The minimal Tjurina number and the 4/3 problem for plane curve singularities

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Milnor and Tjurina numbers

- $f:(\mathbb{C}^n,\mathbf{0})\longrightarrow(\mathbb{C},0)$ be a germ of isolated hypersurface singularity.
- We denote by

$$T_f := \frac{\mathbb{C}\{x_1,\ldots,x_n\}}{(f,\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})} \quad M_f := \frac{\mathbb{C}\{x_1,\ldots,x_n\}}{(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})}$$

The Milnor number is defined as

$$\mu := \dim_{\mathbb{C}} M_f.$$

It is a topological invariant of the singularity

John Milnor 1931-

The Tjurina number is defined as

$$\tau := \dim_{\mathbb{C}} T_f.$$

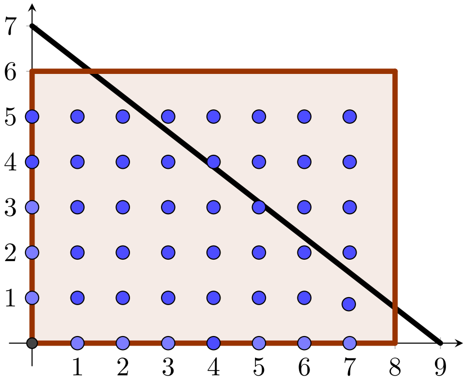
It is an analytic invariant of the singularity

Galina N. Tjurina 1938-1970

Toy example

Let us consider the curve $f(x,y) = y^7 - x^9 = 0$. In this case $M_f = T_f$, by using SINGULAR we calculate $\mu = \tau = 48$ and a basis for this algebra:

$$\{x^7y^5, x^6y^5, x^5y5, x^4y^5, x^3y^5, x^2y^5, xy^5, y^5, x^7y^4, x^6y^4, x^5y^4, x^4y^4, x^3y^4, x^2y^4, xy^4, y^4, x^7y^3, x^6y^3, x^5y^3, x^4y^3, x^3y^3, x^2y^3, xy^3, y^3, x^7y^2, x^6y^2, x^5y^2, x^4y^2, x^3y^2, x^2y^2, xy^2, y^2, x^7y, x^6y, x^5y, x^4y, x^3y, x^2y, xy, y, x^7, x^6, x^5, x^4, x^3, x^2, x, 1\}$$



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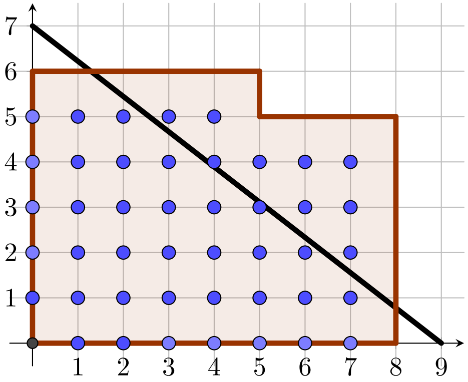
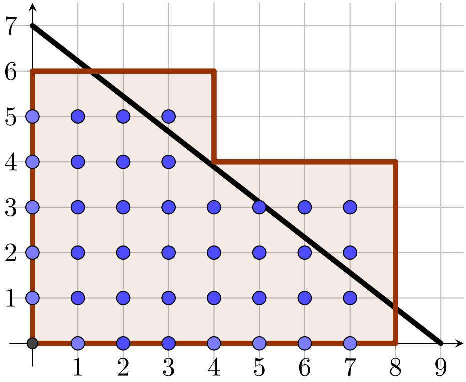


$$f(x,y) = y^7 - x^9 + x^5y^5$$

$$\mu = 48$$

$$\tau = 45$$

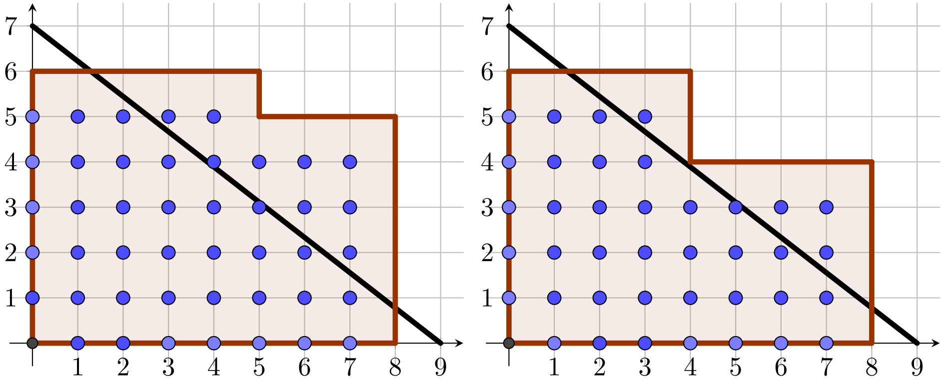


$$f(x,y) = y^7 - x^9 + x^4y^4$$

$$\mu = 48$$

$$\tau = 40$$

Toy example



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Topology vs Analysis

Definition

Two isolated hypersurface singularities defined by f and g have the same topological type if there is a homeomorphism $\varphi : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that $\varphi(V_f) = V_g$.

Topology vs Analysis

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Theorem (Teissier 1973)

If two isolated hypersurface singularities defined by f and g have the same topological type then $\mu(f) = \mu(g)$.

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Theorem (Mather-Yau 1982)

The hypersurface isolated singularities defined by f and g are analytically equivalent if and only if their Tjurina algebras are isomorphic as $\mathbb{C}$–algebras.

In particular, same analytic type $\Rightarrow$ $\tau(f) = \tau(g)$.

Deformation theory

Definition

A deformation of an isolated singularity (X, x) is a germ of flat morphism $(\mathcal{Y}, 0) \to (S, 0)$ whose special fiber is isomorphic to (X, x). We call $(S, 0)$ the base space of the deformation. The deformation is called versal if any other deformation results from it by base change. It is called miniversal if it is versal and S has minimal possible dimension.

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Theorem (Grauert 1972)

Any complex space germ (X, x) with isolated singularity has a miniversal deformation.

Deformation theory

Theorem (Tjurina 1969)

Let $(X,0) \subset (\mathbb{C}^n,0)$ *be an isolated hypersurface singularity defined by* $f \in \mathcal{O}_{\mathbb{C}^n,0}$ *and* $g_1,\ldots,g_k \in \mathcal{O}_{\mathbb{C}^n,0}$ *be a* $\mathbb{C}$*–basis of the Tjurina algebra* T_f *(resp. of the Milnor algebra* M_f*).*
If we set,

$$F(x,\mathbf{t}) := f(x) + \sum_{j=1}^{k} t_j g_j(x), \quad (\mathcal{X},0) := V(F) \subset (\mathbb{C}^n \times \mathbb{C}^k, 0),$$

then $(X,0) \hookrightarrow (\mathcal{X},0) \xrightarrow{\varphi} (\mathbb{C}^k,0)$, *with* φ *the second projection, is a miniversal (resp. versal) deformation of* $(X,0)$.

Problem: What is the minimal possible dimension of T_f if we fix a versal deformation?

A bit of history

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Moduli of irreducible plane curve singularities

Let $C : f(x,y) = 0$ be a germ of irreducible plane curve singularity, $\Gamma = \Gamma(C) := \{v_C(g) \in \mathbb{Z}_{\geq 0} \mid g \in \mathcal{O}_{C,\mathbf{0}} \setminus \{\mathbf{0}\}\} =< \overline{\beta}_0, \ldots, \overline{\beta}_g >$ semigroup of values of C.
Let $(C^\Gamma, \mathbf{0}) \subset (\mathbb{C}^{g+1}, \mathbf{0})$ be the curve defined via the parameterization

$$C^\Gamma : u_i = t^{\overline{\beta}_i}, \qquad 0 \leq i \leq g.$$

Theorem (Teissier 1973)

Every branch $(C, \mathbf{0})$ with semigroup Γ is isomorphic to the generic fiber of a one parameter complex analytic deformation of $(C^\Gamma, \mathbf{0})$.

Moduli of irreducible plane curve singularities

- $G : (X, \mathbf{0}) \longrightarrow (D, \mathbf{0})$ be the miniversal deformation of C^Γ. Moreover, $\mu = \dim D$.
- $D_\Gamma := \{$miniversal constant semigroup deformation of $C^\Gamma\} \subset D$.
- $\widetilde{M}_\Gamma := D_\Gamma / \sim$ moduli space associated to Γ.
- $q_{\min} :=$ minimal possible dimension of the base space of the miniversal constant semigroup deformation of the fiber $(G^{-1}(\boldsymbol{v}), \mathbf{0})$, $\boldsymbol{v} \in D$.

Theorem (Teissier 1973)

The dimension of the generic component M_1 of the moduli space of branches $\widetilde{M}_\Gamma$ with semigroup Γ is

$$\dim M_1 = q_{\min}.$$

Moduli of irreducible plane curve singularities

(Teissier) $D_\Gamma^{(2)} := \{\boldsymbol{v} \in D_\Gamma \mid (G^{-1}(\boldsymbol{v}), \mathbf{0}) \text{ is a plane branch}\}$ is open dense in D_Γ.

Moreover, if $m : D_\Gamma \longrightarrow \widetilde{M}_\Gamma$ is the natural projection then $m(D_\Gamma^{(2)})$ is the moduli space M_Γ of plane branches with semigroup Γ.

Even more, $M_1 \cap M_\Gamma \subset \widetilde{M}_\Gamma$ is a non-empty Zariski open dense set.

Theorem (Teissier 1973)

If we denote by $\tau_- = \dim D_\Gamma$.

$$\tau_{min} = q_{\min} + \mu - \tau_-$$

Minimal Tjurina number

Theorem (Alberich-A.-Blanco-Melle; Genzmer-Hernandes 2019)

For any equisingular class of germs of irreducible plane curve singularity,

$$\tau_{min} = \sigma(n) + \frac{n^2+3n-6}{2} + \sum_{p \text{ free}} \frac{(e_p-1)(e_p+2)+2\sigma(e_p+1)}{2} + \sum_{p \text{ sat.}} \frac{e_p(e_p-1)+2\sigma(e_p+2)}{2},$$

where the summation runs on all points p equal or infinitely near to the origin and $\sigma(k) = \frac{(k-2)(k-4)}{4}$ if k is even and $\sigma(k) = \frac{(k-3)^2}{4}$ if k is odd.

Proof: Genzmer gives a formula for the dimension of the generic component of the moduli space q. Also Wall proved a formula for the codimension $\tau_{min} - q$ of the μ–constant stratum of the miniversal deformation. $\square$

Dimca and Greuel Question

Example (Dimca-Greuel)

Consider the families of curves

$$X_a : x^{2a+1} + x^a y^{a+1} + y^{2a} = 0, \quad X_b : x^{2b+1} + x^{b+1} y^{b+1} + y^{2b+1} = 0.$$

For those families $\tau(X_a) = 3a^2$, $\mu(X_a) = 2a(2a-1)$, $\mu(X_b) = 4b^2$, $\tau(X_b) = 4b^2 - (b-1)^2$. Therefore, it follows that

$$\mu/\tau \xrightarrow[a\to\infty]{} 4/3. \quad \mu/\tau \xrightarrow[b\to\infty]{} 4/3$$

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Conjecture (Dimca-Greuel 2017)

Is for any plane curve singularity $\frac{\mu}{\tau} < \frac{4}{3}$?

Corollary

For any plane branch singularity,

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

Proof: $4\tau_{min} - 3\mu \geq 3n - 4 + \sum_{\substack{p \text{ free} \\ e_p > 0}} (e_p - 1) - \sum_{p \text{ sat.}} e_p > 0.$

Problems of the τ_{min} approach

- Too restrictive: Teissier and Genzmer results do not work for non irreducible curves.
- Too hard: We may find a formula for τ_{min} and not to be able to estimate it.
- Why $4/3$?

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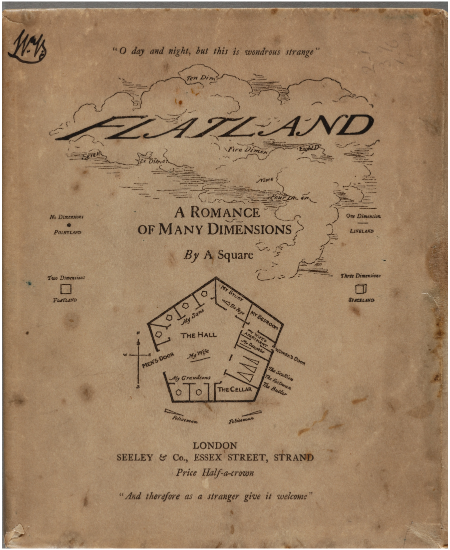

...The little Hexagon meditated on this a while and then said to me; "But you have been teaching me to raise numbers to the third power: I suppose three-to-the-third must mean something in Geometry; what does it mean?" "Nothing at all", replied I, "not at least in Geometry; for Geometry has only Two Dimensions"....

"Flatland, A Romance of Many Dimensions" by Edwin Abbott.

Milnor and Tjurina numbers for surface singularities

Let $(X,0) \in (\mathbb{C}^3,0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. Let $\widetilde{X} \to X$ be a resolution of singularities of X.

Key guest: The geometric genus

$$p_g := \dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}).$$

Theorem (Wahl 1985)

Let $(X,0) \in (\mathbb{C}^3,0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. Then

$$\mu - \tau \leq 2p_g$$

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DG Question from other perspective

Consider the germ of isolated surface singularity

$$\Sigma : z^2 + f(x,y) = 0.$$

Theorem (Tomari 1991)

Let p_g be the geometric genus of Σ and μ its Milnor number. Then

$$8p_g + 1 \leq \mu.$$

Remark: $C : f(x,y) = 0$ then $\tau(C) = \tau(\Sigma)$ and $\mu(C) = \mu(\Sigma)$.

DG Question from other perspective

Theorem (A. 2019)

For any germ of plane curve singularity

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

Proof: Let $f(x,y)=0$ be an equation of a germ of a plane curve singularity. Consider the surface singularity $f(x,y)+z^2=0$. Then, Wahl + Tomari give

$$\mu - \tau \leq 2p_g < \mu/4 \quad \square$$

Consequence: The bound $4/3$ can be inferred from the geometry of the singularity.

Durfee conjecture and the bound for surface singularities

Is $\mu/\tau < 4/3$ for any surface singularity? NO

$$f = x^{14} + y^6 z^8 + z^{14} + x^9 z^5 + (x + y + z)^{15}.$$

We can compute with SINGULAR that the Milnor number is $\mu = 2288$ and the Tjurina number is $\tau = 1660$. Therefore, $\mu/\tau > 4/3$.

What is the bound for surface singularities?

Conjecture (Durfee 1978)

For any isolated surface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$

$$6p_g \leq \mu.$$

Some partial results by: Tomari (91), Ashikaga (93), Némethi (98), Melle-Hernández (2000), Kóllar and Némethi (2017), Enokizono (2018). Still open

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Bound for surface singularities

Proposition

Let $(X,0) \subset (\mathbb{C}^3,0)$ *be an isolated surface singularity of one of the following types:*

(1) *Quasi-homogeneous singularities,*

(2) $(X,0)$ *of multiplicity* 3,

(3) *absolutely isolated singularity,*

(4) *suspension of the type* $\{f(x,y)+z^N=0\}$,

(5) *the link of the singularity is an integral homology sphere,*

(6) *the topological Euler characteristic of the exceptional divisor of the minimal resolution is positive.*

Then

$$\frac{\mu}{\tau} < \frac{3}{2}$$

Remark: All these cases are the cases for which Durfee conjecture is known to be true.

Bound for surface singularities

Is $\frac{\mu}{\tau} < \frac{3}{2}$ sharp? YES. Consider $F(x,y,z) = x^d + y^d + z^d + g(x,y,z) = 0$ with $deg(g) \geq d+1$. Then, Wahl shows that

$$\tau_{min} = (2d-3)(d+1)(d-1)/3.$$

Also, $\mu = (d-1)^3$. Then

$$\frac{\mu}{\tau_{\min}} \xrightarrow[d\to\infty]{} \frac{3}{2}.$$

Conjecture

For any $(X,0) \subset (\mathbb{C}^3, 0)$ *isolated surface singularity:*

$$\frac{\mu}{\tau} < \frac{3}{2}.$$

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Open problem

Theorem (Looijenga-Steenbrink 1985)

If (X,x) is an isolated complete intersection singularity of dimension $n \geq 2$, then

$$\mu - \tau = \sum_{p=0}^{n-2} h^{p,0}(X,x) + a_1 + a_2 + a_3.$$

Problem

Let $(X,0) \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity of dimension $n \geq 2$ and codimension $r = N - n$. Is there an optimal $\frac{b}{a} \in \mathbb{Q}$ with $b < a$ such that

$$\mu - \tau < \frac{b}{a}\mu \text{ ?}$$

Where optimal means that there exist a family of singularities such that μ/τ tends to $\frac{a}{a-b}$ when the multiplicity at the origin tends to infinity.

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Thanks for the attention!!