6). Adding-up these inequalities for all i gives the needed estimate.

Step 3. We prove Cauchy-Crofton's formula.

Given A in $\mathcal{S}_n$ and $\alpha > 0$, we can find an $\mathcal{H}^l$-negligible definable set $E \subset A$ such that $A \setminus E$ can be covered by α-flat disjoint definable sets $B_1, \dots, B_k$ (see Lemma 3.1.13 and Proposition 2.3.4). Applying step 2 to all the B_i's and adding the resulting inequalities provides for any $\alpha \in (0, \frac{1}{2}]$:

$$|\mu(A) - \mathcal{H}^l(A)| \leq C\alpha\mathcal{H}^l(A)$$

(C being given by step 2), and therefore $\mu(A) = \mathcal{H}^l(A)$. □

Set for $j \in \mathbb{N} \cup \{\infty\}$, $P \in \mathbb{G}_l^n$, and $A \in \mathcal{S}_n$:

$$K_j^P(A) := \{x \in P : \operatorname{card}(N_P^x \cap A) = j\}.$$

As emphasized at the beginning of the above proof, Corollary 1.8.9 yields that for every set $A \in \mathcal{S}_n$, there is an integer k such that $K_j^P(A)$ is empty for all integers $j > k$ and all $P \in \mathbb{G}_l^n$. We will call the smallest integer having this property the **multiplicity** of A and will denote it by m_A.

Since $K_j^P(A) = \emptyset$ for any integer $j > m_A$, and because $\mathcal{H}^l(K_\infty^P(A)) = 0$ if $l \geq \dim A$, Cauchy-Crofton's formula may be rewritten for $l \geq \dim A$ as:

$$\mathcal{H}^l(A) = \beta_{l,n} \sum_{j=1}^{m_A} j \int_{\mathbb{G}_l^n} \mathcal{H}^l(K_j^P(A))\, dP. \tag{4.1.7}$$

This formula, together with the uniform finiteness properties of definable sets, provides many bounds for the $\mathcal{H}^l$-measure of these sets. We illustrate this fact with a result about definable families that will be useful later on.

Proposition 4.1.3. *Let $A \in \mathcal{S}_{m+n}$ and let $l := \max_{t \in \mathbb{R}^m} \dim A_t$. There exists a constant C such that for all $t \in \mathbb{R}^m$ and all $r \geq 0$:*

$$\mathcal{H}^l(A_t \cap \mathbf{B}(0,r)) \leq Cr^l. \tag{4.1.8}$$

Proof. In the case $l = n$, since $\mathcal{H}^l(\mathbf{B}(0,r)) = \mathcal{H}^l(\mathbf{B}(0,1))r^l$, the result is clear. For the same reason, (4.1.8) also holds when A_t is for each t a subset of some l-dimensional vector subspace of $\mathbb{R}^n$, which establishes this estimate for the definable family $(K_j^P(A_t))_{P \in \mathbb{G}_l^n, t \in \mathbb{R}^m}$. Since Corollary 1.8.12 yields $\sup\{m_{A_t} : t \in \mathbb{R}^m\} < \infty$, the case $l < n$ thus easily comes down from (4.1.7). □

As a matter of fact, for any $A \in \mathcal{S}_{m+n}$ such that $\dim A_t \leq l$ for all $t \in \mathbb{R}^m$ and $\sup_{t \in \mathbb{R}^m} diam(A_t) < \infty$ (see (3.1.4) for $diam$), we have:

$$\sup\{\mathcal{H}^l(A_t) : t \in \mathbb{R}^m\} < \infty. \tag{4.1.9}$$

4.2 On integration of definable functions

In this section, we introduce the class of $\mathcal{L}_{og}$-functions and show that the integrals of definable functions give rise to $\mathcal{L}_{og}$-functions (Corollary 4.2.7).

Let $X \in \mathcal{S}_n$ and $k \leq n$. For any $f : X \to \mathbb{R}$, we set $|f|_{1,\mathcal{H}^k} := \int_X |f| d\mathcal{H}^k$ (possibly infinite). We denote by $L^1_{\mathcal{H}^k}(X)$ the set of functions $f : X \to \mathbb{R}$ that are L^1 for the measure $\mathcal{H}^k$ (and then say that f is $L^1_{\mathcal{H}^k}$).

Proposition 4.2.1. *Let $f : A \to \mathbb{R}$ be a definable function, $A \in \mathcal{S}_{m+n}$. For each $l \leq n$, the set*

$$\{t \in \mathbb{R}^m : f_t \in L^1_{\mathcal{H}^l}(A_t)\} \tag{4.2.1}$$

is definable.

Proof. Let $B := \{(t, x, y) \in A \times \mathbb{R} : 0 \leq y \leq |f_t(x)|\}$. The function f_t is $L^1_{\mathcal{H}^l}$ if and only if B_t has finite $\mathcal{H}^{l+1}$-measure. The family $(B_t)_{t\in\mathbb{R}^m}$ being definable, it follows from Corollary 3.2.12 that there exists a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $C \in \mathcal{P}$ and any $(t, t') \in C \times C$, the sets B_t and $B_{t'}$ are bi-Lipschitz homeomorphic. The set appearing in (4.2.1) being the union of some elements of $\mathcal{P}$, it must be definable. □

In the situation of the above proposition, the function $g(t) := \int_{A_t} f_t \, d\mathcal{H}^l$, defined on the set appearing in (4.2.1), is of course not always definable. For instance, if $f_t(x) = 1/x$ and $A_t = [1, t]$, for every $t > 1$, then $g(t) = \ln t$, which is not a definable function. We are going to explain (Theorem 4.2.6) that the function g is always a polynomial combination of definable functions and of their logarithms, which leads us to the following definition.

We denote by $x \mapsto \ln x$ the natural logarithm function, that we will consider as defined on $\mathbb{R}$, with $\ln(-x) = \ln x$ and $\ln 0 = 0$.

Definition 4.2.2. A **$\mathcal{L}_{og}$-function on a definable set** X is a function f of type

$$f = P(a_1, \ldots, a_k, \ln a_1, \ldots, \ln a_k), \tag{4.2.2}$$

where P is a polynomial and the a_i's are definable functions on X.

Remark 4.2.3. If $C_1, \ldots, C_l$ is a definable partition of $X \in \mathcal{S}_n$ and if $f : X \to \mathbb{R}$ is a function such that $g_i := f_{|C_i}$ is a $\mathcal{L}_{og}$-function for each i then f is itself a $\mathcal{L}_{og}$-function. This follows from the definition.

The following proposition gives a description of $\mathcal{L}_{og}$-functions which is derived from the Preparation Theorem and can be regarded as a Preparation Theorem for $\mathcal{L}_{og}$-functions.

Proposition 4.2.4. *Given a $\mathcal{L}_{og}$-function $f : X \to \mathbb{R}$, $X \in \mathcal{S}_n$, there exists a cell decomposition of $\mathbb{R}^n$ compatible with X such that on every cell $C \subset X$ we have for $x = (\tilde{x}, x_n) \in X \subset \mathbb{R}^{n-1} \times \mathbb{R}$:*

$$f(x) = \sum_{i,j,k=0}^{N} \mu_i(x)\, c_j(\tilde{x})\, \ln^k(x_n - \theta(\tilde{x})), \tag{4.2.3}$$

where $N \in \mathbb{N}$, θ is a definable function on the basis D of C (independent of $i, j,$ and k), the μ_i's are reduced functions on C with translation θ, and the c_j's are $\mathcal{L}_{og}$-functions on D.

Proof. Consider a function f as in (4.2.2) and apply the Preparation Theorem to the a_i's. This provides a cell decomposition compatible with X such that on every cell $C \subset X$, every a_i can be written (see Lemma 1.6.7)

$$a_i(\tilde{x}, x_n) = |x_n - \theta(\tilde{x})|^{r_i} b_i(\tilde{x}) U_i(\tilde{x}, x_n - \theta(\tilde{x})), \tag{4.2.4}$$

where b_i and θ are analytic definable functions, $r_i \in \mathbb{Q}$, and U_i is an $\mathcal{L}$-unit of the cell C. Let us fix such a cell C and observe that the latter equality implies:

$$\ln a_i(x) = r_i \ln |x_n - \theta(\tilde{x})| + \ln b_i(\tilde{x}) + \ln U_i(\tilde{x}, x_n - \theta(\tilde{x})).$$

Since U_i is a unit, $\ln U_i$ is a definable function. By (4.2.2) and (4.2.4), we thus can see that f can be expressed as a sum of type:

$$\sum_{i,j,k=0}^{N} \mu_i(x) c_j(\tilde{x}) \ln^k |x_n - \theta(\tilde{x})|, \tag{4.2.5}$$

where each μ_i is a definable function, each c_j is a $\mathcal{L}_{og}$-function on the basis of C, and $N \in \mathbb{N}$.

The expression appearing in (4.2.5) is not completely satisfying because we are not sure that each μ_i can be reduced with translation θ. To overcome this problem, we are going to make use of an argument which is similar to the one that we used in the proof of Lemma 1.6.7 (see (1.6.2) and (1.6.3)). Since each μ_i is definable, refining the cell decomposition if necessary, we can assume it to be reduced on C. Denote by θ' the translation of the reduction on C (for all i, see Lemma 1.6.7).

Up to a refinement, we can assume that $(\theta' - \theta)$, $(x_n - \theta')$, and $(x_n - \theta)$ are of constant sign on C and that their respective absolute values are comparable with each other. If $|x_n - \theta| \leq |x_n - \theta'|$ then (see (1.6.2) and (1.6.3)) $|x_n - \theta'|$ is reduced with translation θ. This implies that μ_i is itself reduced with translation θ and we are done. So, we can assume that $|x_n - \theta'| \leq |x_n - \theta|$. We now distinguish two cases:

Case 1: $|x_n - \theta'| \leq |\theta' - \theta|$ on C.

One can easily see that in this case $\frac{x_n-\theta'}{\theta'-\theta}$ takes values in $(-1,+\infty)$ and is bounded away from -1 and $+\infty$ on C (using that $|x_n-\theta'|\le|x_n-\theta|$). Since $u(h):=\frac{\ln(1+h)}{h}$ extends to a nowhere vanishing analytic function on $(-1,+\infty)$, we thus see that

$$\ln(1+\frac{x_n-\theta'}{\theta'-\theta})=\frac{x_n-\theta'}{\theta'-\theta}\,u\left(\frac{x_n-\theta'}{\theta'-\theta}\right)$$

is reduced with translation θ'. As a matter of fact, if we write

$$\ln(x_n-\theta)=\ln(\theta'-\theta)+\ln(1+\frac{x_n-\theta'}{\theta'-\theta}),$$

and plug this equality into (4.2.5), we see that f has the desired form (with translation θ' in this case).

Case 2: $|x_n-\theta'|\ge|\theta'-\theta|$ on C. This case is addressed analogously (see (1.6.3)). □

Proposition 4.2.5. *Let $f:X\times\mathbb{R}\to\mathbb{R}$ be a $\mathcal{L}_{og}$-function, $X\in\mathcal{S}_n$. If f is bounded then $\lim_{\varepsilon\to0^+}f(x,\varepsilon)$ exists for all $x\in X$ and defines a $\mathcal{L}_{og}$-function on X.*

Proof. Let $\mathcal{D}$ be the cell decomposition of $\mathbb{R}^{n+1}$ compatible with $X\times\mathbb{R}$ provided by Proposition 4.2.4 (applied to f). We may assume $\mathcal{D}$ to be compatible with $X\times\{0\}$. Fix a cell $D\in\pi(\mathcal{D})$, where $\pi:\mathbb{R}^{n+1}\to\mathbb{R}^n$ is the canonical projection. There is a unique cell C of $\mathcal{D}$ which is a band $(0,\zeta)$, with $\zeta:D\to(0,+\infty)$ definable function.

By Proposition 4.2.4, there are $\mathcal{L}_{og}$-functions $c_0,\dots,c_N$ on D, a definable function θ on D, as well as some reduced functions $\mu_0,\dots,\mu_N$ on C (with translation θ) such that we have on C

$$f(x,\varepsilon)=\sum_{i,j,k=0}^{N}\mu_i(x,\varepsilon)c_j(x)\ln^k(\varepsilon-\theta(x)). \tag{4.2.6}$$

Up to a refinement of our cell decomposition, we may assume that either $\theta\equiv0$ or θ does not vanish on D. If $\theta(x)$ is nonzero then $\ln(\varepsilon-\theta)$ tends to $\ln\theta$ as $\varepsilon\to0^+$ and $\lim_{\varepsilon\to0^+}\mu_i(x,\varepsilon)$ gives rise to a definable function (this limit is finite for μ_i is reduced with translation θ). The result is thus clear in this case and we will suppose $\theta\equiv0$.

By Proposition 1.8.7, there is a definable partition $\mathcal{P}$ of D into definable manifolds such that for every $B\in\mathcal{P}$ and each i, $\mu_i(x,\varepsilon)$ coincides with a Puiseux series in ε with analytic coefficients on $(0,\xi)$, where $\xi<\zeta_{|B}$ is a positive continuous definable function on B. Fix $B\in\mathcal{P}$. By (4.2.6), f itself then may be expressed on $(0,\xi)$ as a convergent series $\sum_{j=\nu}^{\infty}\sum_{k=0}^{N}\alpha_{jk}(x)\varepsilon^{\frac{j}{p}}\ln^k\varepsilon$, where the α_{jk}'s are $\mathcal{L}_{og}$-functions on B, $p\in\mathbb{N}^*$, $N\in\mathbb{N}$, and $\nu\in\mathbb{Z}$. Every monomial $\alpha_{jk}(x)\varepsilon^j\ln^k\varepsilon$ for which (j,k) is nonzero tends to 0 or $\pm\infty$ as ε tends to 0 (for each x). As f is bounded, we see that $\alpha_{jk}(x) \equiv 0$ for all $(j,k) \in (\mathbb{Z} \setminus \mathbb{N}) \times \mathbb{N}$ and all $(j,k) \in \{0\} \times \mathbb{N}^*$ (since any two distinct such monomials cannot go to $\pm\infty$ at the same speed). We deduce that

$$\lim_{\varepsilon \to 0^+} f(x,\varepsilon) = \alpha_{00}(x),$$

which is a $\mathcal{L}_{og}$-function on $B \in \mathcal{P}$ (see Remark 4.2.3). $\square$

Theorem 4.2.6. *If $f : A \to \mathbb{R}$, $A \in \mathcal{S}_{m+n}$, is a $\mathcal{L}_{og}$-function and if $l \leq n$ is such that $f_t(x) := f(t,x) \in L^1_{\mathcal{H}^l}(A_t)$, for all $t \in \mathbb{R}^m$, then the function*

$$g(t) := \int_{x \in A_t} f_t(x)\, d\mathcal{H}^l(x), \quad t \in \mathbb{R}^m,$$

is a $\mathcal{L}_{og}$-function as well.

Proof. The strategy of the proof will go as follows. The description of $\mathcal{L}_{og}$-functions given in (4.2.3) will provide a convergent expansion of $f(t,x)$ in x_n and $\ln x_n$ (it will suffice to integrate with respect to the last variable x_n for we will argue by induction). Integrating every term of this convergent series will provide an expansion of the same type for g, yielding that g is a $\mathcal{L}_{og}$-function.

Up to a definable diffeomorphism, we can assume that A is included in $[0,1]^{m+n}$. Take a cell decomposition of $\mathbb{R}^{m+n}$ compatible with A such that f is continuous on every cell (see Remark 1.8.3). It is enough to show the result for a cell $C \subset A$ (see Remark 4.2.3).

If $\dim C_t < l$ (for some and hence for all t) then the result is obvious. If $\dim C_t > l$ (for all t in the basis of C) then $f \equiv 0$ on C (for if f were nonzero at some $(t_0,x_0) \in C$ then f_{t_0}, which is continuous, would be bounded below away from zero near x_0 and thus could not be $L^1_{\mathcal{H}^l}$, since every open neighborhood of x_0 in C_{t_0} has dimension bigger than l). We thus can assume $\dim C_t = l$ for all t in the basis of C. Moreover, there is a linear map $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^{m+l}$ preserving the m first coordinates and inducing a diffeomorphism on C onto a cell of $\mathbb{R}^{m+l}$. It means that this is no loss of generality to simply address the case $l = n$.

Thanks to Proposition 4.2.4, we can assume that $f(t,x)$ can be decomposed for $x = (\tilde{x}, x_n) \in C_t \subset \mathbb{R}^{n-1} \times \mathbb{R}$ as:

$$f_t(x) = \sum_{i,j,k=0}^{N} \mu_i(t,x) \cdot c_j(t,\tilde{x}) \cdot \ln^k(x_n - \theta(t,\tilde{x})), \tag{4.2.7}$$

where θ is a definable function on the basis of C, the μ_i's are reduced functions on C (with translation θ), and the c_j's are $\mathcal{L}_{og}$-functions.

Fix $i \leq N$. Up to a change of variables (without changing notations) of type $(t,x) \mapsto (t, \tilde{x}, x_n + \theta(t,\tilde{x}))$, we can assume that $\theta \equiv 0$. As μ_i is a reduced function on C with translation $\theta \equiv 0$, it can be written for $(t,\tilde{x},x_n)$ in C

$$\mu_i(t,x) = b(t,\tilde{x})\, x_n^r\, U(t,x), \quad r \in \mathbb{Q}, \tag{4.2.8}$$

where b is a definable function, $r \in \mathbb{Q}$, and U is a unit (depending on i), i.e., a function that can be written $\psi \circ W$ with W bounded mapping of type

$$W(t,x) = (u_1(t,\tilde{x}), \dots, u_p(t,\tilde{x}), v(t,\tilde{x})x_n^{\frac{1}{q}}, w(t,\tilde{x})x_n^{-\frac{1}{q}}),$$

with $u_1, \dots, u_p, v, w$ definable functions, $q \in \mathbb{N}^*$, and ψ analytic on a neighborhood of $cl(W(C))$.

Before integrating f_t, we need to break the unit U into two series, one with negative powers in x_n and one with nonnegative powers. This is the purpose of the claim below. Set first for simplicity for $(t,x) = (t,\tilde{x},x_n)$ in C:

$$W_1(t,x) := (u_1(t,\tilde{x}), \dots, u_p(t,\tilde{x}), v(t,\tilde{x})\, x_n^{\frac{1}{q}}, v(t,\tilde{x})w(t,\tilde{x}))$$

as well as

$$W_2(t,x) := (u_1(t,\tilde{x}), \dots, u_p(t,\tilde{x}), w(t,\tilde{x})\, x_n^{-\frac{1}{q}}, v(t,\tilde{x})w(t,\tilde{x})).$$

Claim. There is a partition of C into cells such that on each element of this partition, U can be written

$$U(t,x) = \psi(W(t,x)) = \Psi_1(W_1(t,x)) + \Psi_2(W_2(t,x)) \tag{4.2.9}$$

where Ψ_1 and Ψ_2 are two analytic functions on a neighborhood of $cl(W_1(C))$ and $cl(W_2(C))$ respectively.

To prove this, we shall distinguish three cases. Given $\eta > 0$, splitting C into several cells, we can assume that one of the following situations occurs on C:

<u>**Case 1:**</u> $|v(t,\tilde{x})x_n^{\frac{1}{q}}| \geq \eta$. In this case, we can write

$$w(t,\tilde{x})x^{-\frac{1}{q}} = \Lambda(v(t,\tilde{x})x_n^{\frac{1}{q}}, v(t,\tilde{x})w(t,\tilde{x})),$$

with $\Lambda(y,z) = \frac{z}{y}$. Hence, since $U = \psi \circ W$, it is enough to set $\Psi_2 = 0$ and $\Psi_1(u,y,z) := \Psi(u,y,\Lambda(y,z))$, which is analytic on $cl(W_1(C))$ (since $s \mapsto \frac{1}{s}$ is analytic on the complement of the origin).

<u>**Case 2:**</u> $|w(t,\tilde{x})x_n^{-\frac{1}{q}}| \geq \eta$. This case is addressed completely analogously. We set $\Psi_1 = 0$ and Ψ_2 is defined in a similar way as Ψ_1 in case 1.

<u>**Case 3:**</u> We suppose that $|v(t,\tilde{x})x_n^{\frac{1}{q}}|$ and $|w(t,\tilde{x})x_n^{-\frac{1}{q}}|$ are both bounded by η. If η is chosen small enough then the result directly follows from Lemma 1.6.13 (applied to the function ψ). This completes the proof of the claim.

Write now C as (ζ,ζ'), where ζ and ζ' are two functions on the basis D of C satisfying $\zeta < \zeta'$. Set for simplicity $\xi_{t,\varepsilon} := \zeta_t + \varepsilon(\zeta'_t - \zeta_t)$ as well as $\xi'_{t,\varepsilon} := \zeta'_t - \varepsilon(\zeta'_t - \zeta_t)$, for $\varepsilon > 0$, and observe that $[\xi_{t,\varepsilon}, \xi'_{t,\varepsilon}] \subset C_t$. By Proposition 4.2.5 and Lebesgue's Dominated Convergence Theorem, it is enough to show that

$$\lambda(t,\varepsilon) := \int_{[\xi_{t,\varepsilon},\xi'_{t,\varepsilon}]} f_t \, d\mathcal{H}^n$$

is a $\mathcal{L}_{og}$-function. Notice that $\lambda(t,\varepsilon) = \int_{\tilde{x}\in D_t} h(t,\varepsilon,\tilde{x}) d\mathcal{H}^{n-1}(\tilde{x})$ where we have set for $\tilde{x} \in D_t$

$$h(t,\varepsilon,\tilde{x}) := \int_{\xi_{t,\varepsilon}(\tilde{x})}^{\xi'_{t,\varepsilon}(\tilde{x})} f_t(\tilde{x}, x_n) dx_n.$$

Since we can argue by induction on n, it is enough to establish that h is a $\mathcal{L}_{og}$-function. The above claim (see (4.2.9)) implies that f_t can be decomposed on C as the sum of two convergent series (via (4.2.7) and (4.2.8)). As we can integrate these two series by integrating every term, it is enough to deal with each monomial $x_n^{j/p} \cdot \ln^k x_n$, $j \in \mathbb{Z}, k \le N$, $p \in \mathbb{N}^*$, appearing in the convergent expansion.

These monomials may easily be integrated by finitely many integrations by parts. Namely, for $k = 0$ or $\frac{j}{p} = -1$, a straightforward computation of antiderivative yields that $\int_{\xi_{t,\varepsilon}(\tilde{x})}^{\xi'_{t,\varepsilon}(\tilde{x})} x_n^{j/p} \cdot \ln^k x_n \, dx_n$ is a $\mathcal{L}_{og}$-function. For k positive integer or $\frac{j}{p} \neq -1$, after a suitable integration by parts, one gets a new integral of the same type with a lower exponent in $\ln x_n$. $\square$

Together with Proposition 4.2.1, this theorem implies:

Corollary 4.2.7. *If $f : A \to \mathbb{R}$ is definable, where $A \in \mathcal{S}_{m+n}$, and $l \le n$, then*

$$g(t) := \int_{x\in A_t} f_t(x) \, d\mathcal{H}^l(x),$$

defined on $E := \{t \in \mathbb{R}^m : f_t \in L^1_{\mathcal{H}^l}(A_t)\}$, is a $\mathcal{L}_{og}$-function.

In particular, in the case of the constant function $f : A \to \mathbb{R}$, $f \equiv 1$, we get:

Corollary 4.2.8. *Let $A \in \mathcal{S}_{m+n}$ and $l \le n$. The function $g(t) := \mathcal{H}^l(A_t)$, defined on the definable set $\{t \in \mathbb{R}^m : \mathcal{H}^l(A_t) < \infty\}$, is a $\mathcal{L}_{og}$-function.*

4.3 On the $\mathcal{H}^l$-measure of globally subanalytic sets

4.3.1 The function $\psi(X, r)$ and the density θ_X

Given a set $X \in \mathcal{S}_n$, we set for $x \in \mathbb{R}^n$ and $r \ge 0$:

$$\psi(X, x, r) := \mathcal{H}^l(X \cap \mathbf{B}(x, r)),$$

where $l = \dim X$. When $x = 0$, we will shorten $\psi(X, x, r)$ into $\psi(X, r)$.

In this section, we describe some properties of ψ and introduce the notion of density, sometimes called the Lelong number. It is easy to see that if $A \in \mathcal{S}_{m+n}$ then $E^l := \{t \in \mathbb{R}^m : \dim A_t = l\}$ also belongs to $\mathcal{S}_{m+n}$ for every l. Hence, applying Proposition 4.1.3 and Corollary 4.2.8 to E^l for each l, we see:

Proposition 4.3.1. *For any $A \in \mathcal{S}_{m+n}$, the function $(t, x, r) \mapsto \psi(A_t, x, r)$ is a $\mathcal{L}_{og}$-function satisfying $\psi(A_t, x, r) \lesssim r^{l_t}$, where $l_t := \dim A_t$, for $(t, x, r) \in \mathbb{R}^m \times \mathbb{R}^n \times [0, +\infty)$.*

In particular, in the case $m = 0$, thanks to Proposition 1.8.6, this entails that for every $X \in \mathcal{S}_n$ there exist positive integers p and N as well as real numbers $a_{i,j}$, $i \in \mathbb{N}$, $j \leq N$, such that for $r > 0$ small enough:

$$\psi(X, r) = \sum_{i=0}^{\infty} \sum_{j=0}^{N} a_{i,j}\, r^{\frac{i}{p}} \ln^j r. \tag{4.3.1}$$

It is worthy of notice that the above proposition entails that the first term of this expansion is of order at least $l := \dim X$.

Proposition 4.3.2. *Let $X \in \mathcal{S}_n$ be of dimension l. Given $x \in \mathbb{R}^n$, the limit*

$$\theta_X(x) := \lim_{r \to 0} \frac{\psi(X, x, r)}{\mathcal{H}^l(\mathbf{B}(0_{\mathbb{R}^l}, 1)) \cdot r^l}$$

exists and is finite. It is called the **density of** X **at** x.

Proof. By (4.3.1), the limit exists, and, by Proposition 4.3.1, it is finite. □

The density is sometimes called the **Lelong** number. It is easily checked that if X is a smooth manifold then $\theta_X \equiv 1$ on X. If X is a complex analytic subset of $\mathbb{C}^n$ (that we can regard as a definable subset of $\mathbb{R}^{2n}$) and $x \in X$ then $\theta_X(x)$ is equal to the multiplicity of X at x. The notion of density may thus be considered as a real counterpart of the complex notion of multiplicity.

Theorem 4.3.3. *The function θ_X is a $\mathcal{L}_{og}$-function on X, for all $X \in \mathcal{S}_n$.*

Proof. Let $X \in \mathcal{S}_n$. Since Proposition 4.3.1 yields that $f(x, r) := \frac{\psi(X,x,r)}{\mathcal{H}^l(\mathbf{B}(0_{\mathbb{R}^l},1)) \cdot r^l}$ is a bounded $\mathcal{L}_{og}$-function, Proposition 4.2.5 gives the result. □

4.3.2 Uniform bounds

We give some estimates of the $\mathcal{H}^l$-measure of germs of globally subanalytic sets that will be needed in the proof of Theorem 4.4.2. It will be very important that the constants given by the following propositions do not depend on the parameters.

Given $X \in \mathcal{S}_n$ and $\varepsilon \geq 0$, define the ε-**neighborhood of** X as:

$$X_{\leq \varepsilon} := \{x \in \mathbb{R}^n : d(x, X) \leq \varepsilon\}.$$

We also set:

$$X_{=\varepsilon} := \{x \in \mathbb{R}^n : d(x, X) = \varepsilon\}.$$

The proposition below enables us to bound uniformly with respect to ε the measure of the ε-neighborhoods of the fibers of a globally subanalytic family.

Proposition 4.3.4. *Let $A \in \mathcal{S}_{m+n}$ be such that $\sup_{t\in\mathbb{R}^m} diam(A_t) < \infty$ (see (3.1.4) for diam), and let $k < n$ be an integer. There is $C > 0$ such that for all $t \in \mathbb{R}^m$ for which $\dim A_t \leq k$ and all $\varepsilon > 0$ we have*

$$\mathcal{H}^{n-1}(A_{t,=\varepsilon}) \leq C\varepsilon^{n-k-1}. \tag{4.3.2}$$

Moreover, given B in $\mathcal{S}_{m+n}$ and an integer $l > k$, there is $C > 0$ such that for all $t \in \mathbb{R}^m$ for which $\dim B_t \leq l$ as well as $\dim A_t \leq k$, and all $\varepsilon > 0$, we have

$$\mathcal{H}^l(A_{t,\leq\varepsilon} \cap B_t) \leq C\,\varepsilon^{l-k}. \tag{4.3.3}$$

Proof. We establish these two estimates simultaneously by induction on n, both statements being obvious in the case $n = 1$. We start with the induction step of (4.3.2). Given $P \in \mathbb{G}^n_{n-1}$ and a positive integer j, we have $K^P_j(A_{t,=\varepsilon}) \subset \pi_P(A_t)_{\leq\varepsilon}$. Moreover, the family of sets $\pi_P(A_t)$, $t \in \mathbb{R}^m, P \in \mathbb{G}^n_{n-1}$, is definable. Hence, thanks to the induction hypothesis (identifying P with $\mathbb{R}^{n-1}$ and applying (4.3.3)), we can conclude that for $t \in \mathbb{R}^m$ and $\varepsilon \geq 0$

$$\mathcal{H}^{n-1}(K^P_j(A_{t,=\varepsilon})) \leq \mathcal{H}^{n-1}(\pi_P(A_t)_{\leq\varepsilon}) \lesssim \varepsilon^{n-k-1},$$

which means that the result is a direct consequence of (4.1.7).

We now turn to perform the induction step of (4.3.3), starting with the case $l = n$, for which we can assume $B_t = \mathbb{R}^n$, for all t. Let, for $x \in \mathbb{R}^n$, $\rho_t(x) := d(x, A_t)$. It is easy to see that $|\partial_x\rho_t| = 1$ at each x where ρ_t is differentiable. Consequently, $J_x(\rho_t) \equiv 1$ on a definable dense subset of $\mathbb{R}^n$, which, by (4.1.1), yields:

$$\mathcal{H}^n(A_{t,\leq\varepsilon}) \leq \int_0^\varepsilon \mathcal{H}^{n-1}(A_{t,=\alpha})d\alpha \overset{(4.3.2)}{\lesssim} \varepsilon^{n-k},$$

as required.

It remains to show the result in the case $l < n$. Observe that in this case, $K_j^P(A_{t,\le\varepsilon} \cap B_t)$ is included in $\pi_P(A_t)_{\le\varepsilon} \cap B_t$, for every $j \in \mathbb{N}^*$ and $P \in \mathbb{G}_l^n$. Hence, thanks to (4.1.7) and the induction hypothesis (identifying P with $\mathbb{R}^l$), we see that the desired estimate holds. □

Proposition 4.3.5. *Given $A \in \mathcal{S}_{m+n}$, there exists a constant C such that for all $t \in \mathbb{R}^m$, all $\varepsilon \in (0,1]$, and all $r \ge 0$, we have:*

$$\psi(A_{t,\le\varepsilon}, r) \le C\, r^{n-1}\varepsilon + \mathcal{H}^n(A_t \cap \mathbf{B}(0_{\mathbb{R}^n}, r)).$$

In particular, if $\dim A_t < n$*, we then have for all such r, t, and ε:*

$$\psi(A_{t,\le\varepsilon}, r) \le C r^{n-1}\varepsilon. \tag{4.3.4}$$

Proof. We will use a method which is similar to the one we used in the proof of the preceding proposition. We first establish the desired estimate assuming $\dim A_t < n$, for all $t \in \mathbb{R}^m$. As explained in the proof of the preceding proposition, $J_x(d(x, A_t)) \equiv 1$ on a definable dense subset of $\mathbb{R}^n$. Moreover, since $\dim A_{t,=\alpha} \le n-1$ for every $\alpha > 0$ and $t \in \mathbb{R}^m$, by Proposition 4.1.3, there exists $C > 0$ such that for all such t and α we have $\mathcal{H}^{n-1}(A_{t,=\alpha} \cap \mathbf{B}(0,r)) \le Cr^{n-1}$. We thus can write:

$$\begin{aligned}
\psi(A_{t,\le\varepsilon}, r) &= \int_{A_{t,\le\varepsilon}\cap\mathbf{B}(0,r)} d\mathcal{H}^n \\
&\overset{(4.1.1)}{\le} \int_0^\varepsilon \mathcal{H}^{n-1}(A_{t,=\alpha} \cap \mathbf{B}(0,r))\, d\mathcal{H}^1(\alpha) \quad (\text{since } J_x(d(x,A_t)) \equiv 1) \\
&\le Cr^{n-1}\varepsilon,
\end{aligned}$$

establishing (4.3.4). To prove the result in general, let us set $E_t := \delta(A_t)$. Since $A_{t,\le\varepsilon} \subset E_{t,\le\varepsilon} \cup A_t$ and $\dim E_t < n$ for all t, the result follows from (4.3.4) for E_t. □

Proposition 4.3.6. *Let $l \in \mathbb{N}$ and let $A \in \mathcal{S}_{m+n}$ be such that* $\dim A_t = l$ *for all $t \in \mathbb{R}^m$. There is $C > 0$ such that for all $t \in \mathbb{R}^m$, we have for all r and r' small enough positive real numbers satisfying $r' \le r$:*

$$|\psi(A_t, r) - \psi(A_t, r')| \le Cr^{l-1}|r - r'|.$$

Proof. For $t \in \mathbb{R}^m$ let λ_t denote the restriction to $A_{t,reg}$ of the function $\rho : \mathbb{R}^n \to \mathbb{R}$, defined by $\rho(x) := |x|$. We wish to estimate $\psi(A_t, r)$ by means of (4.1.1), which requires to establish that $J_{\lambda_t}(x)$ goes to 1 as $x \in A_{t,reg}$ tends to 0 for every $t \in \mathbb{R}^m$.

Since $J_x(\lambda_t) = |\partial_x \lambda_t|$, and since, for almost each $x \in A_{t,reg}$, the vector $\partial_x\lambda_t$ is the projection of $\partial_x\rho = \frac{x}{|x|}$ onto $T_xA_{t,reg}$, it is enough to check that the angle between x and $T_xA_{t,reg}$ tends to zero as $x \in A_{t,reg}$ tends to zero. Indeed, if otherwise, by Curve Selection Lemma we could find a definable arc $\gamma(s)$ in $A_{t,reg}$ tending to zero (as $s \to 0^+$) such that the angle between $\gamma(s)$ and $T_{\gamma(s)}A_{t,reg}$ is bounded below away from zero. Since definable arcs are Puiseux arcs, $\lim_{s\to 0^+} \frac{\gamma(s)}{|\gamma(s)|} = \lim_{s\to 0^+} \frac{\gamma'(s)}{|\gamma'(s)|}$. As $\gamma'(s) \in T_{\gamma(s)} A_{t,reg}$, this is a contradiction which yields that $J_{\lambda_t}(x)$ goes to 1 when $x \in A_{t,reg}$ tends to 0.

Fix now $t \in \mathbb{R}^m$ and choose a positive real number r_0 (depending on t) sufficiently small for the sets $\mathbf{S}(0_{\mathbb{R}^n}, r) \cap A_t$ to be of dimension $(l-1)$ for all $r \le r_0$. As $A_t \cap \mathbf{S}(0_{\mathbb{R}^n}, r) \subset A_t \cap \mathbf{B}(0_{\mathbb{R}^n}, 2r)$, by Proposition 4.1.3, we know that there is a constant C independent of t and r such that for all $r \le r_0$:

$$\mathcal{H}^{l-1}(A_t \cap \mathbf{S}(0_{\mathbb{R}^n}, r)) \le C r^{l-1}. \tag{4.3.5}$$

We thus can write for $r' \le r \le r_0$:

$$\begin{aligned} |\psi(A_t, r) - \psi(A_t, r')| &= \Big|\int_{A_t \cap \mathbf{B}(0,r) \setminus \mathbf{B}(0,r')} d\mathcal{H}^l\Big| \\ &\le 2 \int_{A_t \cap \mathbf{B}(0,r) \setminus \mathbf{B}(0,r')} J_x(\lambda_t)\, d\mathcal{H}^l(x) \ \text{(since } J_q(\lambda_t) \text{ tends to } 1) \\ &\overset{(4.1.1)}{=} 2 \int_{r'}^{r} \mathcal{H}^{l-1}(A_t \cap \mathbf{S}(0_{\mathbb{R}^n}, s))\, d\mathcal{H}^1(s), \end{aligned}$$

which, by (4.3.5), yields the claimed estimate. □

4.4 $\mathcal{H}^l$-measure and α-approximations of the identity

We are going to show that families of homeomorphisms that are close to the identity can induce some stability of the measure even if they are not Lipschitz. This fact is possible because we work with globally subanalytic families of sets. The considered homeomorphisms will however not be assumed to be globally subanalytic.

Definition 4.4.1. Let A and B in $\mathcal{S}_{m+n}$ and let $\alpha : (0, \eta) \times \mathbb{R}^m \to \mathbb{R}$ be a positive definable function. We call **α-approximation of the identity** a family of germs of homeomorphisms (not necessarily definable) $h_t : (A_t, 0) \to (B_t, 0)$, $t \in \mathbb{R}^m$, such that for all $t \in \mathbb{R}^m$ and all $r > 0$ small enough

$$|h_t(x) - x| \le \alpha(r, t), \tag{4.4.1}$$

for all $x \in \mathbf{B}(0, r) \cap A_t$, and

$$|h_t^{-1}(x) - x| \le \alpha(r, t), \tag{4.4.2}$$

for all $x \in \mathbf{B}(0, r) \cap B_t$.

The main purpose for introducing the notion of α-approximation of the identity is the study of the variation of the density on a definable set (section 4.5). The theorem below that compares the measure of the fibers of two families that are related by an α-approximation of the identity is however of its own interest. It is easy to produce examples of non subanalytic families for which this theorem fails.

Theorem 4.4.2. *Let $(A_t)_{t\in\mathbb{R}^m}$ and $(B_t)_{t\in\mathbb{R}^m}$ be two definable families of l-dimensional subsets of $\mathbb{R}^n$ and let $h_t : A_t \to B_t$, $t \in \mathbb{R}^m$, be an α-approximation of the identity, with $\alpha : (0,\eta) \times \mathbb{R}^m \to \mathbb{R}$ definable positive function. There is $C > 0$ (independent of α) such that for every $t \in \mathbb{R}^m$ we have for all $r > 0$ small enough:*

$$|\psi(A_t, r) - \psi(B_t, r)| \le C\alpha(r,t) \cdot r^{l-1}.$$

The idea of the proof of this theorem is that since Cauchy-Crofton's formula reduces the computation of the Hausdorff measure of a set to the computation of the cardinal of generic fibers of generic projections (restricted to the considered set), comparing the $\mathcal{H}^l$-measures of $A_t \cap \mathbf{B}(0,r)$ and $B_t \cap \mathbf{B}(0,r)$ reduces to compare the respective cardinals of $\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}(0,r)$ and $\pi_P^{-1}(x) \cap B_t \cap \mathbf{B}(0,r)$, for $P \in \mathbb{G}_l^n$ and $x \in P$ generic. We therefore start with a proposition which, assuming that we are in the situation of Theorem 4.4.2, yields that the respective cardinals of generic fibers of projections are the same on the complement of a set whose measure is bounded in terms of α.

Proposition 4.4.3. *Let $(A_t)_{t\in\mathbb{R}^m}$ and $(B_t)_{t\in\mathbb{R}^m}$ be two definable families of l-dimensional subsets of $\mathbb{R}^n$ and let $h_t : A_t \to B_t$ be an α-approximation of the identity, with $\alpha : (0,\eta)\times\mathbb{R}^m \to \mathbb{R}$ definable positive function. There are a constant C (independent of α) and a definable family of sets $Z_t(P,r)$, $P \in \mathbb{G}_l^n$, $r > 0$, $t \in \mathbb{R}^m$, satisfying for each such P and t, for all $r > 0$ small enough:*

(i) $\mathcal{H}^l(Z_t(P,r)) \le C\alpha(r,t) \cdot r^{l-1}$.

(ii) For any $x \in P \cap \mathbf{B}(0,r) \setminus Z_t(P,r)$:

$$\mathit{card}\,(\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}(0,r)) = \mathit{card}\,(\pi_P^{-1}(x) \cap B_t \cap \mathbf{B}(0,r)). \qquad (4.4.3)$$

Proof. We first define $Z_t(P,r)$ (see (4.4.5)). In this proof, we will sometimes identify an element P of $\mathbb{G}_l^n$ with $\mathbb{R}^l$ (mentioning it) so that, given a subset $X \subset P$ and $\varepsilon > 0$, we write $X_{\le\varepsilon}$ for the set of points $x \in P$ satisfying $d(x,X) \le \varepsilon$.

Let A_t' be the set of points of A_t at which A_t fails to be an analytic manifold of dimension l. For every $P \in \mathbb{G}_l^n$ and $t \in \mathbb{R}^m$ let $S_t(P)$ be the union of A_t' together with the set of points of $A_t \setminus A_t'$ at which π_P fails to be a submersion. Let then

$$H_t(P,r) := \pi_P(S_t(P)) \cap \mathbf{B}(0,r).$$

For every t and P, $\dim H_t(P,r) \leq \dim \pi_P(S_t(P)) < l$, by Sard's Theorem. As the family $H_t(P,r)$, $P \in \mathbb{G}_l^n$, $r \in \mathbb{R}_+$, $t \in \mathbb{R}^m$, is definable, by Proposition 4.3.5 (identify P with $\mathbb{R}^l$) there is a constant C such that for all $t \in \mathbb{R}^m$:

$$\mathcal{H}^l(H_t(P,r)_{\leq 2\alpha(r,t)}) \leq C\alpha(r,t)r^{l-1}. \tag{4.4.4}$$

Let now

$$\mathcal{M}_t(P,r) := \pi_P(A_t \cap \mathbf{B}(0,r) \setminus \mathbf{B}(0, r-2\alpha(r,t))),$$

and notice that by Proposition 4.3.6, there is $C > 0$ such that for all $t \in \mathbb{R}^m$:

$$\mathcal{H}^l(\mathcal{M}_t(P,r)) \leq C\alpha(r,t) \cdot r^{l-1}.$$

We then easily derive from Proposition 4.3.5 (again identifying P with $\mathbb{R}^l$) that:

$$\mathcal{H}^l(\mathcal{M}_t(P,r)_{\leq 3\alpha(r,t)}) \leq C\alpha(r,t) \cdot r^{l-1},$$

for some C independent of r and t. Let us now define $Z_t(P,r) \subset P$ as the union

$$H_t(P,r)_{\leq 2\alpha(r,t)} \cup H'_t(P,r)_{\leq 2\alpha(r,t)} \cup \mathcal{M}_t(P,r)_{\leq 3\alpha(r,t)} \cup \mathcal{M}'_t(P,r)_{\leq 3\alpha(r,t)}, \tag{4.4.5}$$

where $H'_t(P,r)$ and $\mathcal{M}'_t(P,r)$ are defined in the same way as $H_t(P,r)$ and $\mathcal{M}_t(P,r)$ but replacing A_t with B_t. By the above estimates (the estimates obtained for $H_t(P,r)$ and $\mathcal{M}_t(P,r)$ clearly hold for $H'_t(P,r)$ and $\mathcal{M}'_t(P,r)$ as well) we see that (i) holds.

Let us now prove (4.4.3). Fix $P \in \mathbb{G}_l^n$, $r > 0$, $t \in \mathbb{R}^m$, as well as an element $x \in P \cap \mathbf{B}(0,r) \setminus Z_t(P,r)$, and set for simplicity

$$j := \text{card}\, \pi_P^{-1}(x) \cap A_t \cap \mathbf{B}(0,r) \quad \text{and} \quad j' := \text{card}\, \pi_P^{-1}(x) \cap B_t \cap \mathbf{B}(0,r).$$

We have to check that $j = j'$. Remark that by definition of $Z_t(P,r)$ we have

$$d(x, \pi_P(S_t(P))) > 2\alpha(r,t),$$

which means that $\pi_P^{-1}(\mathbf{B}(x, 2\alpha(r,t))) \cap A_t \cap \mathbf{B}(0,r)$ is exclusively constituted by nonsingular points at which $\pi_{P|A_t}$ is submersive. By symmetry of the roles of A_t and B_t, the analogous fact holds for $\pi_P^{-1}(\mathbf{B}(x, 2\alpha(r,t))) \cap B_t \cap \mathbf{B}(0,r)$.

By Ehresmann's Theorem[1], the intersection of $\pi_P^{-1}(\mathbf{B}(x, 2\alpha(r,t)))$ with the set $A_{t,reg} \cap \mathbf{B}(0,r)$ (resp. $B_{t,reg} \cap \mathbf{B}(0,r)$) is thus the union of j (resp. j') connected components $C_1, \dots, C_j$ (resp. $D_1, \dots, D_{j'}$) and the restriction of π_P to every C_i (resp. D_i) is a homeomorphism onto $\mathbf{B}(x, 2\alpha(r,t)) \cap P$.

Since x does not belong to $\mathcal{M}_t(P,r)_{\leq 3\alpha(r,t)}$, the ball of radius $\alpha(r,t)$ centered at x does not intersect $\mathcal{M}_t(P,r)_{\leq 2\alpha(r,t)}$. Hence, due to the definition of $\mathcal{M}_t(P,r)$, every

[1]A covering map above a simply connected set is always globally trivial.

point y of $C_{j_0} \cap \pi_P^{-1}(\mathbf{B}(x, \alpha(r,t)))$, with $j_0 \in \{1, \dots, j\}$, must belong to $\mathbf{B}(0, r - 2\alpha(r,t))$, so that, by (4.4.1), the point $h_t(y)$ must belong to $\mathbf{B}(0,r)$. Moreover, again due to (4.4.1), we have $\pi_P(h_t(y)) \in \mathbf{B}(x, 2\alpha(r,t))$ so that $h_t(y)$ actually belongs to one of the D_i's. As C_{j_0} is connected and the D_i's are disjoint, the integer $k \le j'$ for which $h_t(y) \in D_k$ just depends on j_0 and not on the point y in $C_{j_0} \cap \pi_P^{-1}(\mathbf{B}(x, \alpha(r,t)))$. Let us thus denote by $\sigma(j_0)$ this integer.

In this way, we have defined a mapping σ from $\{1, \dots, j\}$ to $\{1, \dots, j'\}$. In order to show $j' \le j$, it suffices to establish that σ is surjective (by the symmetry of the roles of j and j', it is enough to check $j' \le j$. Here, j and j' might be zero but the argument just above has shown that if $j \neq 0$ then $j' \neq 0$ and the argument below will show that if $j' \neq 0$ then $j \neq 0$).

Let i be an integer between 1 and j', take a point $z \in \pi_P^{-1}(x) \cap D_i$, and set $y := h_t^{-1}(z)$. Since $x \notin \mathcal{M}'_t(P, r)$, the point z belongs to $\mathbf{B}(0, r - 2\alpha(r,t))$; this implies, via (4.4.2), that the point y belongs to $\mathbf{B}(0,r)$. By (4.4.2), it is clear that $\pi_P(y) \in \mathbf{B}(x, 2\alpha(r,t))$. Thus, $y \in C_{i_0}$ for some i_0, which implies that $\sigma(i_0) = i$. □

Proof of Theorem 4.4.2. Since $(A_t)_{t \in \mathbb{R}^m}$ and $(B_t)_{t \in \mathbb{R}^m}$ are two definable families, m_{A_t} and m_{B_t} are bounded independently of t (see Corollary 1.8.12). Moreover, by (i) and (ii) of Proposition 4.4.3, there is a positive constant C such that for each $t \in \mathbb{R}^m$, each j, and each $P \in \mathbb{G}_l^n$, we have for all $r > 0$ small enough:

$$|\mathcal{H}^l(K_j^P(A_t \cap \mathbf{B}(0,r))) - \mathcal{H}^l(K_j^P(B_t \cap \mathbf{B}(0,r)))| \le C\alpha(r,t) \cdot r^{l-1}.$$

The result thus directly follows from (4.1.7). □

4.5 Variation of the density

As we noticed, $\theta_X \equiv 1$ on the regular locus of $X \in \mathcal{S}_n$. This raises a natural question: how is the density of a globally subanalytic set affected by the geometry of the singularities? The theorem below provides information on this issue.

Theorem 4.5.1. *Let a closed set $X \in \mathcal{S}_n$ be stratified by a stratification Σ.*

(i) If Σ is (w)-regular then θ_X is locally Lipschitz on the strata of Σ.

(ii) If Σ is Whitney (b) regular then θ_X is continuous on the strata of Σ.

Proof of (i). For simplicity we will do the proof in the case where the stratification Σ is reduced to two strata S and S' with $S \subset cl(S') \setminus S'$. The proof in the general case relies on the same idea and the reader is referred to [gVa08]. Up to a coordinate system, we may assume $S = \mathbb{R}^k \times \{0\}$ and work nearby the origin. We will carry out the proof in the case $k = 1$. In the case where S has higher dimension, one may apply the same argument to establish that the density is locally Lipschitz with respect to each coordinate of S (which yields the local Lipschitzness of the density on S). Given $t \in S$, define

$$A_t := \{x \in \mathbb{R}^n : (x+t) \in X\}, \tag{4.5.1}$$

so that the germ of A_t at the origin is the translation of the germ of X at t.

We claim that, for t and t' in a neighborhood of 0, there is an α-approximation of the identity $h_{t,t'} : A_t \to A_{t'}$, where $\alpha(r,t,t') = Cr|t-t'|$ for some constant $C > 0$ (here A_t and $A_{t'}$ are regarded as families parameterized by two parameters t' and t, constant with respect to t' and t respectively).

We start by defining a vector field. The desired family of homeomorphisms will then be given by the flow of this vector field. Set for x in S'

$$v(x) := \frac{P_x(e_1)}{< P_x(e_1), e_1 >}, \tag{4.5.2}$$

where P_x stands for the orthogonal projection onto T_xS', e_1 for the first vector of the canonical basis of $\mathbb{R}^n$, and $<,>$ for the euclidean inner product. Extend v to S by setting $v(x) := e_1$ for $x \in S$. It easily follows from the (w) condition that there is $C > 0$ such that for any $x \in S'$ and $t \in S$ sufficiently close to 0:

$$|v(x) - v(t)| \leq C|x-t|. \tag{4.5.3}$$

Denote by ϕ the local flow of this vector field (defined on each stratum). By (4.5.2), we see that if $\pi : \mathbb{R}^n \to \mathbb{R} \times \{0_{\mathbb{R}^{n-1}}\}$ denotes the orthogonal projection then

$$\phi(\pi(x), s) = \pi(\phi(x,s)), \tag{4.5.4}$$

for all $x \in S'$ and $s \in \mathbb{R}$ close to zero. Furthermore, by Grönwall's Lemma, the preceding estimate implies that for $x \in S'$ and $t \in S$ close to 0, and s positive small:

$$|x-t| \exp(-Cs) \leq |\phi(x,s) - \phi(t,s)| \leq \exp(Cs)|x-t|. \tag{4.5.5}$$

The first inequality and (4.5.4) establish that an integral curve starting at $x \in S'$ may not fall into S. The second inequality implies that it stays in a little neighborhood of S if x is chosen sufficiently close to S. It also yields that ϕ is continuous (ϕ being smooth on strata, its continuity just needs to be checked at points of S).

Fix now t and t' in S close to 0, and let for $x \in A_t$ (i.e., $(x+t) \in X$) close to the origin:

$$h_{t,t'}(x) := \phi(x+t, t'-t) - t' = \phi(x+t, t'-t) - \phi(t, t'-t),$$

which belongs to $A_{t'}$. Integrating (4.5.3), we see (using (4.5.5) and the Mean Value Theorem) that there is a constant C' such that:

$$|h_{t,t'}(x) - x| \leq C'|t-t'| \cdot |x|. \tag{4.5.6}$$

This establishes that $h_{t,t'}$ is an α-approximation of the identity with $\alpha(r,t,t') := C'r \cdot |t-t'|$. By Theorem 4.4.2, we get that there is a constant C'' independent of t and t' in S such that for r positive small enough:

$$|\psi(A_t, r) - \psi(A_{t'}, r)| \leq C'' r^l \cdot |t-t'|,$$

where $l = \dim X$. This implies:

$$|\theta_X(t) - \theta_X(t')| = |\theta_{A_t}(0) - \theta_{A_{t'}}(0)| \leq C''|t-t'|,$$

which yields the Lipschitzness of the density in the vicinity of 0.

Proof of (ii). As the argument is very similar, we will just provide a sketch of proof. We also restrict ourselves to the case of a stratification constituted by only two strata S and S', with $S \subset cl(S') \setminus S'$, assuming $\dim S = 1$. Using curve selection lemma, one may actually reduce the proof of the key point (see [gVa08] for more details) to the case where the stratum S is one dimensional.

As our problem is local, we may identify S with a neighborhood of the origin in $\mathbb{R} \times \{0_{\mathbb{R}^{n-1}}\}$. Since S is a one-dimensional stratum, by Proposition 2.6.18, (S', S) satisfies Kuo's (r) condition. By Łojasiewicz's inequality, it means that there is a rational number $\mu < 1$ such that on S' we have on a neighborhood of the origin:

$$\angle(T_{\pi(x)}S, T_x S') \lesssim \frac{|x - \pi(x)|}{|x|^\mu}, \tag{4.5.7}$$

where π is the orthogonal projection onto $\mathbb{R} \times \{0_{\mathbb{R}^{n-1}}\}$. Let v be the vector field on X defined in the proof of (i) (see (4.5.2)). Of course, because we no longer assume the (w) condition, (4.5.3) might fail. Nevertheless, (4.5.7) ensures that for $x \in X$ close to the origin:

$$|v(x) - v(\pi(x))| \lesssim \frac{|x - \pi(x)|}{|x|^\mu}.$$

Denote by ϕ the flow of this vector field. One may show existence and uniqueness of the integral curves by a similar argument as in the proof of (i) (see (4.5.5)). We then can define a family of mappings $h_t : A_t \to A_0$ (where A_t is as in (4.5.1)) by $h_t(x) := \phi(x, -t)$. This mapping is an α-approximation of the identity, with $\alpha(r,t) := Ct^{1-\mu} \cdot r$ for some positive constant C (by the same argument as to show (4.5.6)). Again using Theorem 4.4.2, we derive that for t close to 0:

$$|\psi(A_t, r) - \psi(A_0, r)| \leq Ct^{1-\mu} \cdot r^l$$

(where again $l = \dim X$), which implies

$$|\theta_X(t) - \theta_X(0)| = |\theta_{A_t}(0) - \theta_{A_0}(0)| \leq Ct^{1-\mu},$$

which yields the continuity of the density at the origin. □

4.6 Stokes' formula

We end this chapter by proving Stokes' formula on globally subanalytic (possibly singular) sets (Theorem 4.6.7). The formula that we will give applies to a large class of differential forms, called *stratified forms*. These differential forms are not necessarily continuous but are locally bounded (Proposition 4.6.2). What makes them attractive is that the pull-back of a stratified form via a definable Lipschitz (not necessarily differentiable) map is a stratified form (see Definition 4.6.3).

Stratified forms. If ω is a differential k-form on a submanifold $S \subset \mathbb{R}^n$, we denote by $|\omega(x)|$ the norm of the linear form $\omega(x) : \otimes^k T_x S \to \mathbb{R}$, where S is equipped with the Riemannian metric inherited from the ambient space. We denote by $d\omega$ the exterior differential of ω.

Definition 4.6.1. Let $X \in \mathcal{S}_n$ and let Σ be a stratification of X.

A **stratified differential** 0**-form on** (X, Σ) is a collection of functions $\omega_S : S \to \mathbb{R}$, $S \in \Sigma$, that glue together into a continuous function on X.

A **stratified differential** k**-form on** (X, Σ), $k > 0$, is a collection $(\omega_S)_{S \in \Sigma}$ where, for every S, ω_S is a continuous differential k-form on S such that for any $(x_i, \xi_i) \in \otimes^k TS$, with x_i tending to $x \in S' \in \Sigma$ and ξ_i tending to $\xi \in \otimes^k T_x S'$, we have

$$\lim \omega_S(x_i, \xi_i) = \omega_{S'}(x, \xi).$$

The **support of a stratified form** ω on (X, Σ) is the closure in X of the set

$$\bigcup_{S \in \Sigma} \{x \in S : \omega_S(x) \neq 0\}.$$

When this set is compact, ω is said to be **compactly supported**.

We say that a stratified form $\omega = (\omega_S)_{S \in \Sigma}$ is **differentiable** if ω_S is $\mathscr{C}^1$ for every $S \in \Sigma$ and if $d\omega := (d\omega_S)_{S \in \Sigma}$ is a stratified form.

Proposition 4.6.2. *Let (X, Σ) be a stratified subset of $\mathbb{R}^n$ and let $\omega = (\omega_S)_{S \in \Sigma}$ be a stratified form. If the support of ω is closed (in $\mathbb{R}^n$) then, for every $S \in \Sigma$, $|\omega_S(x)|$ is bounded on every bounded subset of S.*

Proof. If ω is a 0-form, this is clear since ω_S is the restriction of a continuous function on X. Take a stratified k-form ω that has closed support (in $\mathbb{R}^n$), with $k > 0$, and let us assume that the result fails for ω. It means that there is a bounded sequence $(x_i, \xi_i) \in \otimes^k TS$, $S \in \Sigma$, such that $\omega_S(x_i, \xi_i)$ goes to infinity. Since x_i is a bounded sequence, extracting a subsequence if necessary, we may assume that it is convergent to some element $x \in S'$, $S' \in \Sigma$ (the point x must belong to X for ω_S is zero near $fr(X)$). Let

$$\xi_i' := \frac{\xi_i}{\omega_S(x_i, \xi_i)},$$

so that $\omega_S(x_i, \xi_i') = 1$, for all i. As $\omega_S(x_i, \xi_i)$ is going to infinity and ξ_i is bounded, ξ_i' goes to zero. As ω is a stratified form, this implies that we must have

$$\lim \omega_S(x_i, \xi_i') = \omega_{S'}(x, 0) = 0,$$

in contradiction with $\omega_S(x_i, \xi_i') \equiv 1$. $\square$

Given a stratification Σ, we denote by $\Sigma^{(k)}$ the collection of all the strata of Σ of dimension k, and by $\cup\Sigma^{(k)}$ the union of all the elements of $\Sigma^{(k)}$.

Definition 4.6.3. Let $\omega = (\omega_S)_{S\in\Sigma}$ be a stratified form, Σ' be a refinement of Σ, and take $T \in \Sigma'$. By definition of refinements, there is a unique $S \in \Sigma$ which contains T. Let ω_T denote the differential form induced by ω_S on T. It is a routine to check that $\omega' := (\omega_T)_{T\in\Sigma'}$ is also a stratified form. We then say that ω' is a **refinement of** ω.

Given a horizontally $\mathscr{C}^1$ stratified mapping $F : (X, \Sigma_1) \to (Y, \Sigma_2)$ and a stratified form $\omega = (\omega_S)_{S\in\Sigma_2}$ on (Y, Σ_2), let us define the **pull-back of the stratified form** ω under (F, Σ_1, Σ_2) as

$$F^*\omega := (F^*_{|S}\omega_{S'})_{S\in\Sigma_1},$$

where $F^*_{|S}\omega_{S'}$ stands for the pull-back of the differential form $\omega_{S'}$ under the smooth mapping $F_{|S} : S \to S'$, $S \in \Sigma_1, S' \in \Sigma_2$, induced by F on S (see Definition 2.6.9). Since F is horizontally $\mathscr{C}^1$, it is easily checked from the definitions that $F^*\omega$ is then also a stratified form.

Thanks to Proposition 2.6.12, we see that if $h : X \to Y$ is a definable mapping (not necessarily smooth) which is locally Lipschitz with respect to the inner metric then every stratified form on Y can be pulled back to a stratified form on X. In particular, if Y is a manifold (that we can endow with the one-stratum stratification), then every smooth form on Y can be pulled-back to a stratified form by such a mapping h.

Integration of stratified forms. Let (X, Σ) be a stratified set, $X \in \mathcal{S}_n$. Take a compactly supported stratified k-form $\omega = (\omega_S)_{S\in\Sigma}$ on (X, Σ), $k \in \mathbb{N}$, and let $Y \subset X$ be a definable subset of dimension k such that Y_{reg} is oriented. We are going to define the integral of ω on Y, denoted $\int_Y \omega$.

Let Σ' be a refinement of Σ compatible with Y_{reg}. This refinement inducing a refinement ω' of ω (as explained just above), we may naturally set

$$\int_Y \omega := \sum_{S\in\Sigma'^{(k)}, S\subset Y} \int_S \omega_S,$$

where every stratum is endowed with the orientation induced by Y_{reg}. That ω_S is integrable on S follows from the fact that ω is compactly supported (and hence bounded by Proposition 4.6.2).

If Σ'' is another refinement of Σ compatible with Y_{reg} then $Y_{reg} \cap (\cup \Sigma'^{(k)})$ and $Y_{reg} \cap (\cup \Sigma''^{(k)})$ coincide outside a set of dimension less than k (which is thus $\mathcal{H}^k$-negligible) so that

$$\sum_{S \in \Sigma'^{(k)}, S \subset Y} \int_S \omega_S = \sum_{S \in \Sigma''^{(k)}, S \subset Y} \int_S \omega_S \ ,$$

which shows that the integral is independent of the chosen refinement. In the case where $\dim Y < k$, we set this integral to be zero.

Given $X \subset \mathbb{R}^n$, a **definable singular k-simplex of X** will be a continuous definable mapping $\sigma : \Delta_k \to \mathbb{R}^n$ such that $|\sigma| \subset X$, Δ_k being the k-simplex spanned by $0, e_1, \dots, e_k$, where $e_1, \dots, e_k$ is the canonical basis of $\mathbb{R}^k$, and $|\sigma|$ the support of σ. We denote by $C_k(X)$ the $\mathbb{R}$-vector space of definable singular k-chains, i.e., finite linear combinations (with real coefficients) of singular definable simplices, and we will write ∂c for the boundary of c.

Integration on definable singular simplices. Let again (X, Σ) be a stratified set, $X \in \mathcal{S}_n$, and let $\omega = (\omega_S)_{S \in \Sigma}$ be a stratified k-form on X.

We are going to define the integral of ω over an oriented definable singular simplex $\sigma : \Delta_k \to X$. As σ is definable, by Proposition 2.6.10, there exist stratifications $\hat{\Sigma}$ of Δ_k and Σ' of X such that for any S in $\hat{\Sigma}$ there is $T \in \Sigma'$ such that the mapping $\sigma_{|S} : S \to T$, induced by the restriction of σ, is a $\mathscr{C}^\infty$ submersion. Moreover, we may assume that Σ' is a refinement of Σ, and hence that ω is a stratified form on (X, Σ'). We now set[2]:

$$\int_\sigma \omega = \sum_{S \in \hat{\Sigma}^{(k)}} \int_S \sigma^* \omega_{\sigma(S)}.$$

Again, since the manifold $\cup \hat{\Sigma}^{(k)}$ is independent of the stratification $\hat{\Sigma}$ up to a negligible set, this definition is clearly independent of the chosen stratifications. The integral over a definable chain $c \in C_k(X)$ is then defined naturally.

Stokes' formula for stratified forms. Our Stokes' formulas, stated in Theorems 4.6.7 (on definable weakly normal manifolds) and 4.6.9 (on definable singular simplices), require some preliminaries. The main difficulty of extending this formula to manifolds that admit singularities within their closure is that there is no nice notion of boundary in this framework. This motivates the following definitions.

Given a $\mathscr{C}^0$ submanifold M of $\mathbb{R}^n$, we define **the $\mathscr{C}^i$ boundary** as:

$$\partial^i M := \{x \in fr(M) : (cl(M), fr(M)) \text{ is a } \mathscr{C}^i \text{ manifold with boundary at } x\}.$$

[2]The form $(\sigma^* \omega_{\sigma(S)})_{S \in \hat{\Sigma}}$ is not necessarily a stratified form on Δ_k. In particular, $\sigma^* \omega_{\sigma(S)}$ is not necessarily bounded (σ is not assumed to have bounded first derivative). It is however $L^1_{\mathcal{H}^k}$.

The manifold M will be said to be **weakly normal** if there is a definable set $E \subset fr(M)$ along which M is connected (see Definition 3.1.25) and such that $\dim(fr(M) \setminus E) \le \dim M - 2$.

Of course, every normal manifold is weakly normal. An interesting feature of weakly normal definable manifolds is the following property.

Lemma 4.6.4. *Let M be a definable $\mathscr{C}^i$ manifold, $i = 0$ or 1. If M is weakly normal then* $\dim(fr(M) \setminus \partial^i M) \le \dim M - 2$.

Proof. If M is weakly normal, it follows from existence of $\mathscr{C}^0$ definable triangulations (that comes down from Theorem 3.2.4) that there is a definable subset $E \subset fr(M)$ satisfying $\dim(fr(M) \setminus E) \le \dim M - 2$, and such that $(cl(M), E)$ is a $\mathscr{C}^0$ manifold with boundary near every point of E. This already shows the result in the case $i = 0$. If $i = 1$, the result then easily follows from Proposition 1.8.6. $\square$

Lemma 4.6.5. *Let $(M, \partial M)$ be a definable $\mathscr{C}^1$ manifold with boundary that we stratify by $\Sigma := \{M \setminus \partial M, \partial M\}$, and let us take a continuous definable function $\rho : M \to [0, +\infty)$, $\mathscr{C}^1$ on $M \setminus \partial M$, satisfying $\rho^{-1}(0) = \partial M$. For any compactly supported differentiable stratified $(k-1)$-form ω on (M, Σ), $k = \dim M$, we have*

$$\lim_{\varepsilon \to 0^+} \int_{\rho = \varepsilon} \omega = \int_{\partial M} \omega. \tag{4.6.1}$$

Proof. Up to a partition of unity we may assume that the support of ω fits in one chart of M and, up to a coordinate system, we may identify M with $\mathbf{B}(0_{\mathbb{R}^k}, \alpha) \cap \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_k \ge 0\}$, $\alpha > 0$. As ω is a stratified form, the coefficients of restriction of the multi-linear form $\omega(x_1, \dots, x_k)$ to $\mathbb{R}^{k-1} \times \{0_{\mathbb{R}}\}$ are continuous with respect to $x_k \ge 0$ (and bounded, see Proposition 4.6.2). Hence, in the case where $\rho(x_1, \dots, x_k) = x_k$ for all $(x_1, \dots, x_k)$, it suffices to pass to the limit inside the integral. As a matter of fact, it is enough to check that the limit always exists and is independent of the function ρ.

For any function ρ satisfying the assumptions of the lemma, by (the classical) Stokes' formula, we have for relevant orientations and $0 < \varepsilon < \varepsilon'$:

$$\int_{\rho = \varepsilon} \omega - \int_{\rho = \varepsilon'} \omega = \int_{\rho^{-1}((\varepsilon, \varepsilon'))} d\omega.$$

Since ω is compactly supported and $d\omega$ is bounded (see again Proposition 4.6.2), by Lebesgue's Dominated Convergence Theorem, we see that the right-hand-side goes to zero as $\varepsilon, \varepsilon' \to 0$ (bounded definable sets have finite measure, see Proposition 4.1.3). Consequently, the limit exists for all such function ρ. That the limit is independent of ρ follows from an analogous argument. $\square$

Lemma 4.6.6. *Let B be a definable compact subset of $\mathbb{R}^m \times \mathbb{R}^n$. If $\dim B_t \le l$, for all $t \in \mathbb{R}^m$, and $\dim B_{0_{\mathbb{R}^m}} < l$ then there are a constant C and a positive rational number θ such that for all $t \in \mathbb{R}^m$:*

$$\mathcal{H}^l(B_t) \le C\,|t|^\theta.$$

Proof. Let for $t \in \mathbb{R}^m$, $f(t) := \sup_{x \in B_t} d(x, B_{0_{\mathbb{R}^m}})$, with $f(t) = 0$ if B_t is empty. We wish to apply Łojasiewicz's inequality (Theorem 2.2.5) to f. We need for this purpose to check that $f(\gamma(s))$ tends to zero for every definable arc $\gamma : (0, \varepsilon) \to \mathbb{R}^m$ satisfying $\lim_{s \to 0^+} \gamma(s) = 0$. If γ is such an arc, let $x(s)$ be a definable arc satisfying $x(s) \in B_{\gamma(s)}$ and $d(x(s), \gamma(s)) = f(\gamma(s))$, for every s positive small. Because B is compact, the arc $x(s)$ must end at a point of $B_{0_{\mathbb{R}^m}}$, which shows that $d(x(s), B_{0_{\mathbb{R}^m}})$ tends to zero, as required.

We thus can derive from Theorem 2.2.5 that there are $C > 0$ and a positive rational number θ such that $f(t) \le C|t|^\theta$. Since $B_t \subset (B_{0_{\mathbb{R}^m}})_{\le f(t)}$, we have for $t \in \mathbb{R}^m$ and $P \in \mathbb{G}^n_l$:

$$\mathcal{H}^l(\pi_P(B_t)) \le \mathcal{H}^l\left(\pi_P(B_{0_{\mathbb{R}^m}})_{\le f(t)}\right) \overset{(4.3.4)}{\lesssim} f(t) \lesssim |t|^\theta.$$

In virtue of Cauchy-Crofton's formula, this yields the desired estimate. □

We first establish Stokes' formula for stratified forms on weakly normal manifolds.

Theorem 4.6.7. *Let M be an oriented k-dimensional weakly normal definable $\mathscr{C}^0$ manifold (without boundary) and let Σ be a stratification of $cl(M)$. For any compactly supported differentiable stratified $(k-1)$-form ω on $(cl(M), \Sigma)$, we have:*

$$\int_M d\omega = \int_{fr(M)} \omega, \tag{4.6.2}$$

where $fr(M)_{reg}$ is endowed with the induced orientation.

Proof. Take such a stratified form ω and let V be a definable neighborhood in $cl(M)$ of the support on ω. Let $h : |K| \to cl(M)$ be a $\mathscr{C}^0$ definable triangulation such that V, $fr(M)$, as well as the elements of Σ are unions of images of open simplices. Note that if $\sigma \in K$ is such that $\dim \sigma = \dim M$ then $h(\sigma)$ is an open subset of a stratum, and hence a smooth manifold. As $h(cl(\sigma))$ is a $\mathscr{C}^0$ manifold with boundary, $h(\sigma)$ is clearly weakly normal. Since it is enough to prove the result for the sets $h(\sigma)$, $\sigma \in K$ such that $\sigma \subset h^{-1}(V)$, it means that we can assume that ω refines a form which is $\mathscr{C}^1$ on M. Moreover, taking a refinement if necessary we can assume $fr(M)$ to be a union of strata.

By Proposition 2.7.1, there is a $\mathscr{C}^1$ definable function ρ on M satisfying $|\rho(x) - d(x, fr(M))| < \frac{d(x,fr(M))}{2}$, which means that ρ is positive and extends continuously (by 0) on $fr(M)$. Let us then set

$$T_\varepsilon := \{x \in M : \rho(x) \geq \varepsilon\}.$$

Note that for $\varepsilon > 0$ small enough, by Sard's theorem, T_ε is a $\mathscr{C}^1$ manifold with boundary and ω_M is a smooth form on it. Thus, by the classical Stokes' formula:

$$\int_{T_\varepsilon} d\omega_M = \int_{\partial T_\varepsilon} \omega_M.$$

As $d\omega_M$ is bounded (by Proposition 4.6.2) and compactly supported, it easily follows from Lebesgue's Dominated Convergence Theorem (bounded definable sets have finite measure, see Proposition 4.1.3) that:

$$\lim_{\varepsilon \to 0} \int_{T_\varepsilon} d\omega_M = \int_M d\omega_M.$$

We thus only have to show that:

$$\limsup_{\varepsilon \to 0} \int_{\partial T_\varepsilon} \omega_M = \int_{fr(M)} \omega. \tag{4.6.3}$$

As M is weakly normal, by Lemma 4.6.4, there is a definable subset $E \subset fr(M)$ satisfying $\dim(fr(M) \setminus E) \leq \dim M - 2$ and such that $cl(M)$ is a $\mathscr{C}^1$ manifold with boundary at every point of $fr(M) \setminus E$. Let $(\varphi_\delta, \psi_\delta)$ denote a $\mathscr{C}^\infty$ partition of unity subordinated to the covering of $\mathbb{R}^n$ constituted by the two open sets $int(E_{\leq \delta})$ and $\mathbb{R}^n \setminus E_{\leq \frac{\delta}{2}}$ (see section 4.3.2 for $E_{\leq \delta}$). As ω_M is bounded (see Proposition 4.6.2) and because φ_δ has support in $E_{\leq \delta}$, we can write for some $C > 0$:

$$\int_{\partial T_\varepsilon} \varphi_\delta \, \omega_M \leq C \mathcal{H}^{k-1}(E_{\leq \delta} \cap \partial T_\varepsilon).$$

Thanks to Lemma 4.6.6 (applied to the definable family $B_{\delta,\varepsilon} := E_{\leq \delta} \cap \partial T_\varepsilon$, $B_{0,0} := cl(E)$), this entails that:

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \int_{\partial T_\varepsilon} \varphi_\delta \, \omega_M = 0.$$

As a matter of fact, since for each $\delta > 0$ we have $\omega = \varphi_\delta \omega + \psi_\delta \omega$, equality (4.6.3) reduces to show that:

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\partial T_\varepsilon} \psi_\delta \, \omega_M = \int_{fr(M)} \omega. \tag{4.6.4}$$

It follows from Lemma 4.6.5 (applied to $\psi_\delta\,\omega$ which induces a compactly supported stratified form on the manifold with boundary $cl(M)\setminus E$) that

$$\lim_{\varepsilon\to 0}\int_{\partial T_\varepsilon}\psi_\delta\,\omega=\int_{fr(M)\setminus E}\psi_\delta\,\omega=\int_{fr(M)}\psi_\delta\,\omega.$$

Passing to the limit as $\delta>0$ tends to zero and applying Lebesgue's Dominated Convergence Theorem, we get (4.6.4). □

Corollary 4.6.8. *Let $M\in\mathcal{S}_n$ be a definable $\mathscr{C}^0$ manifold with boundary ∂M and let Σ be a stratification of M. For any compactly supported differentiable stratified $(k-1)$-form $\omega:=(\omega_S)_{S\in\Sigma}$ on M, where $k=\dim M$, we have:*

$$\int_M d\omega=\int_{\partial M}\omega.$$

Proof. Take such a stratified form ω as well as a definable compact neighborhood V in $cl(M)$ of the support of ω. Let $h:|K|\to cl(M)$ be a triangulation such that $fr(M)$, V, M, ∂M, as well as the elements of Σ are unions of images of open simplices. As in the proof of Theorem 4.6.7, we just need to prove the result for $M=cl(h(\sigma))$, for each k-dimensional simplex $\sigma\in K$ included in $h^{-1}(V)$. But, as $h(\sigma)$ is normal, this follows from this theorem. □

We now give a version of Stokes' formula on definable singular simplices, which generalizes the classical Stokes' formula for smooth simplices.

Theorem 4.6.9. *Let (X,Σ) be a definable stratified set. If ω is a differentiable stratified $(k-1)$-form on (X,Σ) then we have for all $c\in C_k(X)$:*

$$\int_c d\omega=\int_{\partial c}\omega.$$

Proof. It is of course enough to carry out the proof for one single definable simplex $\sigma:\Delta_k\to X$. We denote by $\pi_1:\Gamma_\sigma\to\Delta_k$ and $\pi_2:\Gamma_\sigma\to X$ the natural projections (where Γ_σ is the graph of σ). By Proposition 2.6.12, there are stratifications $\tilde{\Sigma}$ of Γ_σ and Σ' of X making of π_2 a horizontally $\mathscr{C}^1$ stratified mapping. We may assume that Σ' refines Σ, and hence that ω is a stratified form on (X,Σ'). Note that $\beta:=\pi_2^*\omega$ is a stratified form on $(\Gamma_\sigma,\tilde{\Sigma})$, which, by Corollary 4.6.8, entails that

$$\int_{\Gamma_\sigma}d\beta=\int_{\partial\Gamma_\sigma}\beta. \tag{4.6.5}$$

As $\pi_{1|S}$ is a diffeomorphism onto its image for all $S\in\tilde{\Sigma}$ and since $\pi_2\circ\pi_1^{-1}=\sigma$, we have $\sigma^*\omega_S=\pi_1^{-1*}\beta_{\tilde{S}}$ and $\sigma^*d\omega_S=d\pi_1^{-1*}\beta_{\tilde{S}}$, for every $S\in\Sigma'$ and $\tilde{S}\in\tilde{\Sigma}$ satisfying $\pi_2(\tilde{S})\subset S$. Hence,

$$\int_{\Delta_k}\sigma^*d\omega=\int_{\Delta_k}\pi_1^{-1*}d\beta=\int_{\Gamma_\sigma}d\beta\overset{(4.6.5)}{=}\int_{\partial\Gamma_\sigma}\beta=\int_{\partial\Delta_k}\pi_1^{-1*}\beta=\int_{\partial\Delta_k}\sigma^*\omega,$$

yielding the formula claimed in the statement of the theorem. □

Historical notes. The results about integration of globally subanalytic functions are taken from [Lio-Rol97, Lio-Rol98, Com-Lio-Rol00] (see also [Par01]). The density was originally introduced for complex analytic sets by P. Lelong, and studied in the subanalytic category by K. Kurdyka and G. Raby [Kur-Rab89]. The stability of the measure under α-approximations of the identity (Theorem 4.4.2) was studied in [gVa08] in order to show the continuity of the density on Whitney stratified sets. This problem was actually investigated earlier by G. Comte in [Com00] who established the continuity of the Lelong number under the slightly stronger Kuo-Verdier's (w) condition. Stokes' formula on subanalytic sets was proved by W. Pawłucki in [Paw85]. The present form is indeed a variation that was established in [gVa15].

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