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# THE WU METRIC IS NOT UPPER SEMICONTINUOUS

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H. Wu introduced in [Wu] a new invariant metric which was to combine invariant properties of the Kobayashi–Royden metric and regularity properties of Kähler metrics. The metric depends on some initial (pseudo)metric $\eta$. We call it the Wu (pseudo)metric associated with $\eta$ and write $\mathbb{W}\eta$.

Jarnicki and Pflug asked ([Jar–Pfl 2], [Jar–Pfl 1]) whether $\mathbb{W}\eta$ is upper semicontinuous, if $\eta$ is one of the well-known pseudometrics: Kobayashi–Royden ($\kappa$), Azukawa ($A$) or Carathéodory–Reiffen pseudometric of $k$–th order ($\gamma^k$). The question of the upper semicontinuity appears in a natural way, for instance in the definition of the integrated form $\int(\mathbb{W}\eta)$. Wu (cf. [Wu]) and Cheung and Kim (cf. [Che–Kim]) claimed (but without proof) that $\mathbb{W}\kappa$ was upper semicontinuous.

We give an example of a bounded pseudoconvex domain in $\mathbb{C}^n$ which contradicts that statement.

## 1. Definition

We present the sketch of the definition of the Wu metric in an abstract setting ([Jar–Pfl 2]). For detailed discussion we refer the reader to [Jar–Pfl 2] (or [Jar–Pfl 1]) and [Wu].

For a domain $D \subset \mathbb{C}^n$, denote by $\mathcal{M}(D)$ the space of all pseudometrics

$$\eta : D \times \mathbb{C}^n \to \mathbb{R}_+, \quad \eta(a; \lambda X) = |\lambda|\eta(a; X), \quad \lambda \in \mathbb{C}, (a, X) \in D \times \mathbb{C}^n,$$

such that

$$\forall\, a \in D\ \exists\, M, r > 0 : \eta(z; X) \le M\|X\|, \quad z \in \mathbb{B}(a, r) \subset D, X \in \mathbb{C}^n,$$

where $\mathbb{B}(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$ and $\|\cdot\|$ is the standard Euclidean norm.

Fix a domain $D \subset \mathbb{C}^n$, a point $a \in D$, a pseudometric $\eta_D \in \mathcal{M}(D)$ and put:

$$I_{\eta_D}(a) := \{X \in \mathbb{C}^n : \eta_D(a; X) < 1\},$$
$$V_{\eta_D}(a) := \{X \in \mathbb{C}^n : \eta_D(a; X) = 0\},$$
$U_{\eta_D}(a) :=$ the orthogonal complement of $V$ with respect to the standard scalar product in $\mathbb{C}^n$.

For any pseudo–Hermitian scalar product $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, define

$$q_s(X) := \sqrt{s(X, X)}, \quad X \in \mathbb{C}^n.$$

Let $\mathcal{F}(\eta_D, a)$ be a set of all pseudo–Hermitian scalar products $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that $q_s \le \eta_D(a;\cdot)$ (or, equivalently, $I_{\eta_D}(a) \subset I(q_s)$). There exists a unique (!) element $s(\eta_D, a) \in \mathcal{F}(\eta_D, a)$ which is maximal with respect to the partial ordering $\prec$ defined for $a, b \in \mathcal{F}(\eta_D, a)$:

$$a \prec b \quad \text{if} \quad \det[a(e_j, e_k)]_{j,k=1,\dots,m} \le \det[b(e_j, e_k)]_{j,k=1,\dots,m},$$

for any basis $(e_1, \dots, e_m)$ of $U_{\eta_D}(a)$.

We define

$$\widetilde{\mathbb{W}}\eta_D(a;X) := q_{s(\eta_D,a)}(X), \quad X \in \mathbb{C}^n;$$
$$\mathbb{W}\eta_D(a;X) := \sqrt{m}\,\widetilde{\mathbb{W}}\eta_D(a;X), \quad X \in \mathbb{C}^n,$$

where $m := \dim U_{\eta_D}(a)$.

## 2. Main result

**Theorem 1.** *Let $(\alpha_D)_D$ be a family of holomorphically contractible pseudometrics, such that $A_D \le \alpha_D \le \kappa_D$ for any $D \subset \mathbb{C}^2$.*

*Then there exists a bounded pseudoconvex domain $G \subset \mathbb{C}^2$ such that neither $\mathbb{W}\alpha_G$ nor $\widetilde{\mathbb{W}}\alpha_G$ is upper semicontinuous.*

**Corollary 2.** *For any $n \ge 2$ there exists a bounded pseudoconvex domain $G \subset \mathbb{C}^n$ such that metrics $\mathbb{W}\kappa_G$, $\widetilde{\mathbb{W}}\kappa_G$, $\mathbb{W}A_G$, $\widetilde{\mathbb{W}}A_G$ are not upper semicontinuous.*

We define for $R > 1$ a domain

$$G_R := \{(z,w) \in \Delta \times R\Delta :\ 2R|w|e^{u(z)} < 1\},$$

where

$$u(z) := 1 + \sum_{j=4}^{\infty} \frac{1}{2^j} \max\{\log \frac{|2^{-j} - z|}{2}, -2^{2j}\}$$

is a subharmonic function on the unit disk $\Delta \subset \mathbb{C}$.

The domain $G$ from Theorem 1 can be any domain $G_R$ for $R > 1$ sufficiently large.

## References


[Che–Kim] C. K. Cheung, K. T. Kim, *Analysis of the Wu metric. I: The case of convex Thullen domains*, Trans. Amer. Math. Soc. 348 (1996), 1421–1457.

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