1 Knots in contact 3-manifolds

1.1 The front and the lagrange projection

In this section, let us denote the standard contact form in $\mathbb{R}^3$ by $\alpha_{st} = dz + xdy$, inducing the standard contact structure $\xi_{st} = \ker \alpha_{st}$.

Definition 1. *Let $\gamma : (a,b) \to (\mathbb{R}^3, \alpha_s t)$ be a differentiable map with coordinates $\gamma(t) = (x(t), y(t), z(t))$. We say $\gamma_F(t) = (y(t), z(t)) \in \mathbb{R}^2$ is the front projection for the knot γ. In a similar fashion we define the lagrange projection to be $\gamma_L(t) = (x(t), y(t)) \in \mathbb{R}^2$.*

Observe that if we changed the standard structure (for example, by switching the roles of the variables), we would change the definition of the projections accordingly.

1.1.1 The front projection

Now, consider the case where γ is a legendrian immersion, that is, $\alpha_{st}(\dot\gamma(t)) = 0$ for every t and $\dot\gamma(t) \neq 0$ (equivalently, the tangent vector to the curve is non-vanishing and is contained in the contact structure for every time). Then, we have the following proposition:

Proposition 1. *If γ is a legendrian immersion then the frontal projection has no vertical tangencies.*

Proof. Vertical tangencies in the (y, z) plane appear if $\dot y(t) = 0$ and $\dot z(t) \neq 0$. It is easy to see that this cannot happen.

By applying the contact form, we have an equality for the tangent vector $(\dot x(t), \dot y(t), \dot z(t))$ to γ at t:

$$0 = \alpha_{st}(\dot\gamma(t)) = (dz + xdy)(\dot x(t), \dot y(t), \dot z(t)) = \dot z(t) + x(t)\dot y(t)$$

which implies that if $\dot y = 0$ then $\dot z = 0$.

In addition, since γ is an immersion, $\dot\gamma(t)$ cannot vanish identically, so in the points where $\dot y = 0$ we must have $\dot x \neq 0$, so by the implicit function theorem, we can parametrise the curve γ around such a point using the x coordinate. □

Even more can be said about γ around such points. The following curve:

$$(x(t), y(t), z(t)) = (t, t^2, -\frac{2}{3}t^3)$$

is a legendrian immersion where the point $t = 0$ is a cusp. By a small deformation, any legendrian curve can be taken to have such a form in the singular points of the frontal projection. That is the content of the following theorem.

Theorem 1. *By a C^2-small perturbation of γ, one can ensure that the points where $\dot y = 0$ are isolated and γ looks locally like the model above.*

Note: By a C^2-small perturbation we mean that there are arbitrarily close deformations of γ in the C^2 topology satisfying the consequences of the theorem.

Making use of the theorem, we can easily see that the curve γ is uniquely determined by the frontal projection γ_F. This can be seen by writing $x(t) = -\frac{\dot z(t)}{\dot y(t)}$. This means that, by seeing the curve in the (y, z) plane, we can recover the x coordinate studying the slope of the projection.

In light of the theorem, we will only distinguish the singularities in the front projection into left and right cusps.

Remark: The curve γ is embedded if and only if there are not two points in the front projection with the same coordinates z, y and the same slope $\frac{\dot{z}}{\dot{y}}$, which is simply a condition on γ_F. In addition, there is no need to specify crossings in the planar representation since they can only look as in figure 1.

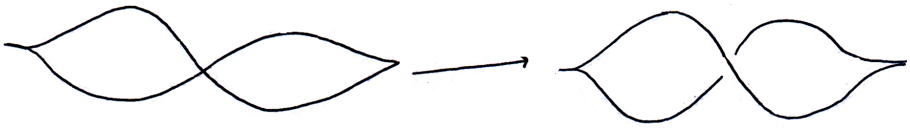

Figure 1: Only possible interpretation of a crossing in the front projection

1.1.2 The lagrange projection

Asumme again the curve $\gamma : (-\epsilon, \epsilon) \to (\mathbb{R}^3, \xi_{st})$ is a legendrian immersion. Then a number of things can be said about its lagrange projection.

Proposition 2. *The lagrange projection γ_L is an immersion as well. Furthermore, γ is determined by the lagrange projection up to translation.*

Proof. Indeed, as we noticed in the previous section, $\dot{x}(t)$ cannot be zero when $\dot{y}(t) = 0$, being γ an immersion, proving the first claim.

Clearly $z(t) = z(0) - \int_0^t x(s)\dot{y}(s)ds$, which proves the second claim. This has some interesting consecuences. If γ is a closed loop (a knot), parametrised by the interval $[0, 1]$, then we have a condition on the integral:

$$\int_0^1 x(s)\dot{y}(s)ds = 0$$

By applying Stokes' theorem, we have that, if β is a section of γ_L enclosing a domain D (all selected with suitable orientations):

$$\int_a^b x(s)\dot{y}(s)ds = \int_\beta xdy = \int_D dxdy = \text{Area}(D)$$

which means that every crossing in the lagrange projection is determined by the oriented area enclosed by the curve up to that intersection. $\square$

1.1.3 Positively transverse curves

Definition 2. *We say that a curve $\gamma : (a, b) \to (\mathbb{R}^3, \xi_{st})$ is positively transverse to the standard contact structure if $\alpha_t(\dot{\gamma}(t)) > 0$ for every t. Explicitely:*

$$\dot{z} + x\dot{y} > 0$$

In the same fashion as above, we can study a positively transverse knot making use of its projections. In particular, in the front projection we can distinguish three cases depending on the value of $\dot{y}(t)$ (that is, whether the knot has a cusp in that given point or if is oriented to the left or to the right).

$$\begin{cases} \text{if } \dot{y} = 0 \text{ then } \dot{z} > 0 \\ \text{if } \dot{y} > 0 \text{ then } x(t) > -\dfrac{\dot{z}}{\dot{y}} \\ \text{if } \dot{y} < 0 \text{ then } x(t) < -\dfrac{\dot{z}}{\dot{y}} \end{cases}$$

This implies that not every possible crossing can appear in the front projection of such a knot. The first line states that vertical tangencies of the knot have to go upwards necessarily. The second and third mean that in a crossing where the two branches are oriented in opposite directions (regarding $y(t)$) and $\dot{z} < 0$, the one oriented towards the right is above (has greater value of $x(t)$). See figure 2 below.

It is easy to check that any other crossing can be achieved and any other curve not containing these two crossings is the projection of a positively transverse curve.

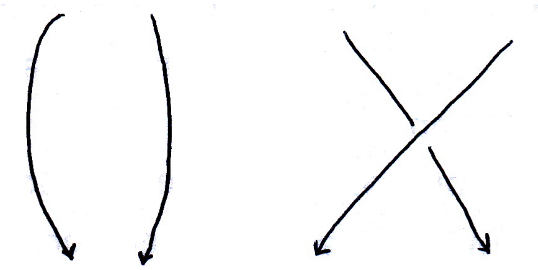

Figure 2: The two types of crossings that cannot appear in the front projection of a positively transverse knot

1.2 Approximation theorems

Theorem 2. *Let $\gamma : \mathbb{S}^1 \to (M^3, \xi)$ be a knot. Then γ can be approximated C^0-closely by a topologically isotopic legendrian knot.*

Proof. We would like to work with a local model that looks like $\mathbb{R}^3$ with the standard contact structure. To achieve this, we can apply Darboux' theorem and obtain finitely many charts covering the knot, since $\mathbb{S}^1$ is compact.

Using local coordinates now, let $(x(t), y(t), z(t))$ be the knot γ. We want to perturb $\dot{y}$ and $\dot{z}$ so that $(-\frac{\dot{z}}{\dot{y}}(t), y(t), z(t))$ is close to γ. This can be achieved by using a zigzaging motion around the curve, introducing cusps such that the slope is mantained arbritrarily close to the desired value of $x(t)$. See figure 3 below.

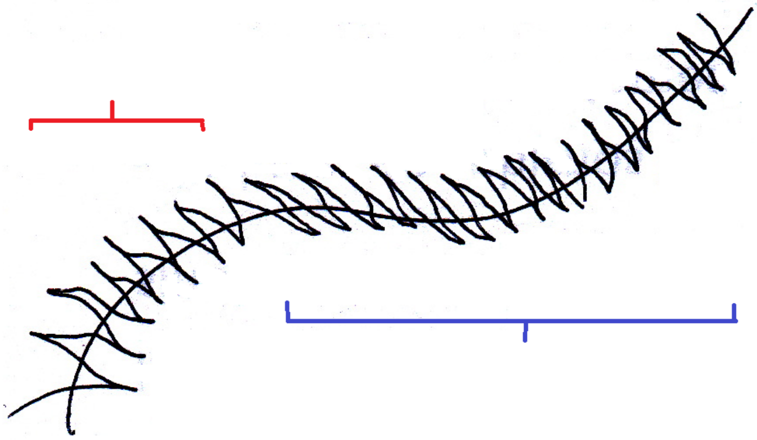

Figure 3: C^0-approximation by a legendrian knot. In the first part, the slope changes from around 0 to -45 degrees, changing thus the value of $x(t)$ from zero to positive. Later on, the slope stabilises, mantaining more or less the value of $x(t)$. This is achieved introducing many cusps

The first remark is that this can be achieved preserving transversality with itself in the front projection, so the resulting curve will still be embedded if the original one was (and it clearly is immersed).

The topological condition of being isotopic can be seen by observing that we can follow this procedure without performing any Reidemeister move, so the resulting knot will indeed be isotopic to the original one. □

Note: Observe that the result cannot be improved to achieve C^1-closedness. This is obvious, since we can create knots that are arbitrarily away from the legendrianity condition. The method relies precisely on deforming the slope of the curve heavily.

Another way of proving the result above would be by using the lagrangian projection. Remembering the remark before about the relationship between the area enclosed by the curve and the value of $z(t)$, we could approximate the projection of the knot by another one that realises small loops to achieve the desired values of $Z(t)$.

We have an analogous result for positively/negatively transverse knots:

Theorem 3. *Let $\gamma : \mathbb{S}^1 \to (M^3, \xi)$ be a knot. Then γ can be approximated C^0-closely by a topologically isotopic positively/negatively transverse knot.*

Proof. Using the theorem above we can assume that γ is legendrian. The idea now makes use of the fact that around such knots we have a local model. Namely, we have that, if we abuse notation and call γ its own image, γ has a tubular neighbourhood $\gamma \subset N_\gamma = \mathbb{S}^1 \times \mathbb{R}^2$, where the contact form looks like $\alpha = \cos(\theta)dx - \sin(\theta)dy$.

Now we can define two new curves that coil around γ in such a way that they are transverse as desired:

$$\gamma_\pm^\delta(\theta) = (\theta, \pm\delta\sin(\theta), \pm\delta\cos(\theta))$$

$$\dot{\gamma}_\pm^\delta(\theta) = (1, \pm\delta\cos(\theta), \mp\delta\sin(\theta))$$

$$\alpha(\dot{\gamma}^{\delta}_{\pm}(\theta)) = \delta\cos^2(\theta) + \delta\sin^2(\theta) = \delta$$

And the isotopy is obtained by interpolating between zero and $\pm\delta$. □

1.3 Classical invariants of legendrian knots

Definition 3. *Let K be a homologically trivial legendrian knot in a contact 3-manifold (M, ξ). Let K' be a knot parallel to K, obtained by pushing K with a vector field ν transverse to the contact structure. Then we define the Thurston-Bennequin invariant (TB invariant from now on) $tb(K)$ as the linking number $lk(K, K')$ of K and K'.*

Recall that homologically trivial means, in the 3-dimensional case, that it bounds an orientable surface Σ, which we call a Seifert surface for the knot. If we choose an orientation for K, we inherit automatically orientations for Σ and K'. That of Σ will be compatible with the orientation of K by the outward normal vector first rule. See figure 4.

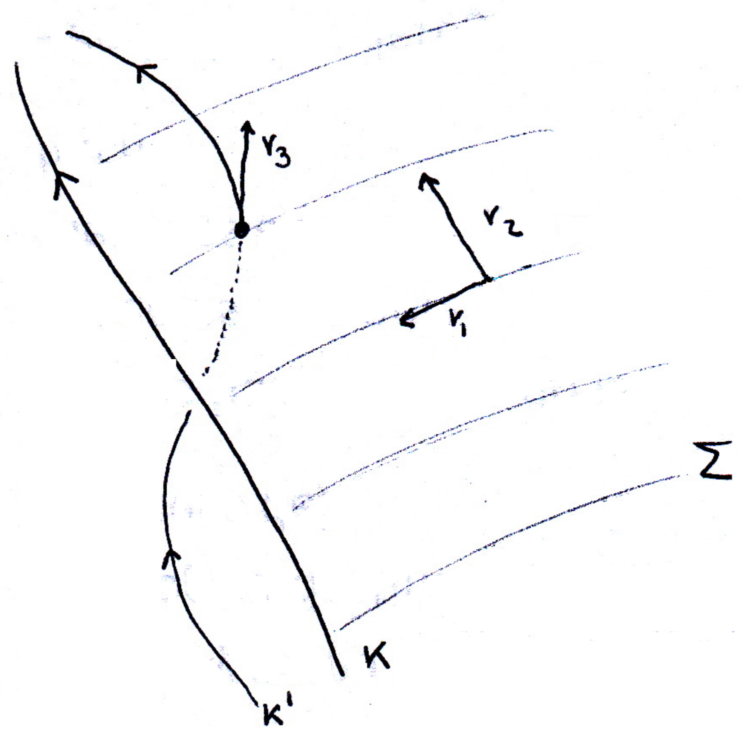

Figure 4: A knot K, a Seifert surface Σ and the auxiliary knot K'. In this case, the basis (v_1, v_2, v_3) is negatively oriented with respect to the standard orientation of $\mathbb{R}^3$.

Definition 4. *By a small perturbation, we can ensure that K' is transverse to Σ, and thus define the linking number of K and K' as:*

$$lk(K, K') = \#(K' \cap \Sigma)$$

where the points in the right hand side are counted taking into account orientations. What we mean by this is that, if at a given point of intersection a positively-oriented basis for the tangent space to Σ *is given by* $\{v_1, v_2\}$ *and the tangent vector to* K' *is* v_3 *then the intersection will be positive if the basis* $\{v_1, v_2, v_3\}$ *has positive orientation in* T_pM*.*

Remark: Being M 3-dimensional, we have a canonical choice of orientation given by the contact form. If α is the contact form, any other form $\alpha_f = f\alpha$, with f a smooth function in M, will describe the same contact structure. A simple computation shows:

$$d(\alpha_f) = df \wedge \alpha + f d\alpha$$

$$(\alpha_f) \wedge d(\alpha_f) = f^2 \alpha \wedge d\alpha$$

so any choice of f will yield the same orientation for M. Obviously, this applies whenever the dimension of our contact manifold is $4n - 1$.

In the definition of the TB invariant we picked a certain Seifert surface, a deformation K' of K and an orientation of K. For the definition to make sense, we need to make sure it is independent of all these choices.

Indeed, the linking number is usually defined for any link in euclidean 3-space as:

Definition 5. *The linking number* $lk(K, K')$ *of two knots* K *and* K' *in* $\mathbb{R}^3$ *is half the sum over all their crossings of:*

$$\begin{cases} +1 & \text{if the oriented overcrossing is obtained from the oriented undercrossing by clockwise rotation} \\ -1 & \text{if the oriented overcrossing is obtained from the oriented undercrossing by counterclockwise rotation} \end{cases}$$

In particular, the linking number is symmetric.

Which is easily seen to be invariant under Reidemeister moves. It is a basic fact in knot theory that this combinatorial definition corresponds to the definition given above in terms of Seifert surfaces. Observe that we are abusing notation and considering knot projections.

Taking this as a fact, it only remains to prove that some other election of K' does not change the linking number. We shall see in the exercises that this linking number is related to the twisting to the Reeb vector field around K and is, therefore, independent of the specific choice of K'.

Naturally, we are interested in the knots with respect to legendrian isotopy (in the same fashion as one is interested, in non-contact knot theory, in isotopy). Indeed, it is straighforward to see that the linking is invariant under contactomorphism and also under legendrian isotopy, which follows from the isotopy extension theorem.

1.4 Computation of the Thurston-Bennequin invariant

Definition 6. *The writhe of the knot* K *is defined as the sum over all crossings of:*

$$\begin{cases} +1 & \text{if the oriented upper crossing is obtained from the oriented under crossing by clockwise rotation} \\ -1 & \text{if the oriented upper crossing is obtained from the oriented under crossing by counterclockwise rotation} \end{cases}$$

as seen in figure 5.

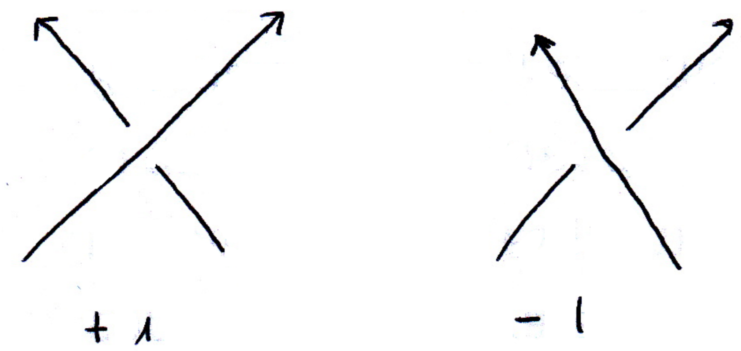

Figure 5: Sign convention for the writhe

Proposition 3. *Let K be a legendrian knot in $(\mathbb{R}^3, \xi_{st})$ and let K_F be its frontal projection. Then the TB invariant can be computed as follows:*

$$tb(K) = \omega(K) - \frac{1}{2}\#cusps$$

with $\omega(K)$ the writhe of the knot.

Proof. In the definition of the TB-invariant we obtained K' from K by pushing with a vector field transverse to the contact structure. An example of such a vector field is the Reeb vector field, which is simply ∂_z for $\mathbb{R}^3$ with the standard contact structure. This implies that we can depict K' in the frontal projection as an upwards displacement of K and study the crossings between K and K'.

Observe first the similarities between the definition of writhe and the combinatorial version of linking number. In every crossing of K with itself, a writhe of -1 would give two crossings of K' with K with value -1, and similarly a writhe of 1 would give two crossings of value 1. See figure 6

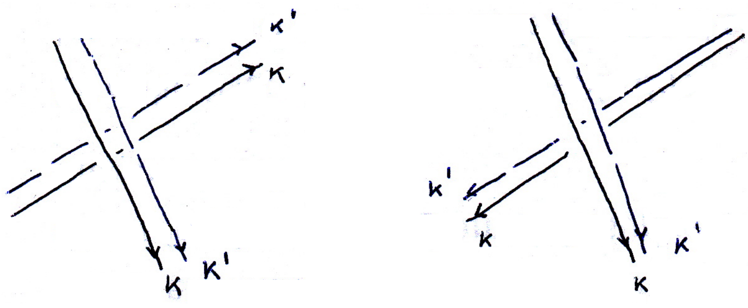

Figure 6: K and K' at a crossing of K with itself

Cusps also introduce crossings of K' with K. By systematic inspection of all posibilities (left and right handed cusps, each with 2 possible orientations), one can see that they produce a single crossing of K' and K with value -1.

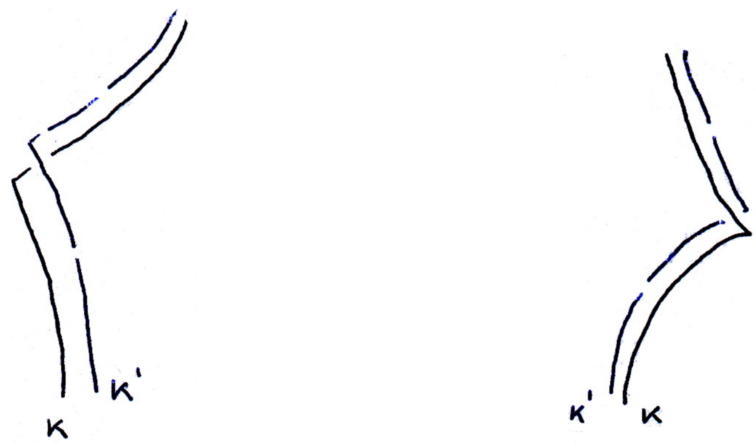

Figure 7: K and K' at cusps of K

These two remarks imply that the formula is indeed true:

$$lk(K, K') = \frac{2\omega(K) - \#\text{cusps}}{2} = \omega(K) - \frac{1}{2}\#\text{cusps}$$

□

1.5 The rotation number

Definition 7. *Let K be an oriented legendrian knot in $\mathbb{R}^3_t$, and γ the map from the interval such that K is its oriented image. Since $\dot{\gamma}(t)$ belongs to ξ_{st} at the point $\gamma(t)$, and it is non-vanishing, it makes sense to consider the degree of the map $\dot{\gamma}$ with respect to the framing given by ∂_x and $\partial_y - x\partial_z$. We define its rotation number to be this degree. Observe that it does depend on the orientation selected.*

This definition can be extended to any homotopically trivial legendrian knot in a contact 3-manifold M, in which we would define the rotation number as the degree of the map $\dot{\gamma}$ with respect to some trivialisation of the contact structure over a Seifert surface c of K, so it would be an invariant depending on both K and c. One would have to prove that different trivialisations yield the same value for $\text{rot}(K, c)$.

Proposition 4. *If we denote $\lambda_\pm$ as the number of left cusps oriented upwards and downwards respectively and with $\mu_\pm$ the right cusps oriented upwards and downwards, then the rotation number verifies:*

$$rot(K) = \lambda_- - \mu_+ = \mu_- - \lambda_+$$

Proof. Since $v_1 = \partial_x$ and $v_2 = \partial_y - x\partial_z$ give the framing, all we have to do is count the number of times $\dot{\gamma}$ crosses v_1, for instance (any other given vector would do, as we are counting the turns). If we write:

$$\dot{\gamma}(t) = g(t)v_1 + f(t)v_2$$

then the vanishing of $f(t)$ gives the points where $\dot{\gamma}$ is a multiple of v_1. In the front projection this happens only at the cusps (those are the only points where the partial with respect to y of γ vanishes).

At right cusps oriented upwards, $f(t)$ changes from positive to negative and $g(t)$ is positive (since x is given by $-\frac{\dot{z}}{\dot{y}}$ and the slope changes from being positive to being negative), so there $\dot{\gamma}$ crosses v_1 with a negative sign. A similar reasoning shows that at left cusps oriented downwards, it crosses v_1 with a positive sign. This yields the first expression.

The second expression is the same computation, but this time considering crossings with $-v_1$. By averaging both expressions, we can obtain:

$$\text{rot}(K) = \frac{\lambda_- - \mu_+ + \mu_- - \lambda_+}{2} = \frac{1}{2}(c_- - c_+)$$

where $c_\pm$ denote the number of cusps oriented upwards and downwards respectively. □

2 Exercises

2.1 Computing the Thurston-Bennequin invariant in an specific example

Let us consider the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, with the one form given by $\alpha = x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2$, which is a contact form when restricted to the sphere. What is the value of the TB-invariant for the knot $K = \{x_1^2 + x_2^2 = 1,\, y_1 = y^2 = 0\} \subset \mathbb{S}^3$?

We are going to make use of the first definition we gave for the linking number, in terms of Seifert surfaces. One such a surface could be the disc:

$$D = \{x_1^2 + x_2^2 + y_1^2 = 1,\, y_2 = 0,\, y_1 \geq 0\}$$

In this case, the Reeb vector field of α is:

$$R_\alpha = x_1\partial_{y_1} - y_1\partial_{x_1} + x_2\partial y_2 - y_2\partial_{x_2}$$

Indeed, we can check that:

$$d\alpha = 2dx_1dy_1 + 2dx_2dy_2$$
$$\alpha(R_\alpha) = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$$
$$d\alpha(R_\alpha, -) = -2x_1dx_1 - 2y_1dy_1 - 2x_2dx_2 - 2y_2dy_2$$

and this last expression is zero when pullbacked to the sphere, since it is the differential of $f = -2|z|^2$, $z = (x_1, y_1, x_2, y_2)$, with level sets the spheres.

Now we can use R_α to push K to K'. Let $\gamma(t) = (\cos(t), 0, \sin(t), 0)$ and $R_\alpha(t) = (0, \cos(t), 0, \sin(t))$. Therefore, for some small δ, we can solve the easy differential equation set by the Reeb vector field and define K' as the image of:

$$\gamma_\delta(t) = (\cos(t)\cos(\delta), \cos(t)\sin(\delta), \sin(t)\cos(\delta), \sin(t)\sin(\delta))$$

which indeed lies on the 3-sphere.

We can already see that K' turns around K just once if δ is small, but we should compute the signed intersection, taking into account orientations. In a neighbourhood of K, a basis for the tangent space to D is given by the vector field v_2, extending the tangent vector field to K:

$$v_2 = -x_2\partial_{x_1} + x_1\partial_{x_2}$$

and the vector field v_1, which in K points outwards:

$$v_1 = x_1 y_1 \partial_{x_1} + x_2 y_1 \partial_{x_2} - (x_1^2 + x_2^2)\partial_{y_1}$$

We now can compute the intersection point of K' with D, by substitution of the parametrisation γ_δ of K' into the defining equation for D. This yields the (only) point:

$$P = (\cos(\delta), \sin(\delta), 0, 0)$$

where the tangent vector to K' is

$$v_3 = \cos(\delta)\partial_{y_1} + \sin(\delta)\partial_{y_2}$$

Thus far we have obtained the desired basis for the tangent space at the intersection point P, so it only remains to substitute said basis into the volume form given by α to determine whether it is positively oriented. The volume form in the sphere, when restricted to the disc is:

$$\alpha \wedge d\alpha|_D = 2x_1 dy_1 dx_2 dy_2 - 2y_1 dx_1 dx_2 dy_2 + 2x_2 dy_2 dx_1 dy_1 - 2y_2 dx_2 dx_1 dy_1 =$$

$$2x_1 dy_1 dx_2 dy_2 - 2y_1 dx_1 dx_2 dy_2 + 2x_2 dy_2 dx_1 dy_1$$

Now we substitute the first vector:

$$(\alpha \wedge d\alpha)(v_1, -, -) = -2x_1 y_1^2 dx_2 dy_2 - 2x_2 x_1 y_1 dy_2 dy_1$$

$$-2x_1 x_2 y_1 dy_1 dy_2 + 2y_1^2 x_2 dx_1 dy_2 - 2x_1(x_1^2 + x_2^2) dx_2 dy_2 - 2x_2(x_1^2 + x_2^2) dy_2 dx_1 =$$

$$-2x_1 dx_2 dy_2 + 2x_2 dx_1 dy_2$$

The second vector:

$$(\alpha \wedge d\alpha)(v_1, v_2, -) = -2(x_2^2 + x_1^2) dy_2$$

And the third:

$$(\alpha \wedge d\alpha)(v_1, v_2, v_3) = -2(x_2^2 + x_1^2)\sin(\delta)$$

which is a negative number, being δ small. This proves that the linking number is -1 and concludes our computation.

Before finishing, we should observe that computing the actual equation for some K' was not necessary at all. We could have just studied the turn of the Reeb vector field with respect to the framing given by v_1 and v_2 when restricted to K.

2.2 Computing the rotation number in an specific example

Compute the rotation number in the example above.

First we need to obtain a trivialisation of the contact structure in the disc D. By inspection, one such trivialisation would be:

$$v_1 = y_2 \partial_{x_1} - y_1 \partial_{x_2} - x_2 \partial_{y_1} + x_1 \partial_{y_2}$$

$$v_2 = x_2 \partial_{x_1} - x_1 \partial_{x_2} - y_2 \partial_{y_1} + y_1 \partial_{y_2}$$

indeed, we can insert them into α and check that are in the kernel and then observe that they are indeed in the tangent space to $\mathbb{S}^3$. Also, although they are not independent globally, they are when restricted to D, as desired.

We are now left with checking the twisting of the tangent vector field to K, $v_3 = -x_2\partial_{x_1} + x_1\partial_{x_2}$ with respect to this frame. But it turns out that v_2 is $-v_3$ when we restrict ourselves to K, as it can be readily seen. This implies the rotation number is 0.

2.3 Standard contact structure on the 3-sphere

Check that the standard contact structure on $\mathbb{S}^3 \subset \mathbb{C}^2$ is a J-invariant subbundle of $T\mathbb{S}^3$.

Let us precompose J with α. The action of J in $T\mathbb{C}^2$ is:

$$J(\partial_{x_i}) = \partial_{y_i}$$

$$J(\partial_{y_i}) = -\partial_{x_i}$$

So now, we have:

$$\alpha \circ J = x_1 d_{x_1} + y_1 d_{y_1} + x_2 d_{x_2} + y_2 d_{y_2} = rdr$$

which precisely describes the tangent space to the sphere, so we are done.

2.4 Realisations of the left and right handed trefoils as legendrian knots.

As we saw above, it is possible to realise any knot as a legendrian knot. However, we remarked that the number of cusps one obtains with the procedure given above is very high. Below, we show how to obtain the left and right handed trefoil with a small number of cusps:

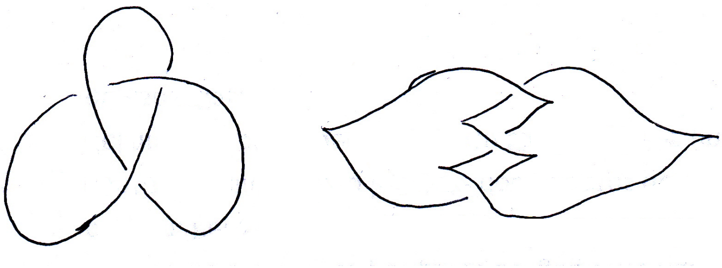

Figure 8: Legendrian realisation of a left trefoil

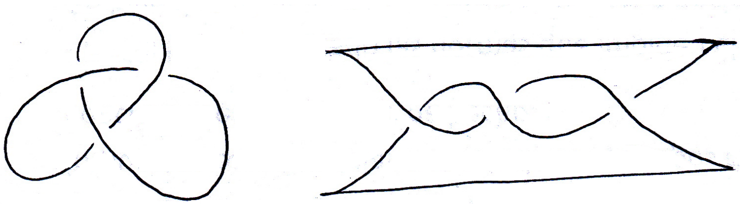

Figure 9: Legendrian realisation of a right trefoil

2.5 Lagrange projection

Using the lagrange projection, we studied the value of the z component by studying the area of each of the planar areas the projection γ_L delimits. Let us study a particular example by studying a knot that is topologically isotopic to the unknot and has a couple of twists in the lagrange projection. This is seen in figure 10.

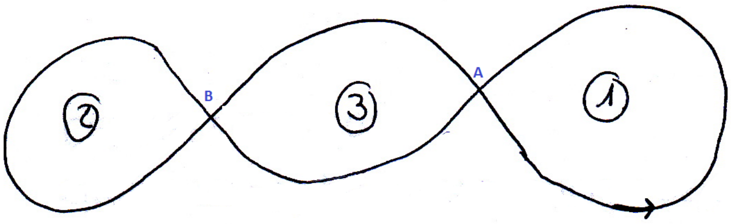

Figure 10: Lagrange projection of a knot

Let us start transversing the knot in the vertex A. Since the oriented oriented area of domain 1 is positive, it means that the value of $z(t)$ when it gets to A after surrounding domain 1 is less than it was in the beginning. Similarly, when we start in B and go around domain 2, we obtain the same condition, that $z(t)$ decreases during the process. Furthermore, since the knot must close, we have that the area of domain 3 (without signs now) must be the sum of the areas of domains 1 and 2. The resulting diagram is seen in figure 11

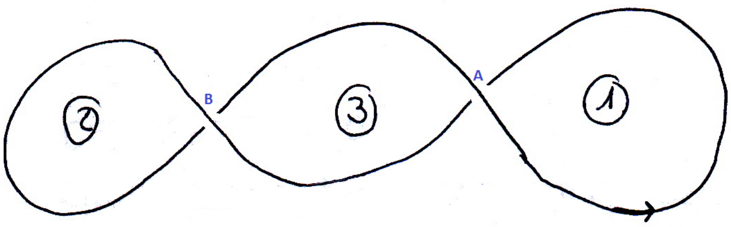

Figure 11: Lagrange projection of the knot with crossings. The over crossings are those with greater value for z