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Problem 1 *Let $ABCD$ be a convex quadrilateral of area 1. Denote $E = AC \cap BD$, $M = \max(AB, BC, CD, DA, AC, BD)$, $m = \min(AB, BC, CD, DA, AC, BD)$. Prove that:*

- $1 \geq \sqrt{P_1} + \sqrt{P_2}$, *where P_1, P_2 are areas of AEB,CED, respectively,*
- $M \geq \sqrt{2}m$,
- $AB + BC + CD + DA + AC + BD \geq 4 + \sqrt{8}$.

Problem 2 *Let $ABCD$ be a convex quadrilateral inscribed in a circle such that $\angle ABC = 90°$. Let M, N be an orthogonal projections on AC and AD, respectively.*

Prove that the center of the segment BD lies on the line MN.

Let ABC be a triangle and $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$ be such that $AA_1 \cap BB_1 \cap CC_1 \neq \emptyset$. Let us denote $AA_1 \cap BB_1 \cap CC_1 = M$.

Problem 3 *Prove that:*

$$\frac{AM \cdot BM \cdot CM}{A_1M \cdot B_1M \cdot C_1M} \geq 8.$$

Problem 4 *Prove that there exists incircle of A_1MB_1C if there are incircles of B_1MC_1A, C_1MA_1B.*

Problem 5 *Let ABC be a equilateral triangle. Prove that:*

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A.$$

Problem 6 *Let us consider a square of area 1 with n points inside. Prove that one can number this points by $P_1, P_2, \dots, P_n$ such that:*

$$|P_1P_2|^2 + |P_2P_3|^2 + \cdots + |P_nP_1|^2 \leq 4.$$

Problem 7 *Let $ABCDEF$ be a hexagon such that*

$$AB = BC, \quad CD = DE, \quad EF = FA.$$

Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

Problem 8 *Let $ABCDEF$ be a hexagon such that*

$$\angle ABC = \angle BCD = \angle CDE = \angle DEF = \angle EFA = \angle FAB.$$

Prove that

$$AB + BC = DE + EF.$$

Problem 9 *Let A, B, C, D, E lie on a semicircle with radius 1.*

Prove that

$$AB^2 + BC^2 + CD^2 + DE^2 + AB \cdot BC \cdot CD + BC \cdot CD \cdot DE < 4.$$

Problem 10 *Let* $A_1, A_2, \dots A_n$ *be points on a circle (with radius* 1 *and center* O*) such that*

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n} = 0.$$

Prove that for any point B

$$A_1B + A_2B + \dots A_nB \geq n.$$

Problem 11 *Every point on the plane has been colored by blue or yellow.*
Prove that there is a equilateral triangle with all vertices colored by the same colour.

Problem 12 *Let* $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \mathbb{C}$ *such that* $\Omega_i \cap \Omega_j = \emptyset$*, for* $i \neq j$.
Prove that at least one set Ω_i *has the following property:*

for any $d > 0$ *there are* $x, y \in \Omega_i$ *such that* $|x - y| = d$.

Problem 13 *Rozważmy* n *okręgów* $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ *przecinajacych sie wspólnie w punkcie* O *o środkach odpowiednio w punktach* $O_1, O_2, \dots, O_n$ *(przy czym* O, O_i, O_{i+1} *nie sa współliniowe). Niech* $A_1, A_2, \dots, A_n$ *oznaczaja odpowiednio punkty przeciacia okregów* Γ_1 *i* Γ_2*,* Γ_2 *i* Γ_3 *itd. Niech* B_1 *bedzie dowolnym punktem należacym do okregu* Γ_1 *różnym od* O *i* A_1*. Dalej niech* B_2 *oznacza punkt przeciecia okregu* Γ_2 *i prostej* B_1A_1*. Niech* B_3 *oznacza punkt przeciecia okregu* Γ_3 *i prostej* B_2A_2 *itd.*
Pokazać, że ostatni spośród punktów B *pokryje sie z pierwszym, czyli* $B_1 = B_{n+1}$.