Seminar on Complex Analysis
Meeting 1915, 11th December 2006
lecture: **Convexity of symmetrized ellipsoids**
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A domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-*convex* if $D \cap L$ is connected and simply connected for any complex affine line L such that $D \cap L$ is not empty.

For $p > 0$ let $\mathcal{E}_{p,n} := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^p + \dots + |z_n|^p < 1\}$, $E := \mathcal{E}_{1,1}$ and let $\pi_n = (\pi_{n,1}, \dots, \pi_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as follows

$$\pi_{n,k}(z) = \sum_{1 \leqslant j_1 < \dots < j_k \leqslant n} z_{j_1} \dots z_{j_k}, \quad 1 \leqslant k \leqslant n, \ z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

The set $\mathbb{E}_{p,n} := \pi_n(\mathcal{E}_{p,n})$ is called the *symmetrized* (p,n)-*ellipsoid*.

Proposition 1 (cf. Theorem 1 in [3]). (i) *If* $p \geqslant 2$ *and* $n \geqslant 3$ *then* $\mathbb{E}_{p,n}$ *is not* $\mathbb{C}$-*convex.*

(ii) *Let* $1 < p < 2$. *If* $n \geqslant k = k(p) := \min\{l \in \mathbb{N} : 2 \log_{l(l-1)} l < p\}$, *then* $\mathbb{E}_{p,n}$ *is not* $\mathbb{C}$-*convex.*

Since $2 \log_{n(n-1)} n \searrow 1$ as $n \to +\infty$, we have the following

Corollary 2. *For any* $p > 1$ *there exists* $n \in \mathbb{N}$ *such that* $\mathbb{E}_{p,n}$ *is not* $\mathbb{C}$-*convex.*

Proposition 3. (i) *For any* $p \in (0, \log_2 \frac{5}{4}) \cup (2, +\infty)$ *and* $n \geqslant 2$ *the set* $\mathbb{E}_{p,n}$ *is not convex.*

(ii) *The set* $\mathbb{E}_{2,2}$ *is convex.*

Corollary 4. (i) *There exists a non-convex set in* $\mathbb{C}^n$, $n \geqslant 2$, *such that its image through* π_n *is not convex, either.*

(ii) *There exists a convex set in* $\mathbb{C}^2$ *such that its image through* π_2 *is convex, too.*

In the proof of Proposition 3 (ii) we will use the following

Lemma 5. *Let* $z_1 \in a_1 + r_1 E$, $z_2 \in a_2 + r_2 E$. *Then* $\frac{1}{2}(z_1 + z_2) \in \frac{1}{2}(a_1 + a_2) + \frac{1}{2}(r_1 + r_2)E$.

Recall that $h_a \in \operatorname{Aut}(\mathcal{E}_{2,n})$, where

$$h_a(z) := \frac{\sqrt{1 - \|a\|^2}(\|a\|^2 z - \langle z, a \rangle a) - \|a\|^2 a + \langle z, a \rangle a}{\|a\|^2 (1 - \langle z, a \rangle)},$$
$$z, a \in \mathcal{E}_{2,n}, \ a \neq 0.$$

If $a = 0$ then $h_0 := \mathrm{id}_{\mathcal{E}_{2,n}}$.

Let Σ_n denote the group of all permutations of the set $\{1, 2, \dots, n\}$. For $\sigma \in \Sigma_n$, $z = (z_1, \dots, z_n)$ denote $z_\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$. Thus we have the following

Lemma 6. *Let $a \in \mathcal{E}_{2,n}$ is such that $a = a_\sigma$ for any $\sigma \in \Sigma_n$. If $f : \mathbb{E}_{2,n} \to \mathbb{E}_{2,n}$ is a holomorphic mapping such that $f \circ \pi_n = \pi_n \circ h_a$ then $f \in \mathrm{Aut}(\mathbb{E}_{2,n})$.*

Proposition 7. *$\mathbb{E}_{2,2}$ is Lu Qi-Keng domain.*

In the proof of Proposition 7 we use the following

Lemma 8. *For any $z \in \mathcal{E}_{2,2}$ there exists $\tilde{a} = (a, a) \in \mathcal{E}_{2,2}$ such that $h_{\tilde{a}2}(z) = 0$, where $h_{\tilde{a}} = (h_{\tilde{a}1}, h_{\tilde{a}2}) \in \mathrm{Aut}(\mathcal{E}_{2,2})$.*

References

[1] S. R. Bell *The Bergman kernel function and proper holomorphic mappings*, Trans. Amer. Math. Soc. **270** (1982) 685–691.

[2] A. Edigarian, W. Zwonek *Geometry of the symmetrized polydisc*, Arch. Math. **84** (2005) 364–374.

[3] N. Nikolov, P. Pflug, W. Zwonek *An example of a $\mathbb{C}$-convex domain which is not biholomorphic to a convex domain*, preprint—arXiv:math.CV/0608662, (2006).