0.1 Contact structures on 3-manifolds

Theorem 0.1.1 (Martinet, 1971)**.** Every closed, orientable 3-manifold admits a contact structure.

Hence all manifolds M in the lecture are tacitly assumed to be closed and orientable.

Proof. Let us begin with result on the structure of 3-manifolds. By result of Lickorish and Wallace, every connected, closed, oriented manifold can be obtained from S^3 via surgery.

To be more precise, there is a link

$$\coprod^n S^1 \hookrightarrow S^3,$$

and an extension to a framed embedding

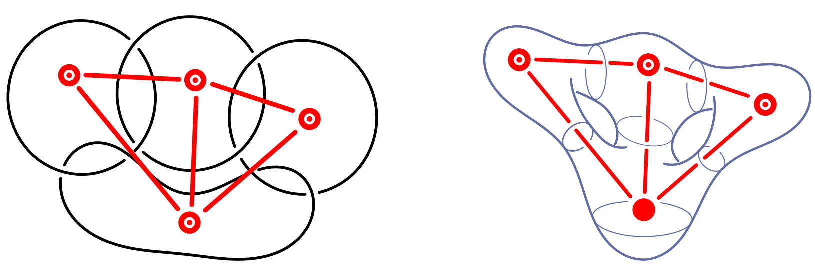
$$\coprod S^1 \times D^2 \hookrightarrow S^3.$$

Hence we may cut each circle in the link along its tubular neighbourhood to obtain

$$S^3 \setminus \left(\coprod S^1 \times D^2 \right).$$

Now we glue in $D^2 \times S^1$s on the boundary

$$\partial(S^1 \times D^2) = S^1 \times S^1 = \partial(D^2 \times S^1).$$



The other way to see it is to create a graph based on the link. For each circle in the link we set a vertex. Two vertices are joined by an edge if the appropriate circles are linked. A tubular neighbourhood of the graph in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is a (filled) n-torus (n-fold connected sum of tori)Its complement in S^3 is again a (filled) n-torus. The construction finishes by adding a Dehn twists on appropriate handles before gluing boundaries of these tori. However we have to take care of a contact structure as we carry the surgery steps.

Contact surgery Of course S^3 admits standard contact structure ξ_{st}. By the isotopy extension theorem (in the first lecture) we may isotope the embedded S^1s until they become transverse to ξ_{st} (actually, we proved appropriate theorem for isotopy to Legendrian knots). Again using results presented before (the neighbourhood theorem which we again proved only for Legendrian knots) we may choose δ appropriately small radius and framed embeddings

$$f\colon S^1 \times D^2_\delta \hookrightarrow S^3$$

such that the pullback of ξ_{st} is the standard form.

$$f^*\xi_{st} = d\overline{\theta} + r^2 d\overline{\phi}.$$

($\overline{\theta}$ denotes a coordinate on S^1, whereas $(r, \overline{\phi})$ are polar coordinates on D^2). From now on we identify $(S^1 \times D^2_\delta, d\overline{\theta} + r^2 d\overline{\phi})$ with $(f(S^1 \times D^2_\delta), f^*\xi_{st})$.

We will perform surgery inside the tubular neighbourhood

$$S^1 \times D^2_\delta.$$

Choose $0 < \varepsilon < \delta$ and cut $S^1 \times D^2_\varepsilon \subset S^1 \times D^2_\delta \subset S^3$. Now glue $D^2 \times S^1_\varepsilon$ using the following identification on the boundary $\partial(D^2 \times S^1_\varepsilon) = S^1 \times S^1_\varepsilon$ (for clarity we use $(\overline{\theta}, r, \overline{\phi})$ for coordinates on the tube $S^1 \times D^2_\varepsilon$ **inside** S^3, whereas (ϕ, r, θ) live on $D^2 \times S^1_\varepsilon$.)

$$(\phi, r = \varepsilon, \theta) \longmapsto (\overline{\theta}, r = \varepsilon, -\overline{\phi}).$$

Observe, that the minus sign in the formula takes care of the orientation of the resulting manifold.

Now we need to extend the contact form on $S^3 \setminus S^1 \times D^2_\varepsilon$ to the interior of glued $D^2 \times S^1_\varepsilon$. Of course the easiest way would be to choose constant extension of the from on the boundary, $d\overline{\theta} + r^2 d\overline{\phi}$. However under our glueing identification it becomes

$$d\phi - r^2 d\theta \tag{1}$$

and as we approach $r = 0$ we encounter problems with smoothness. To avoid it we would like the extension to behave as

$$d\theta + r^2 d\phi \tag{2}$$

in a neighbourhood of $r = 0$.

So we bring an **ansatz**[1] in.

$$\alpha \overset{\text{def.}}{=} h_1(r)d\theta + h_2(r)d\phi.$$

Additionaly we have the boundary conditions on h_1 and h_2:

$$\begin{aligned} h_1(r) &= -r^2 & h_2(r) &= 1 & &\text{near } r = \delta \quad \text{by (1)} \\ h_1(r) &= 1 & h_2(r) &= r^2 & &\text{near } r = 0 \quad \text{by (2)} \end{aligned}$$

Of course it has to be a contact form, so we have to check when $\alpha \wedge d\alpha \neq 0$.

$$\begin{aligned} \alpha \wedge d\alpha &= (h_1 d\theta + h_2 d\phi) \wedge (h_1' dr \wedge d\theta + h_2' dr \wedge d\phi) \\ &= h_1 h_2' d\theta \wedge dr \wedge d\phi + h_2 h_1' d\phi \wedge dr \wedge d\theta \\ &= (h_1 h_2' - h_2 h_1') d\theta \wedge dr \wedge d\phi. \end{aligned}$$

Hence such α is a contact form for all $r \neq 0$ if and only if

$$h_1 h_2' - h_2 h_1' = \begin{bmatrix} h_1 & h_1' \\ h_2 & h_2' \end{bmatrix} \neq 0.$$

We may regard $h_1(r)$ and $h_2(r)$ as coordinates of a curve $c\colon [0, \delta] \to \mathbb{R}^2$, $c(r) = (h_1(r), h_2(r))$.

Then the condition above simply means, that the tangent vector to the curve and vector defined by the curve itself are linearly independent (i.e. not parallel). An example of such curve is presented in figure 1.

$\square$

Theorem 0.1.2 (Lutz, 1971)**.** Let M be a 3-manidold. Every cooriantable, tangent 2-plane field $\eta \subset TM$ is homotopic to a contact structure.

[1]**ansatz** – first approach, a starting equation; *etwas in **Ansatz** bringen* – to use something for the calculation; to start the calculation using something.

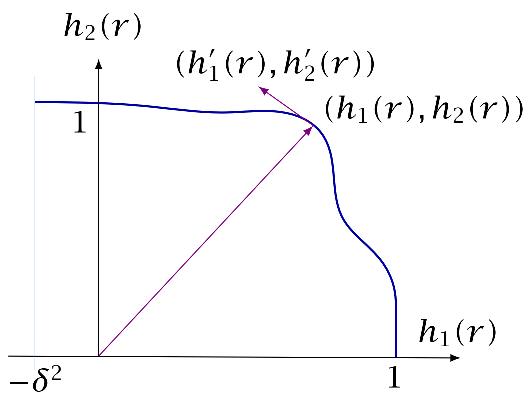


Figure 1: Example of curve $c(r) = (h_1(r), h_2(r))$

0.2 Tight and overtwisted

Example 1. Recall that $\alpha = \cos r dz + r \sin r d\phi$ is a contact form on $\mathbb{R}^3$ in cylindrical coordinates (r, ϕ, z).

The disk $\Delta \overset{\text{def.}}{=} \{z = 0, r \leq \pi\}$ has as boundary a Legendrian curve and

$$T\Delta|_{\partial\Delta} = \ker \alpha|_{\partial\Delta}.$$

Hence the Thurston-Bennequin invariant vanishes

$$tb(\partial\Delta) = 0.$$

However, such a disk bounding a Legendrian curve can not be embedded into $\mathbb{R}^3_{st}$.

Definition 0.2.1. The characteristic foliation of a surface $\Sigma \subset (M, \xi)$ in a contact 3-manifold is the singular 1-dimensional foliation defined locally as a curves tangent to subspaces

$$T_p\Sigma \cap \xi_p.$$

Singularities occur when $T_p\Sigma = \xi_p$.

Example 2. Consider our disk Δ in $(\mathbb{R}^3, \alpha)$. Its characteristic foliation is the radial foliation, with singular points the origin and the boundary.

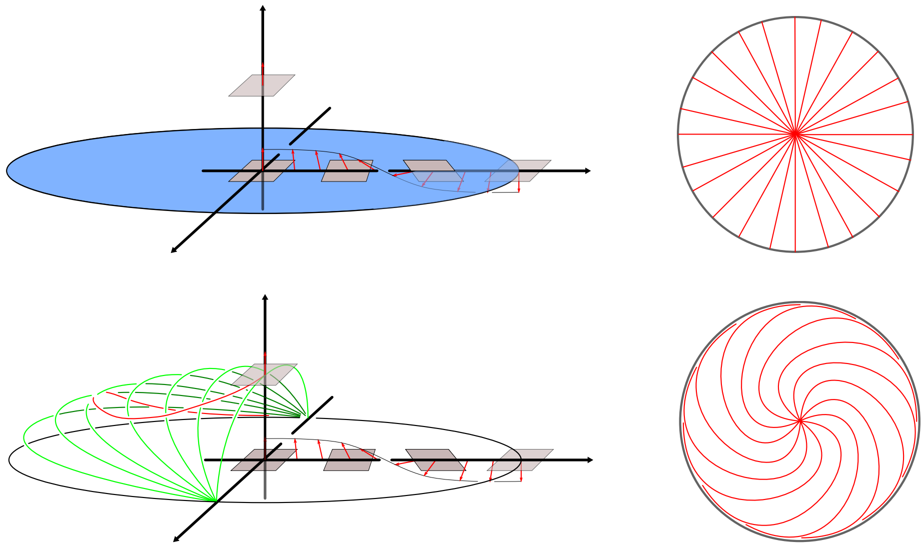


Figure 2: Δ and $\Delta' \subset (\mathbb{R}^3, \ker(\alpha))$ and their characteristic foliations

Now keep the boundary of Δ fixed and push the interior of the disk slightly away from $(z = 0)$-plane to obtain Δ'. Then the singular set consists of just one point. The surfaces and their characteristic foliations are pictured in figure 2.

Definition 0.2.2 (Eliashberg)**.** A contact structure on a 3-manifold is **overtwisted** if there exist an **embedding** of a disk whose characteristic foliation looks like the one on Δ. Equivalently, if there exist an embedded disk with boundary a Legendrian curve γ with $tb(\gamma) = 0$.

If there is no such disk, then the contact structure is **tight**.

Theorem 0.2.1 (Bennequin, 1983)**.** On $\mathbb{R}^3$ the standard contact structure $\xi_{st} = \ker(dz + xdy)$ is tight.

Remark 1. Actually in $(\mathbb{R}^3, \xi_{st})$ one can find an **immersed** disc with boundary a Legendrian curve with Thurston-Bennequin invariant equal 0.

Lemma 0.2.2. Let ξ be a contact structure on a 3-manifold M. Then there is an overtwisted contact structure ξ' homotopic to ξ as a tangent 2-plane field.

Proof. Choose a knot k transverse to ξ and its neighbourhood $S^1 \times D^2$ on which the contact structure is standard,

$$\xi|_{S^1 \times D^2} = \ker(d\theta + r^2 d\phi).$$

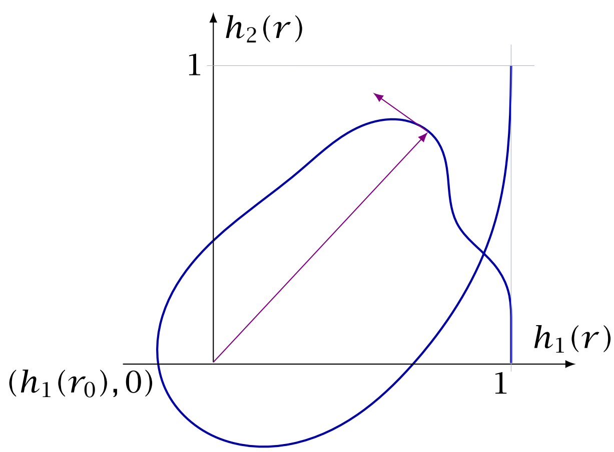
Now as before, on $S^1 \times D^2$ replace ξ by

$$\xi' \overset{\text{def.}}{=} \ker\left(h_1(r)d\theta + h_2(r)d\phi\right)$$

with additional boundary conditions

$$h_1(r) = 1 \qquad h_2(r) = r^2 \qquad \text{for } r \in [0, \varepsilon) \cup (1-\varepsilon, 1].$$

Of course we must just make sure that the curve $c(r) = (h_1(r), h_2(r))$ is not parallel to its tangent vector, because otherwise ξ' is not a contact structure anymore. An example of such curve is pictured below.



Note that as we cross the x-axis for the first time (e.g. for parameter $r = r_0$), contact planes are perpendicular to the normal planes to the curve and

$$\Delta = \{\theta = \theta_0, r \le r_0\}$$

is an overtwisted disk. Note also, that the curve has to go around origin, otherwise there will exist a point when curve is parallel to its tangent.

Now we need a homotopy between contact structures ξ and ξ'. This can be rephrased as finding a homotopy between paths in figure 3. However if we want to conduct such a homotopy on the (h_1, h_2) plane (e.g. linear interpolation depending on the parameter t), then we have to pass through the origin (e.g. for $t = t_1$ and $r = r_1$), and at this moment

$$\xi'_t = \ker(f_(t_1)h_1(r_1)d\theta + f_2(t_1)h_2(r_1)d\phi \equiv 0)$$

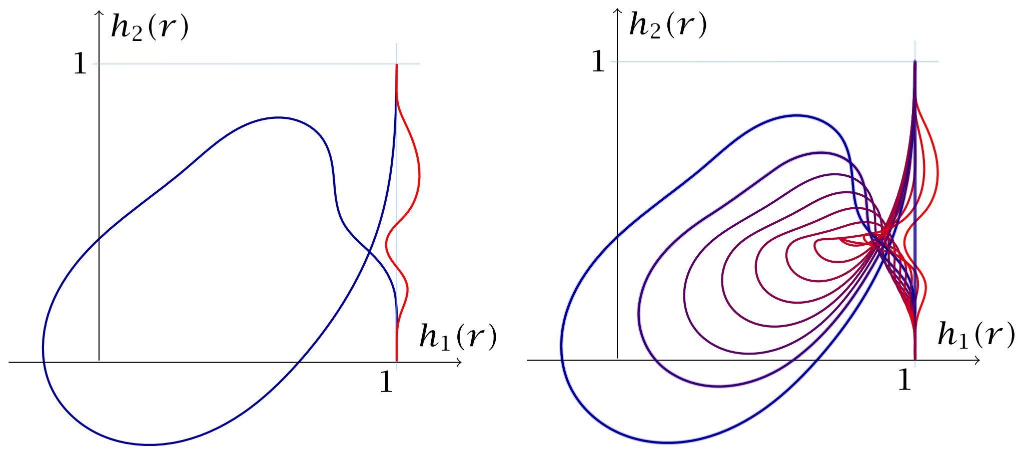


Figure 3: Homotopy of paths. Blue is the overtwisted structure, red is (isotopic to) the standard one.

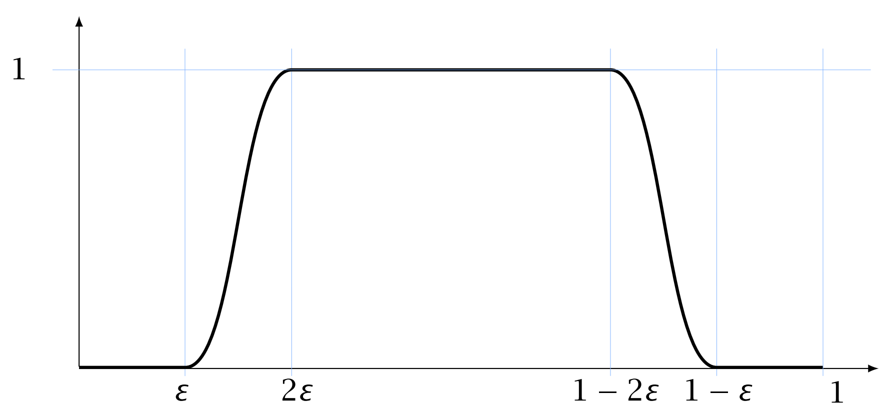
is not a 2-plane field. We may fix it by adding a little of dr direction when passing through $(0,0)$.

Let $\chi_\varepsilon(r)$ be a smooth characteristic function of $[2\varepsilon, 1-2\varepsilon]$ vanishing outside $[\varepsilon, 1-\varepsilon]$. An example of such function can be found below.

Then we claim that

$$\alpha_t = t(1-t)\chi(r)dr + h_1(r)d\theta + h_2(r)d\phi$$

is the desired homotopy. Indeed if $2\varepsilon < r_1 < 2\varepsilon$ then α_t contains dr summand, hence regardless of h_1 and h_2 $\alpha_t$does not vanish. □



Theorem 0.2.3. Let M be a 3-manifold.

Lutz: Every homotopy class of tangent 2-plane bundles contains an overtwisted contact structure.

Eliashberg, 1989: Every two overtwisted contact structures that are homotopic as 2-plane fields are homotopic as contact structures and hence – by Gray stability – isotopic.

0.3 Symplectic fillings

Definition 0.3.1. Let (W, ω) be a compact symplectic 4-manifold bounded by a contact 3-manifold (M, ξ). We assume, that the orientation on M is induced from W.

- W is called a **weak symplectic filling** of a M if $\omega|_\xi > 0$.
- W is called a **strong symplectic filling** it there exists a Liouville vector field Y for ω defined *near and transverse* to $M \subset W$, such that
$$\ker(i_Y \omega|_{TM}) = \xi.$$

Remark 2.
- Every strong filling is a weak filling.
- Not every weakly fillable contact 3-manifold is strongly fillable.

Theorem 0.3.1 (Eliashberg-Gromov)**.** Every weakly fillable contact 3-manifold is tight.

Example 3.
- $(\mathbb{R}^3, \xi_{st})$ is tight (Bennequin).
- (S^3, ξ_{st}) is tight by a strong filling: $(D^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$, $Y = x_1\partial_{x_1} + y_1\partial_{y_1} + x_2\partial_{x_2} + y_2\partial_{y_2}$.
- $(T^3, \ker(\cos\theta dx - \sin\theta dy))$ is tight by a weak filling: $(S^1 \times S^1 \times S^2, \xi_{st})$.

  Exercise: find a strong filling.

Remark 3. Observe, that $\mathbb{R}^3 \xrightarrow{\pi} T^3$ is a covering by a tight contact manifold. Moreover $\pi^*(\cos\theta dx - \sin\theta dy) = \xi_{st}$ on $\mathbb{R}^3$. Hence T^3 is tight.