Self-indexing Energy function for Morse-Smale diffeomorphisms

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We show that necessary and sufficient conditions to existence of self-indexing energy function for gradient-like diffeomorphism on M^3 are equivalent to existence of special Heegaard splitting. The result was obtained in collaboration with V. Grines and F. Laudenbach.

Let M^n be a smooth closed orientable n-manifold. A diffeomorphism $f : M^n \to M^n$ is called *Morse-Smale diffeomorphism* if its nonwandering set $\Omega(f)$ consists of finitely many hyperbolic periodic points ($\Omega(f) = Per(f)$) whose invariant manifolds have a transversal intersection. D. Pixton [**?**] defined *a Liapunov function* for a Morse-Smale diffeomorphism f as a Morse function $\varphi : M^n \to \mathbf{R}$ such that $\varphi(f(x)) < \varphi(x)$ when x is not a periodic point and $\varphi(f(x)) = \varphi(x)$ when it is. He claimed that such function there is for any Morse-Smale diffeomorphism. If φ is a Liapunov function for a Morse-Smale diffeomorphism f, then any periodic point of f is a critical point of φ. The opposite is not true in general since a Liapunov function may have critical points which are not periodic points of f. Then Pixton defined an *energy function* for a Morse-Smale diffeomorphism f as a Liapunov function φ such that critical points of φ coincide with periodic points of f. He proved that for any Morse-Smale diffeomorphism given on a surface there is an energy function and construct an example of a Morse-Smale diffeomorphism on S^3 which has no energy function.

If φ is a Liapunov function of a Morse-Smale diffeomorphism $f : M^n \to M^n$ then any periodic point p is a maximum of the restriction of φ to the unstable manifold $W^u(p)$ and a minimum of its restriction to the stable manifold $W^s(p)$. If these extremums are non-degenerate then invariant manifolds of p are transversal to all regular level sets of φ in some neighborhood U_p of p. This local property is useful for the construction of a (global) Liapunov function. So we introduce the following definition.

Definition 1. A Liapunov function $\varphi : M^n \to \mathbb{R}$ for a Morse-Smale diffeomorphism $f : M^n \to M^n$ is called a *Morse-Liapunov function* if any periodic point p is a non-degenerate maximum of the restriction of φ to the unstable manifold $W^u(p)$ and a non-degenerate minimum of its restriction to the stable manifold $W^s(p)$.

Next definition follows to S. Smale [**?**] who introduced similar one for gradient-like vector fields.

Definition 2. A Morse-Liapunov function φ is called a *self-indexing energy function* when the following conditions are fulfilled:

1) the set of the critical points of function φ coincides with the set $Per(f)$ of the periodic points of f;

2) $\varphi(p) = dim\ W^u(p)$ for any periodic point $p \in Per(f)$.

It is possible to prove that if a Morse-Smale diffeomorphism $f : M^3 \to M^3$ has a self-indexing energy function then the condition $W^u(x) \cap W^s(y) \neq \emptyset$ implies $\dim W^s(x) < \dim W^s(y)$ for any pair of periodic points x, y ($x \neq y$), that is f is gradient-like.

Now let $f : M^3 \to M^3$ be a gradient-like diffeomorphism. Let us denote by Ω^+ (resp. Ω^-) the set of all sinks (resp. sources), by Σ^+ (resp. Σ^-) the set of all saddle points having one-dimensional unstable (resp. stable) invariant manifolds, by L^+ (resp. L^-) the union of the unstable (resp. stable) one-dimensional separatrices. We set $\mathcal{A}(f) = \Omega^+ \cup L^+ \cup \Sigma^+$, $\mathcal{R}(f) = \Omega^- \cup L^- \cup \Sigma^-$ and $L = L^- \cup L^+$. By the construction $\mathcal{A}(f)$ (resp. $\mathcal{R}(f)$) is a connected set which is attractor (repeller) of f. We set $g(f) = \frac{|\Sigma^+ \cup \Sigma^-| - |\Omega^+ \cup \Omega^-| + 2}{2}$, where $|.|$ stands for the cardinality.

Theorem. A gradient-like diffeomorphism $f : M^3 \to M^3$ has a self-indexing energy function if and only if there is a smooth Heegaard surface F of genus $g(f)$ with tubular neighborhood $V(F)$ such that $M^3 \setminus int\ V(F) = P^+ \cup P^-$, where:

1) P^+ (P^-) is f-compressed (f^{-1}-compressed) handlebody and $\mathcal{A}(f) \subset P^+$ ($\mathcal{R}(f) \subset P^-$);

2) the set $W^s(\sigma^+) \cap P^+$ ($W^u(\sigma^-) \cap P^-$) consists of exactly one two-dimensional closed disc for each saddle point $\sigma^+ \in \Sigma^+$ ($\sigma^- \in \Sigma^-$).

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References

[1] D. Pixton, "Wild unstable manifolds", *Topology*, 16, 2, 167-172 (1977).

[2] S. Smale, "On gradient dynamical systems", *Annals of Math*, 74, 199-206 (1961).