Contents

1 Other topological constructions of contact structures on 3-dimensional manifolds

1.1 Branchel covers

Theorem 1 (**Hilden, Montesinos, 1976**). *Let M be a closed, connected, orientable 3-dimensional manifold. Then there exists a knot $k \subset S^3$ and a map $p : M \to S^3$ with the following properties:*

- $p\Big|_{M \setminus p^{-1}(k)} : M \setminus p^{-1}(k) \to S^3 \setminus k$ *is a smooth 3-fold covering,*
- $p^{-1}(k) = k_1 \cup k_2$,
- *p is a diffeomorphism of a neighbourhood of k_1 onto a neighbourhood of k,*
- *in suitable coordinates (θ, r, φ) on neighbourhoods of k_1 and k_2 the map p is given by $(\theta, r, \varphi) \mapsto (\theta, r, 2\varphi)$.*

Remark 1. *This map is not smooth along k_2. Its smooth map would be given by $(\theta, r, \varphi) \mapsto (\theta, r, 2\varphi)$. For lifting of contact structure the non-smooth version is better.*

2nd proof of Martinet theorem:

Proof. We may assume that k is transverse to ξ_{st} on S^3 and $\alpha = d\theta + r^2 d\varphi$ near k.

$$p : M \setminus k_2 \to S^3$$

is a local diffeomorphism. $p^*\alpha$ is a contact form on $M \setminus k_2$.

In a neighbourhood of k_2 we have (for $r > 0$) $p^*\alpha = d\theta + 2r^2 d\varphi$. This expresssion defines a smooth extension of of $p^*\alpha$ over k_2 as a contact form. □

This proof is due to J. Gonzalo Pérez.

1.2 Open books

Definition 1 (**Open book decomposition**). *A manifold M admits an open book decomposition (Σ, Φ) if there exists a compact, orientable surface Σ with $\partial\Sigma = S^1$ and a diffeomorphism $\Phi : \Sigma \to \Sigma$ equal to identity near $\partial\Sigma$, such that $M \simeq \Sigma(\Phi) \cup (\partial\Sigma \times D^2)$.*
Here $\Sigma(\Phi)$ is the mapping torus defined by: $\Sigma(\Phi) := \Sigma \times [0, 2\pi] \big/ _{(x,2\pi)\sim(\Phi(x),0)}$.

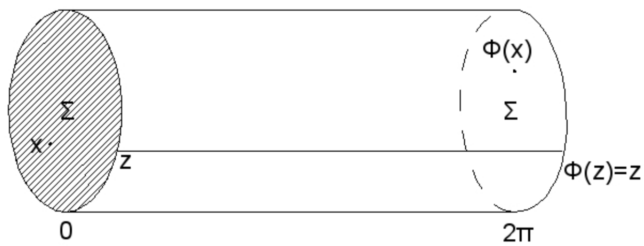

Figure 1: $\Phi\Big|_{\partial\Sigma} = id$, hence the identification near the boundary is as in a torus with cross section Σ. In general $\Phi\Big|_{\Sigma} \neq id$ so the identification might twist the inside of the torus. Note that $\partial(\Sigma(\Phi)) \simeq \partial\Sigma \times S^1$. To each S^1 in $\partial(\Sigma(\Phi))$ we glue in a disc D^2.

The details of the construction of open book decomposition are depicted and discussed in Figures 1. and 2.

Theorem 2 (**Alexander, 1923**). *Let M be a closed, connected, orientable 3-dimensional manifold. Then there exist an open book decomposition of M.*

Theorem 3 (**Thurston - Winkelnkemper**). *Every open book decomposition (Σ, Φ) supports a contact structure ξ_Φ.*

The begining of the proof of Thurston - Winkelnkemper theorem: Let $\theta \in S^1$ be the coordinate along $\partial\Sigma$ (take Σ oriented from now on). Let s be a collar parameter of $\partial\Sigma$ in Σ, such that $\partial\Sigma = \{s = 0\}$ and $s < 0$ in the interior of Σ. Σ is a surface with a boundary, so there exists an exact area form $d\beta$. Let φ be a parameter in S^1. Let us define a 1-form α by

$$\alpha = \beta + d\varphi.$$

Then

$$\begin{aligned} d\alpha &= d\beta, \\ \alpha \wedge d\alpha &= d\beta \wedge d\varphi \quad \text{- area form on } \Sigma \times S^1 \end{aligned}$$

Such a form α is well defined on $\Sigma \times S^1$, so it is defined in the interior of the pages of the open book decomposition. Now we would like to glue it corectly to define it also near the binding.

By assumption Φ is equal to identity near $\partial\Sigma$, so we can choose the collar parameter s in such a way, that $\Phi = id$ on $[-2, 0] \times \partial\Sigma \subset \Sigma$. Recall that by assumption

$$M \simeq \Sigma(\Phi) \cup (\partial\Sigma \times D^2).$$

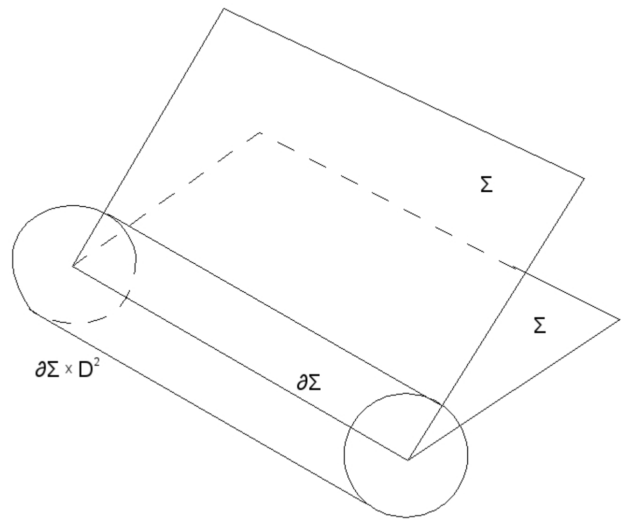

Figure 2: The pages of the open book are the identical copies of Σ, whereas the binding is the $\partial\Sigma = S^1$.

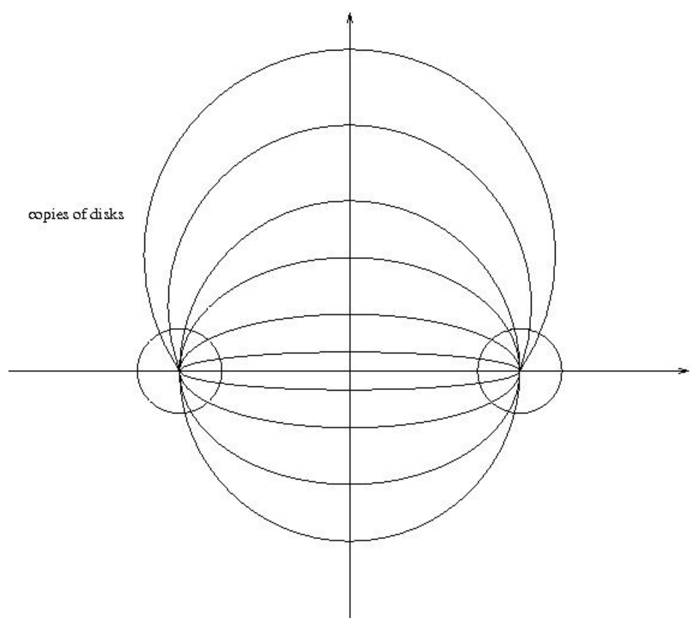

Figure 3: $S^3 = \mathbb{R}^3 \cup \{\infty\}$ admits an open book decomposition with page $\Sigma \simeq D^2$ and monodromy $\Phi = id_{D^2}$. The decomposition is obtained by gluing in disks/hemispheres into a circle S^1. One can see the decomposition in the above figure - it is obtained by rotation around the vertical axis.

Recall that $\partial(\Sigma(\Phi)) \simeq \partial\Sigma \times S^1$, so near the boundary of $\partial(\Sigma(\Phi))$ we can define a mapping:

$$\begin{aligned} (s,\theta,\varphi) &\in [0,1]\times\partial\Sigma\times S^1 = [0,1]\times\partial(\Sigma(\Phi)) \\ (s,\theta,\varphi) &\mapsto (\theta, 1-s, \varphi) \in \partial\Sigma\times D^2_2 = \partial\Sigma\times[0,2]\times S^1 \end{aligned}$$

Lemma 1. *The set of 1-forms β on Σ with the following propeties:*

- $\beta = e^s d\theta$ *on* $[-\frac{3}{2}, 0]\times\partial\Sigma \subset \Sigma$
- $d\beta$ *is an area form on* Σ *of total area* $\int_\Sigma d\beta = 2\pi$

is non-empty and convex.

The proof of the lemma:

Proof. Let β_0 be a 1-form on Σ with $\beta_0 = e^s d\theta$ on $[-2,0]\times\Sigma$. Then

$$\int_\Sigma d\beta_0 = \int_{\partial\Sigma}\beta_0 = \{s = 0 \text{ on } \partial\Sigma\} = \int_{\partial\Sigma} d\theta = 2\pi.$$

Let ω be an area form on Σ with $\int_\Sigma \omega = 2\pi$ and $\omega = e^s ds\wedge d\theta$ on $[-2,0]\times\partial\Sigma$. This choice of ω is always possible. Note that then

$$\int_{[-2,0]\times\partial\Sigma} e^s ds\wedge d\theta = 2\pi(1-e^{-2}) < 2\pi.$$

Moreover, $\int_\Sigma \omega - d\beta_0 = 0$ and $\omega - d\beta_0 \equiv 0$ on $[-2,0]\times\partial\Sigma$.

Extend $\omega - d\beta_0$ to a 2-form Ω on $\hat{\Sigma} = \Sigma\cup_{\partial\Sigma} D^2$ with $\Omega\big|_{D^2} \equiv 0$.

$$\text{Since} \quad \int_{\hat{\Sigma}}\Omega = 0, \quad \text{by de Rham theorem} \quad \Omega = d\hat{\beta}.$$

Moreover, in view $d\hat{\beta} = \Omega$ and $\Omega\big|_{([-2,0]\times\partial\Sigma)\cup D^2} \equiv 0$, then by Poincaré lemma $\hat{\beta} = d\hat{f}$ on $([-2,0]\times\partial\Sigma)\cup D^2$.

We claim that

$$\beta = \beta_0 + \hat{\beta}\big|_\Sigma - d(\psi f)$$

is the desired form, where ψ is defined as on Figure 4.

For such defined β the following holds

$$d\beta = d\beta_0 + d\hat{\beta}\Big|_\Sigma = d\beta_0 + \Omega\Big|_\Sigma = d\beta_0 + \omega - d\beta_0 = \omega.$$

This proves that the desired set of 1-forms is non-empty.

Let us now prove the convexity. Suppose, β' is further 1-form with these properties, then it is easy to check that so is the form

$$(1-t)\beta + t\beta', \quad t\in[0,1].$$

□

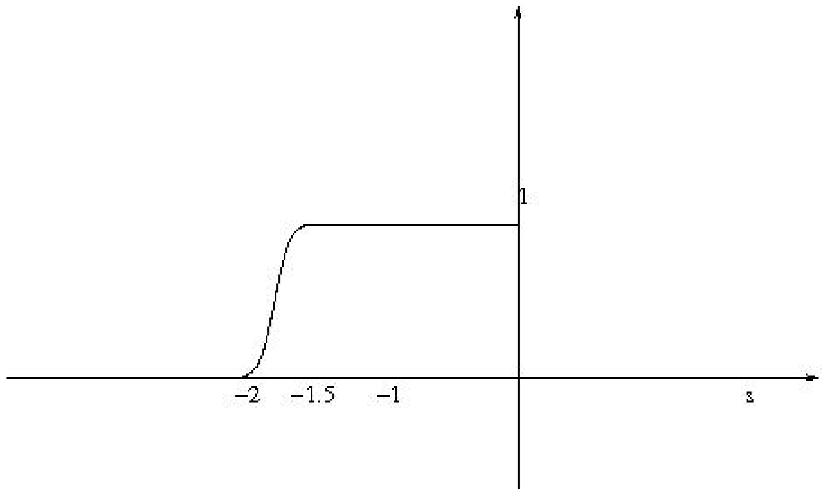

Figure 4: A ψ is a smooth function on $\mathbb{R}_-$, such that $\psi\Big|_{(-\infty,-2]} \equiv 0$ and $\psi\Big|_{[-1.5,0]} \equiv 1$.

The continuation of the proof of Thurston - Winkelnkemper theorem:

Proof. Let $\tilde{\beta}$ be a 1-form on Σ as just described in the lemma. Let μ be a smooth function on $[0, 2\pi]$, such that μ is 1 in the neighbourhood of 0 and μ is 0 in the neighbourhood of 2π.

Then $\beta = \mu(\varphi)\tilde{\beta} + (1-\mu(\varphi))\Phi^*\tilde{\beta}$ is a 1-form on $\Sigma \times [0, 2\pi]$, whose restriction to each fibre $\Sigma \times \{\varphi\}$ has the properties as in the lemma. This induces a 1-form on $\Sigma(\Phi)$.

Note that $d\varphi$ on $\Sigma \times \mathbb{R}$ is invariant under $(x, \varphi) \sim (\Phi(x), \varphi - 2\pi)$, so it induces a 1-form on $\Sigma(\Phi)$.

Let us define

$$\alpha = \beta + Cd\varphi \quad \text{on} \quad \Sigma(\Phi), \quad \text{for some} \quad C \in \mathbb{R}_+.$$

Let us calculate

$$\alpha \wedge d\alpha = (\beta + Cd\varphi) \wedge (d\beta) = \beta \wedge d\beta + Cd\varphi \wedge d\beta$$

Recall, that by the assumption $d\beta$ is an area form on each fibre $\Sigma \times \{\varphi\}$, so $d\varphi \wedge d\beta$ is an area form on $\Sigma(\Phi)$. This means that for large enough $C > 0$, $\alpha \wedge d\alpha > 0$ everywhere on $\Sigma(\Phi)$. This proves that α is indeed a contact form on $\Sigma(\Phi)$. □

Ansatz:
On $\partial\Sigma \times D^2$ the 1-form α can be defined as follows

$$\alpha = h_1(r)d\theta + h_2(r)d\varphi.$$

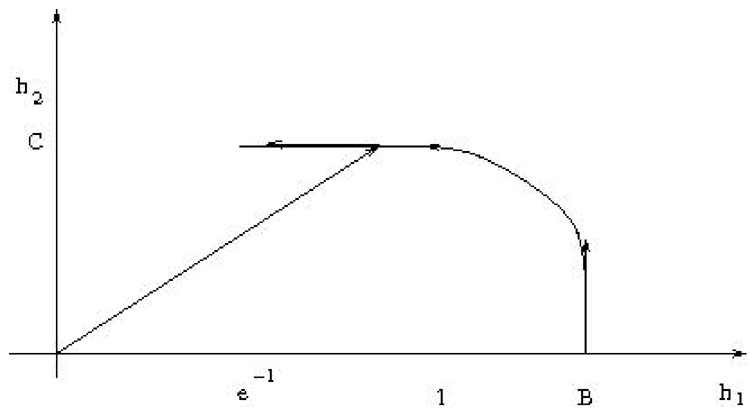

Figure 5: The figure depicts the algorithm for finding the solutions for functions h_1, h_2. One first draws a curve which satisfies the boundary conditions and then interpolates as in Lutz twist.

for some functions h_1, h_2, which have to satisfy the following boundary conditions:

$$h_1(r) = e^{1-r}, \qquad h_2(r) = C, \quad \text{on} \quad \partial\Sigma \times D^2_{[1,2]}$$
$$h_1(r) = B > 1, \qquad h_2(r) = r^2.$$

We define $D^2_{[1,2]} = \{(r,\varphi) \big| r \in [1,2]\}$. Functions h_1, h_2 also have to satisfy

$$Det \begin{pmatrix} h_1 & h_1' \\ h_2 & h_2' \end{pmatrix} \neq 0.$$

The solution to this set of equations is depicted on Figure 5.

Teaser: It turns out that for a given manifold M there exists a contact structure (M, ξ) if and only if there exists an open book decomposition (Σ, Φ) of M.