Hamiltonian reduction by stages and structure of co-adjoint orbits

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A symplectic reduction may be described as follows: given the symplectic action of a Lie group on a symplectic manifold having a momentum map, one divides a level set of the momentum map by the action of a suitable subgroup to form a new symplectic manifold. Before the division step, one has a manifold carrying a degenerate closed 2-form. Removing such a degeneracy by passing to a quotient space one obtains the reduced symplectic manifold [**?**, **?**].

The issue of performing reduction by stages arises already in the paper of Marsden and Weinstein [**?**] and may be formulated as follows: one wants a framework in which repeated reduction by two successive symmetry groups can be performed and the result is the same as that of a single larger group ([**?**]). One of the nicest examples of reduction by stages is the theory of semidirect product reduction that is due to Guillemin and Sternberg [**?**] and Marsden, Ratiu and Weinstein [**?**, **?**].

The issue of performing reduction by stages in its final and most general form was formulated by Marsden and Ratiu (see [**?**]). They decided that the framework of starting with a big group M with a normal subgroup N and trying to reduce first by N and then by some kind of a quotient group M_ν/N_ν (not exactly by M/N, $M_\nu \subset M$, $N_\nu \subset N$) was the right framework for reduction by stages theory.

In this talk we represent a short proof [**?**] of the Stages Hypothesis of Marsden-Misiołek-Ortega-Perlmutter-Ratiu (MMOPR), which is a sufficient condition for a general reduction by stages theorem [**?**],[**?**]. In the book [**?**] this hypothesis was verified for all split group extensions M of a Lie group N. In particular, both central extensions and semidirect products with a vector space fit into this class. We give the short Lie-algebraic proof of this hypothesis in a general case, for arbitrary pair (M, N) of a Lie group M and its normal not-necessary closed subgroup N. Our proof of the hypothesis is based on changing of the approach and the point of view: reformulating the Stages Hypothesis we obtain this hypothesis as a general fact in the structure theory of co-adjoint orbits of Lie groups: each M-coadjoint orbit contains some affine subspace determined by the normal subgroup N and this subspace is a N_ν-orbit.

Moreover, changing the point of view again we solve the non-equivariance problem arising in [**?**]: starting with an equivariant moment map J for the Lie group M, trying to first reduce by N (to obtain the first reduced space) and then by the quotient group M_ν/N_ν (to obtain the second reduced space), in general the action of this quotient group on the first reduced space is non-Hamiltonian, i.e. the corresponding moment map for the group M_ν/N_ν induced by J is non-equivariant. We solve this problem replacing the quotient group M_ν/N_ν by the group M_ν which acts equivariantly and the corresponding quotient space (second reduced space) is the same since the subgroup N_ν acts trivially. Moreover, in this case we use weaker conditions to carry out the reduction by stages procedure.

Contents:

3. The Stages Hypothesis.
4. Structure of co-adjoint orbits of Lie groups.
5. Strongly symplectic actions are Hamiltonian.

References

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