On infinitesimal orbit type theorems

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It is known, that the Principal Orbit Type (POT) Theorem is not true in the Lorentzian case, i.e. if $\alpha : G \times M \to M$ is a smooth isometric action of a Lie-group on a smooth Lorentzian manifold, then there is not necessarily a maximal orbit type. (Consider *MSO* (3) on the 3-dimensional Minkowski-space $R_{2,1}$). This inspired D. Alekseevsky and J. Szenthe to introduce the infinitesimal type of an orbit, where they changed the isotropy subgroups to their Lie-algebras in the definitions, thus:

Definition 1 *The orbit $G(x)$ has greater or equal infinitesimal orbit type as $G(y)$ (in notion $G(x) \geq G(y)$), if there are orbit points $x_0 \in G(x)$, $y_0 \in G(y)$, such that $\mathfrak{g}_{x_0} \subseteq \mathfrak{g}_{y_0}$, where $\mathfrak{g}_{x_0}$, $\mathfrak{g}_{y_0}$ are the Lie-algebras of the isotropy subgroups G_{x_0}, G_{y_0}.*

This definition allows us to prove the Innitesimal Principal Orbit Type (IPOT) Theorem, where we do not not need that the action is proper, which is used in (POT).

Theorem 2 *(IPOT Riemann) Let $\alpha : G \times M \to M$ be a smooth isometric action of a Lie-group G on a Riemannian manifold M, then there is a unique maximal infinitesimal orbit type (called principal), and the orbits which have the infinitesimal principal orbit type build an open, dense and connected set in M.*

Definition 3 *An orbit $G(x)$ is normalizable, if there is a bunle $\widetilde{N}G(x) \subset TM$ over $G(x)$ such that $NG(x)$ is G-invariant (under α), and for every $z \in G(x)$ $T_zM = T_zG(x) + \widetilde{N}_zG(x)$.*
The action α is normalizable if every orbit is normalizable.

In the Lorentzian case an orbit with spacelike or timelike tangent spaces is always normalizable.

Theorem 4 *(IPOT Lorentz) Let $\alpha : G \times M \to M$ be a smoot normalizable isometric action of a Lie-group G on a Lorentz manifold M, then there is a maximal infinitesimal principal orbit type (called principal), and the orbits which have the infinitesimal principal orbit type build an open dense set in M.*

An example shows, that connectedness is not always true. So there can be more maximal infinitesimal orbit types if and only if there are orbits which are not normalizable.

Theorem 5 *An orbit in the Lorentz case which is not normalizable, has lightlike tangent spaces, and the lightlike integral curves of the orbit are lightlike geodesics.*