# 1 Contact structures and Reeb vector fields

**Definition 1.1.** Let $M$ be a smooth $(2n+1)$-dimensional manifold. A contact structure on $M$ is a maximally non-integrable hyperplane field $\xi \subset TM$. Suppose that there exists a global 1-form $\alpha$ such that $\xi = \ker\alpha$ (then $\xi$ is called coorientable). Then the non-integrability condition is equivalent to

$$\alpha \wedge (d\alpha)^n > 0,$$

i.e. the form $\alpha \wedge (d\alpha)^n$ is a volume form on $M$. The pair $(M, \xi)$ is called a contact manifold and $\alpha$ is called a contact form.

**Example 1.2.** Consider the euclidean space $\mathbb{R}^{2n+1}$ and a 1-form

$$\alpha = dz + \sum_{i=1}^{n} x_i dy_i.$$

Then

$$d\alpha = \sum_{i=1}^{n} dx_i \wedge dy_i$$
$$\alpha \wedge (d\alpha)^n = dz \wedge dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n,$$

thus $\alpha$ is a contact form. The contact structure determined by $\alpha$ is called the standard contact structure.

**Remark 1.3.** The form $\alpha \wedge (d\alpha)^n$ is a volume form consequently $M$ is necessarily orientable.

**Remark 1.4.** Any other contact form, which determines the same contact structure, has to be of the form $f\alpha$ for some non-vanishing $f \in C^\infty(M)$.

**Lemma 1.5.** Suppose that $\dim M = 3$ and let $\xi = \ker\alpha$ be a hyperplane field on $M$. Then $\xi$ is a contact structure if and only if for all pointwise linearly independent vector fileds $X, Y \in \xi$ we have $[X, Y] \notin \xi$.

*Proof.* Let $X, Y \in \xi$ be two vector fields on $M$. Then

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = -\alpha([X, Y]).$$

Contact condition for $\alpha$ is satisfied if $d\alpha|_{\ker\alpha} \neq 0$. Thus $\alpha([X, Y]) \neq 0$ as claimed. □

**Example 1.6.** Consider $\mathbb{R}^3$. Then $\alpha = dz + xdy$ and $\xi$ is spanned by vector fields $\partial_x$ and $\partial_y - x\partial_z$.

$$[\partial_x, \partial_y - x\partial_z] = -\partial_z.$$

**Definition 1.7.** The Reeb vector field $R$ of a contact form $\alpha$ is defined by the following equations

$$\begin{cases} d\alpha(R, \cdot) \equiv 0, \\ \alpha(R) \equiv 1. \end{cases}$$

**Remark 1.8.** If a contact form $\alpha$ is fixed, then the Reeb vector field is uniquely defined.

**Definition 1.9.** A Liouville vector field $Y$ on a symplectic manifold $(W^{2n+2}, \omega)$ is a vector field satisfying $L_Y \omega = \omega$.

**Remark 1.10.** By the Cartan formula $L_Y \beta = d(i_Y \beta) + i_Y d\beta$, thus is $\beta$ is closed, then $L_Y \beta = d(i_Y \beta)$.

**Lemma 1.11.** Let $Y$ be a Liouville vector field on a symplectic manifold $(W, \omega)$. Suppose that $M$ is a codimension one submanifold of $W$ transverse to $Y$. Then $\alpha = i_Y \omega$ is a contact form on $M$.

*Proof.* From the definition of the Liouville vector field it follows that

$$\alpha \wedge (d\alpha)^n = i_Y \omega \wedge (d(i_y \omega))^n = i_Y \omega \wedge \omega^n = \frac{1}{n+1} i_Y \omega^{n+1}.$$

Consequently $\alpha \wedge (d\alpha)^n$ is a volume form on any hypersurface transverse to $Y$. □

**Example 1.12.** Let $(M, \xi)$ be a contact manifold and let $\alpha$ be a contact form. Let $W = M \times \mathbb{R}$ and $\omega = d(e^t \alpha)$. The contact condition implies that $\omega$ is a symplectic form on $W$. Additionally $\partial_t$ is a Liouville vector field transverse to $M$. Thus every contact manifold can be obtained as a hypersurface of a symplectic manifold transverse to some Liouville vector field. Symplectic manifold $(W, \omega)$ is called symplectization of $(M, \alpha)$.

# 2 Gray stability and the Moser trick

**Lemma 2.1.** Let $\omega_t$ be a smooth 1-parameter family of $k$-forms on some manifold $M$. Let $\psi_t$ be an isotopy of $M$. Define a vector field $X_t$ by the equality

$$X_t \circ \psi_t = \dot{\psi}_t,$$

i.e. $\psi_t$ is a flow of $X_t$. Then the following equality holds

$$\frac{d}{dt}(\psi_t^* \omega_t) = \psi_t^*(\dot{\omega}_t + L_{X_t} \omega_t).$$

**Proposition 2.2.** *Let $\xi_t$, $t \in [0,1]$ be a smooth family of contact structures ona closed manifold $M$. Then there exists an isotopy $\psi_t$ of $M$ such that*

$$T\psi_t(\xi_0) = \xi_t.$$

*Proof.* Let $\alpha_t$ be a family of contact 1-forms corresponding to $\xi_t$. Our aim is to find an isotopy of $M$ satisfying

$$\psi_t^* \alpha_t = \lambda_t \alpha_0 \tag{1}$$

for some 1-parameter family of positive smooth functions $\lambda_t$.

Let's assume that $\psi_t$ is a flow of a time-dependent vector field $X_t$. Now differentiating (1) and using lemma 2.1 we obtain

$$\psi_y^* \left(\dot{\alpha}_t + L_{X_t}\alpha_t\right) = \dot{\lambda}_t \alpha_0$$
$$\psi_t^* \left(\dot{\alpha}_t + d(\alpha_t(X_t)) + i_{X_t} d\alpha_t\right) = \frac{\dot{\lambda}_t}{\lambda_t}\psi_t^* \alpha_t = \psi_t^* \left(\mu_t \alpha_t\right),$$

where $\mu_t = \frac{\dot{\lambda}_t}{\lambda_t} \circ \psi_t^{-1}$. Each $\psi_t$ is a diffeomorphism, thus the above equality is equivalent with the following

$$\dot{\alpha}_t + d(\alpha_t(X_t)) + i_{X_t} d\alpha_t = \mu_t \alpha_t. \tag{2}$$

Assume now that $X_t \in \xi_t$. Then (2) implies that

$$\dot{\alpha}_t + i_{X_t} d\alpha_t = \mu_t \alpha_t. \tag{3}$$

Plugging in the Reeb vector field $R_t = R_{\alpha_t}$ yields the following

$$\dot{\alpha}_t(R_t) = \mu_t. \tag{4}$$

If we define $\mu_t$ by the equality 4, then

$$\left(\dot{\alpha}_t - \mu_t \alpha_t\right)(R_t) \equiv 0. \tag{5}$$

The non-degeneracy of $d\alpha_t|_{\xi_t}$ impllies that there exists a unique $X_t \in \xi_t$ such that (3) is satisfied. Consequently if $\psi_t$ is a flow of $X_t$, which is globally defined since $M$ is closed, then $\psi_t$ is an isotopy with the desired property. □

**Theorem 2.3** (Darboux Theorem)**.** *Let $(M, \alpha)$ be a contact manifold. Given $p \in M$, there exists a coordinate chart around $p$ such that in the local coordinates the following equality holds*

$$\alpha = dz + \sum_{i=1}^{n} x_i dy_i.$$

*Proof.* Without loss of generality we can ussume that $M = \mathbb{R}^{2n+1}$ and $p = 0$. Using linear changes of coordinates we can arrange that at 0 $\partial_{x_i}|_0, \partial_{y_i}|_0 \in \xi_0$ and following equalities hold

$$\alpha(\partial_z)|_0 = 1,$$
$$\left(i_{\partial_z} d\alpha\right)|_0 = 0,$$
$$(d\alpha)_0 = \sum_{i=1}^{n} \left(dx_i \wedge dy_i\right)|_0.$$

Define

$$\alpha_0 = dz + \sum_{i=1}^{n} x_i dy_i,$$
$$\alpha_t = (1-t)\alpha_0 + t\alpha.$$

Notice that $\alpha_t|_0 = \alpha_0|_0$. Now we would like to apply the Moser trick to find an isotopy $\psi_t$ such that

$$\psi_t^* \alpha_t = \alpha_0. \tag{6}$$

Differentiating 6 yields

$$\psi_t^* \left(\dot{\alpha}_t + L_{X_t} \alpha_t\right) = 0, \tag{7}$$

$$\dot{\alpha}_t + d(\alpha(X_t)) + i_{X_t} d\alpha_t = 0. \tag{8}$$

Vector field $X_t$ can be written as a sum $X_t = H_t R_t + Y_t$, where $Y_t \in \ker \alpha_t$ and $H_t$ is a smooth function. Then (7) becomes

$$\dot{\alpha}_t + dH_t + i_{Y_t} d\alpha_t = 0. \tag{9}$$

Pluging in the Reeb vector field gives

$$\dot{\alpha}_t(R_t) + dH_t(R_t) = 0.$$

This equation can be solved locally for $H_t$. Then the proof proceeds analogously as the proof of the theorem 2.2. □

# 3 Neighbourhood theorems

**Definition 3.1.** A knot $k$ in a contact 3-manifold $(M^3, \xi)$ is called legendrian if for every point $p \in k$ we have $T_p k \subset \xi_p$.

**Theorem 3.2.** *Let $k \subset (M^3, \xi)$ be a legendrian knot. Then there are coordinates $(\theta, x, y)$ in a neighbourhood of $k = S^1 \times \{0\} \subset S^1 \times \mathbb{R}^2 = \nu(k, M)$ such that*

$$\xi|_{\nu(k,M)} = \ker(\cos\theta dx - \sin\theta dy).$$

*Here $\nu(k, M)$ denotes the tubular neighbourhood of $k$ in $M$.*

*Proof.* Without loss of generality we can assume that $M = S^1 \times \mathbb{R}^2$ and $k = S^1 \times \{0\}$. Let $\alpha$ be a contact form on $M$ and define a 1-form $\alpha_0 = \ker(\cos\theta dx - \sin\theta dy)$. Assume that $\theta$ is a coordinate on $S^1 \times \{0\}$. Define an automorphism

$$\phi\colon \nu(S^1 \times \{0\}, M) \to \nu(S^1 \times \{0\}, M)$$

as follows. Choose vector fields $X_0, X$ such that $\alpha(X) = \alpha_0(X_0) = 0$ and $d\alpha(\partial_\theta, X) = d\alpha_0(\partial_\theta, X_0) = 1$. Then $\phi$ acts as follows

$$R_{\alpha_0} \longmapsto R_\alpha$$
$$X_0 \longmapsto X.$$

This definition determines $\phi$ uniquely. Now let $\tau\colon \nu(S^1 \times \{0\}, M) \to S^1 \times \mathbb{R}^2$ be a tubular map. This $\tau$ is a diffeomorphism on a neighbourhood of the zero section of $\nu(S^1 \times \{0\}, M)$. Furthermore

$$\tau|_{S^1 \times \{0\}} = \mathrm{id},$$
$$D\tau|_{S^1 \times \{0\}} = \mathrm{id}\,.$$

The composition $\tau \circ \phi \circ \tau^{-1}$ is a diffeomorphism of a neighbourhood of $S^1 \times \{0\}$, thus $\alpha_0$ and $\alpha_1 = (\tau \circ \phi \circ \tau^{-1})^* \alpha$ are contact forms on a neighbourhood of $S^1 \times \{0\}$ with

$$\alpha_0|_{TS^1 \times \{0\}} = \alpha_1, d\alpha_0|_{TS^1 \times \{0\}} = d\alpha_1.$$

To finish the proof it is sufficient to apply Moser trick to $\alpha_t = (1-t)\alpha_0 + t\alpha_1$. □

# 4 Contact vector fields and Hamiltonian functions

Let $(W, \omega)$ be a symplectic manifold. Choose a smmoth function $H \in C^\infty(W)$. It defines a unique vector field $X_h$ by the relation

$$-dH = \omega(X_H, \cdot).$$

Vector field $X_H$ is called a hamiltonian vector field corresponding to the Hamiltonian $H$. The flow of $X_H$ preserves the symplectic form $\omega$, because

$$L_{X_H}\omega = d(i_{X_H}\omega) + i_{X_H} d\omega = d(i_{X_H}\omega) = -d^2 H = 0.$$

Now our goal is to give a similar construction in contact geometry.

**Definition 4.1.** A vector field $X$ on a contact manifold $(M, \xi)$ is called a contact vector field if its flow preserves the contact structure $\xi$.

**Example 4.2.** Let $M = S^1 \times \mathbb{R}^2$ and $\alpha = \cos\theta dx - \sin\theta dy$. Then $\alpha$ is a contact form on $M$. Consider a vector field $X = x\partial_x + y\partial_y$. We have

$$\begin{aligned} L_X\alpha &= i_X d\alpha + d(i_X\alpha) = \\ &= i_X(-\sin\theta d\theta \wedge dx - \cos\theta d\theta \wedge dy) + d(x\cos\theta - y\sin\theta) = \\ &= x\sin\theta d\theta + y\cos\theta d\theta + \cos\theta dx - \sin\theta dy - x\sin\theta d\theta - y\cos\theta d\theta = \\ &= \alpha. \end{aligned}$$

Thus $\psi_t^*\alpha = e^t\alpha$, so $T\psi_t(\xi) = \xi$. Consequently $X$ is a contact vector field.

**Remark 4.3.** In general if $X$ is a contact vector field for $\xi = \ker\alpha$, then $L_X\alpha = \mu\alpha$ for some function $\mu$.

**Proposition 4.4.** *Let $(M, \xi)$ be a contact manifold and let $\xi = \ker\alpha$. Then there exists a bijection*

$$\{\textit{Contact vector fields on } M\} \to C^\infty(M).$$

*The correspondence is given by the following formulas*

$$\begin{aligned} X &\longmapsto H_X = \alpha(X) \in C^\infty(M) \\ C^\infty(M) \ni H &\longmapsto X_H, \end{aligned}$$

*where $X_H$ is a vector field defined by the following two equations*

$$\begin{cases} \alpha(X_H) = H, \\ i_{X_H} d\alpha = dH(R_\alpha)\alpha - dH. \end{cases}$$

*Proof.* Let's check first that $X_H$ is a contact vector field.

$$L_{X_H}\alpha = i_{X_H} d\alpha + d(i_{X_H}\alpha) = \\ = dh(R_\alpha)\alpha - dH + dH = dH(R_\alpha)\alpha.$$

Thus $X_H$ is a contact vector field indeed.

In order to check that the map given in the theorem is bijective one can compute the compositions of both maps. For the first composition we obviously have

$$H_{X_H} = H.$$

To check the other composition notice that

$$i_{X_{H_X}} d\alpha = dH_X(R_\alpha)\alpha - dH_X = L_X - d(i_X\alpha).$$

Using Cartan formula we obtain the following equality

$$i_{X_{H_X}} d\alpha = i_X d\alpha. \tag{10}$$

Additionally

$$\alpha(X_{H_X}) = \alpha(X). \tag{11}$$

Equalities 10 and 11 imply that $X = X_{H_X}$. □

# 5 Isotopy extension theorem

**Theorem 5.1.** *Let $(M, \xi)$ be a contact 3-manifold and let $J_t \colon S^1 \to M$ be an isotopy of legendrian embeddings. Then there is a contact isotopy $\psi_t$ of $M$ such that*

$$\psi_t \circ j_0 = j_t.$$

*Proof.* Define $X_t$ on $j_t(S^1)$ by $X_t \circ j_t = \frac{d}{dt} j_t$. Let $H_t = \alpha(X_t)$ be a contact Hamiltonian corresponding to $X_t$. In order to solve the isotopy extension problem, we have to find a compactly supported function

$$\widetilde{H}_t \colon M \to \mathbb{R},$$

which extends $H_t$ defined on $j_t(S^1)$, then $X_{\widetilde{H}_t}$ is a contact vector field extending $X_t$. The flow of $X_{\widetilde{H}_t}$ gives the desired isotopy. Recall that $\widetilde{X}_t = X_{\widetilde{H}_t}$ is defined in terms of $\widetilde{H}_t$ by

$$\alpha(\widetilde{X}_t) = \widetilde{H}_t, \qquad i_{\widetilde{X}_t} d\alpha = d\widetilde{X}_t(R_\alpha)\alpha - d\widetilde{H}_t.$$

The $\widetilde{X}_t$ and $\widetilde{H}_t$ agree with $X_t$ and $H_t$ on $j_t(S^1)$, respectively. Hence we need

$$\alpha(X_t) = \widetilde{H}_t, \qquad i_{X_t} d\alpha = d\widetilde{H}_t(R_\alpha)\alpha - d\widetilde{H}_t \tag{12}$$

along $j_t(S^1)$. The first condition states that $\widetilde{H}_t = H_t$ on $j_t(S^1)$. Since $j_t$ is a legendrian embedding, so $T(j_t(S^1)) \subset \xi$. Hence the second condition implies that for any $v \in Tj_t(S^1)$

$$0 = d\alpha(X_t, v) + d\widetilde{H}_t(v) = d\alpha(X_t, v) + dH_t(v),$$

thus

$$\begin{aligned} j_t^*(d\alpha(X_t, \cdot) + dH_t) &= j_t^*(i_{X_t} d\alpha + d(i_{X_t}\alpha)) = \\ &= j_t^*(i_{X_t} d\alpha) + d(j_t^* i_{X_t}\alpha) = \frac{d}{dt}(j_t^*\alpha) = 0, \end{aligned}$$

since $j_t$ is the isotopy of contact embeddings so the above condition is satisfied.

To sum up, to define $\widetilde{H}_t$ it is enough to define it on $j_t(S^1)$ and define its differential $d\widetilde{H}_t$ along $j_t(S^1)$. The first part is done by the first condition from 12. To fullfill the second condition define $d\widetilde{H}_t(R) = 0$ and for $v \in \ker\alpha$ the value of $d\widetilde{H}_t$ is defined by the second condition from 12. □