Surfaces in contact 3-manifolds.

Let again $(M, \xi = \ker \alpha)$ be oriented and cooriented contact 3-manifold, let $S \subset M$ be a closed surface. Identify a neighbourhood of S with a normal bundle $S \times \mathbb{R}$. In local coordinates - z coming from the fiber - write $\alpha = \beta_z + u_z dz = a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz$, with β_z understood as a 1-form <u>on S</u> and u_z - as a function on S. Now

$$d\alpha = d\beta_z - \dot{\beta_z} \wedge dz + du_z \wedge dz$$

with $\dot{F}$ meaning the partial derivative along z. The contact condition for α now reads

$$0 \neq \alpha \wedge d\alpha = (\beta_z + u_z d_z)(d\beta_z - \dot{\beta_z} \wedge dz + du_z \wedge dz)$$

and amounts to say that

$$u_z d\beta_z + \beta_z \wedge (du_z - \dot{\beta_z}) > 0$$

as a two form on S. Notice the clever use of the fact that we work on a surface to get rid of some terms.

Characteristic foliation. Let Ω be an area form on S. Define S_ξ by the unique vector field X that

$$\iota_X \Omega = \beta_0$$

where β_0 is of course α restricted to the tangent bundle TS. Observe that X lies in $\ker \beta_0$. X itself is 0 precisely in points p where α looses its β_0 term, that is $\alpha_p = u_0(p)dz$, i.e. $T_pS = \xi_p$.

Recall that the divergence of a filed is a measure, how much a vector field changes the volume form: the Lie derivative along X of the 2-form Ω is again a 2-form and so must be proportional to Ω itself:

$$div(X)\Omega = L_X \Omega$$

and recall that by Cartan Magic Formula you don't need to know what Lie derivative is, because

$$L_X \Omega = d\iota_X \Omega$$

which happens to be $d\beta$ in our case.

Lemma. *A vector field on S defines a characteristic foliation of some contact structure in $S \times \mathbb{R}$ iff $div(X)$ is nonzero in every point in which X is zero.*

Proof. Assume that we have a contact structure α as before, therefore

$$u_z d\beta_z + \beta_z \wedge (du_z - \dot{\beta_z}) > 0$$

holds. Suppose in a point p we have $X_p = 0$, then $(\beta_0)_p = 0$ and so $\xi_p = T_pS$. Compute

$$d(\iota_X \Omega)_p = (d\beta_0)_p = d\alpha|_{T_pS} = d\alpha|_{\xi_p} \neq 0$$

and we are done.

Other way around, assume X a vector field satisfying $X_p = 0 \Rightarrow div(X)(p) \neq 0$. Define $\beta = \iota_X \Omega$ and a function u by $d\beta = u\Omega$. By our assumption, whenever β_p is zero, $u(p)$ is not.

Pick any 1-form γ that gives $\beta \wedge \gamma \geq 0$ and $\beta \wedge \gamma_p > 0$ where β was nonzero.

Define $\beta_z = \beta + z(du - \gamma)$, with z again coming from vertival direction. Define $\alpha = \beta_z + udz$. On the surface, at the zero level,

$$ud\beta_0 + \beta_0 \wedge \gamma = u^2\Omega + \beta_0 \wedge \gamma > 0$$

and since this condition is open, it holds in some neighbourhood, which concludes the proof.

Definition. *A vector field X on a surface is of Morse-Smale type if*

(1) *X has at moste finitely many singularities, all nondegenerate;*
(2) *X has at most finitely many closed orbits, all nondegenerate;*
(3) *the limit sets of the orbits can only be singular points or closed orbits;*
(4) *there are no orbits connecting hyperbolic points.*

Proposition. *By a $\mathcal{C}^\infty$-small perturbation of S (in $S \times \mathbb{R}$) we can make S_ξ of Morse-Smale type.*

Proposition. *The characteristic foliation of a surface determines the germ of the contact structure near the surface.*

Convex Surfaces.

Definition. *A surface S in a contact 3-manifold is called convex iff there exists (at least in small neighbourhood) a contact vector field transverse to S.*

Lemma. *A surface S of is convex iff there exists an embedding $\psi : S \times \mathbb{R} \hookrightarrow M$ such that $\psi(\cdot, 0)$ is the inclusion and the pullback of the contact form, $\psi^*\alpha$ is an $\mathbb{R}$-invariant contact form.*

Proof. The transverse flow gives - locally - a tubular neighbourhood of S with the flow tangent to the fibers, leaving the structure invariant. Conversely, $\mathbb{R}$-invariant embedding is equiped with a contact transverse field - image of a vertical field, concluding the proof.

Notice that in the invariant case, the contact condition in neighbourhood of S with contact form $\alpha = \beta + udz$ and with a characteristic foliation given by X becomes

$$udiv(X) - Xu > 0$$

Definition. *The dividing set* Γ_S *on a convex surface* S *is*

$$\{p \in S \mid Y(p) \in \xi_p\}$$

for some transverse contact field Y*. This set depends on* Y *chosen.*

Example. *For a 3-torus* $\mathbb{T}^3$ *with contact structure* $\ker(\cos\theta dx - \sin\theta dy)$*, the field* ∂_x *makes any "vertical slice" two torus* $S = \{x = x_0\}$ *into a convex surface. The dividing set are then two "horizontal slices"* $\Gamma_S = \{\theta = \frac{\pi}{2} \vee \theta = \frac{3\pi}{2}\}$*.*

We will now describe, how a dividing set on a convex surface can look like. By the previous lemma, we are always in a situation of a contact structure

$$\alpha = \beta + udz$$

$$ud\beta + \beta \wedge du > 0$$

and with a transverse field ∂_z. Γ_S is then the zero set of u, but du is nonzero there, by the contact condition. Hence, by Implicit Functions Theorem, Γ_S is a collection of circles on S.

More than that, each of this circles are transverse to ξ. Suppose to the contrary. Then we have a tangent vector v such that

- $\iota_v ud\beta = 0$, since u vanishes along Γ_S;
- $\iota du = 0$, since it is tangent to $\{u = 0\}$;
- $\iota_v \beta = 0$, since v lies in ξ

but then

$$\iota_v(ud\beta + \beta \wedge du) = 0$$

contradicting the contact condition. This in particular means, that Γ_S meets no singular points of S_ξ, as those are where $T_pS = \xi_p$.

Theorem. Γ_S *is determined by* S_ξ *up to isotopy transverse to* S_ξ*.*

Proposition. *If the field defining* S_ξ *is Morse-Smale, then* S *is convex.*

Those combined with previous lemmas produce

Theorem. *Given a convex surface* S*, any singular foliation divided by* Γ_S *can be realised as the characteristic foliation by a* $\mathcal{C}^\infty$*-small perturbation of* S*.*