Bergman kernel vanishing on a divisor

Julien Keller
C.M.I., Univ. de Provence, Marseille, France

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Abstract. Let M be a smooth projective manifold, and L an ample line bundle over M. We fix $h \in Met^{\infty}(L)$ a smooth hermitian metric on L with positive curvature $\omega = c_1(h) > 0$. The Bergman kernel is the integral kernel of the L^2-orthogonal projection π from the space of smooth sections of L^k to the space $H^0(M, L^k)$ of holomorphic sections of L^k, that is

$$\pi(f)(x) = \int_M B(x,y) f(y) dV_y$$

where $B(x,y) = \sum_{i=1}^N S_i(x)^{*_h} S_i(y)$ with $N = \dim H^0(M, L^k)$ and (S_i) is an orthonormal basis with respect to $\int_M h^k(.,.)\frac{\omega^n}{n!}$. A well-known result of G. Tian states that one can understand the asymptotic behavior of $B(x,y)$ for large k (and this has various nice applications for geometers). Actually, far from the diagonal the Bergman kernel vanishes exponentially fast, while on the diagonal there is the following asymptotic formula :

$$B_k(x) = B(x,x) = k^n + k^{n-1}\frac{scal(\omega)}{2} + O(k^{n-2}) \tag{1}$$

where $scal(\omega)$ stands for the scalar curvature of the Kähler form ω.

We consider now an effective irreducible divisor $D = \{s_D = 0\}$ on M and the *Bergman function vanishing on* D, i.e

$$B_{k,c}(x) = \sum_{i=1}^{N'_k} |S_i(x)|_h^2 \in C^{\infty}(M, \mathbb{R}_+)$$

where c is a constant small enough to ensure the existence of global holomorphic sections of $L^k - ckD$, and $N'_k = \dim H^0(M, L^k - ckD)$. We study the asymptotic behavior of the function $B_{k,c}$ and obtain the following result.

Theorem 1 *Let us denote*

$$\mathcal{NV}_c = \Big\{ x \in M,\ s.t.\ \exists h_D \in Met^{\infty}(\mathcal{O}(D)), with \\ \omega + i\partial\bar\partial \log |s_D|^{2c}_{h_D} > 0 \ and \\ |s_D|_{h_D} \ attains\ its\ maximum\ at\ x \Big\}$$

Then, for $p \in \mathcal{NV}_c$,

$$\lim_{k\to+\infty} \frac{B_{k,c}(p)}{k^n} = 1$$

and for $p \in M\backslash\overline{\mathcal{NV}_c}$,

$$\lim_{k\to+\infty} \frac{B_{k,c}(p)}{k^n} = 0.$$

In particular, the volume of the *non-vanishing set* $\mathcal{NV}_c$ is given in terms of topological constants :

$$Vol_\omega(\mathcal{NV}_c) = \int_{\mathcal{NV}_c} \frac{\omega^n}{n!} = \int_M \frac{c_1(L - cD)^n}{n!} = Vol(L - cD)$$

and $M\backslash\overline{\mathcal{NV}_c}$ is a *canonical* neighborhood of D depending only on h and c. In particular we obtain a generalisation of (1) in that context, which appears to be related to some geometric problem of algebraic stability of the couple (M, L) (to be more precise, to slope stability defined by J. Ross and R. Thomas).