Cross theorem with singularities pluripolar vs. analytic case

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1. Introduction. Main result

We fix an integer $N \geq 2$ and let D_j be a (connected) *Riemann domain of holomorphy* over $\mathbb{C}^{n_j}$, $j = 1, \dots, N$. Let $\varnothing \neq A_j \subset D_j$ be *locally pluriregular*, $j = 1, \dots, N$.

We will use the following conventions. For arbitrary $B_j \subset D_j$, $j = 1, \dots, N$, we write $B'_j := B_1 \times \dots \times B_{j-1}$, $j = 2, \dots, N$, $B''_j := B_{j+1} \times \dots \times B_N$, $j = 1, \dots, N-1$. Thus, for each $j \in \{1, \dots, N\}$, we may write $B_1 \times \dots \times B_N = B'_j \times B_j \times B''_j$ (with natural exceptions for $j \in \{1, N\}$). Analogously, a point $a = (a_1, \dots, a_N) \in D_1 \times \dots \times D_N$ will be frequently written as $a = (a'_j, a_j, a''_j)$, where $a'_j := (a_1, \dots, a_{j-1})$, $a''_j := (a_{j+1}, \dots, a_N)$ (with obvious exceptions for $j \in \{1, N\}$).

We define an *N–fold cross*

$$\boldsymbol{X} = \boldsymbol{X}((D_j, A_j)_{j=1}^N) := \bigcup_{j=1}^N A'_j \times D_j \times A''_j.$$

One may prove that $\boldsymbol{X}$ is connected.

More generally, for arbitrary *pluripolar* sets $\Sigma_j \subset A'_j \times A''_j$, $j = 1, \dots, N$, we define an *N–fold generalized cross*

$$\boldsymbol{T} = \boldsymbol{T}((D_j, A_j, \Sigma_j)_{j=1}^N) := \bigcup_{j=1}^N \Big\{(a'_j, z_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j\Big\} \subset \boldsymbol{X}.$$

We say that $\boldsymbol{T}$ *is generated by* $\Sigma_1, \dots, \Sigma_N$. Obviously, $\boldsymbol{X} = \boldsymbol{T}((D_j, A_j, \varnothing)_{j=1}^\infty)$.

Observe that any 2–fold generalized cross is in fact a 2–fold cross.

Let h_{A_j, D_j} denote the relative extremal function of A_j in D_j, $j = 1, \dots, N$. Recall that

$$h_{A,D} := \sup\{u \in \mathcal{PSH}(D) : u \leq 1,\ u|_A \leq 0\}.$$

Put $\widehat{\boldsymbol{X}} := \{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : h^*_{A_1, D_1}(z_1) + \dots + h^*_{A_N, D_N}(z_N) < 1\}$, where * stands for the upper semicontinuous regularization. One may prove that $\widehat{\boldsymbol{X}}$ is a (connected) domain of holomorphy and $\boldsymbol{X} \subset \widehat{\boldsymbol{X}}$.

Let $M \subset \boldsymbol{T}$ be *relatively closed*. We say that a function $f : \boldsymbol{T} \setminus M \longrightarrow \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{T} \setminus M$ (we write $f \in \mathcal{O}_s(\boldsymbol{T} \setminus M)$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, the function $D_j \setminus M_{(a'_j, \cdot, a''_j)} \ni z_j \longmapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$ is holomorphic in $D_j \setminus M_{(a'_j, \cdot, a''_j)}$, where $M_{(a'_j, \cdot, a''_j)} := \{z_j \in D_j : (a'_j, z_j, a''_j) \in M\}$ is the *fiber of M over* (a'_j, a''_j).

We are going to discuss the following extension theorem with singularities proved in [Jar-Pfl 2003a], [Jar-Pfl 2003b], see also [Jar-Pfl 2007].

Theorem 1.1 (Extension theorem with singularities for crosses)**.** *Under the above assumptions, let $\boldsymbol{T} \subset \boldsymbol{X}$ be an N–fold generalized cross and let $M \subset \boldsymbol{X}$ be a relatively closed set such that*

(†) *for all $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, the fiber $M_{(a'_j, \cdot, a''_j)}$ is pluripolar.*

Then there exist an N–fold generalized cross $\boldsymbol{T}' \subset \boldsymbol{T}$ (generated by pluripolar sets $\Sigma'_j \subset A'_j \times A''_j$ with $\Sigma'_j \supset \Sigma_j$, $j = 1, \dots, N$) and a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that:

(A) $\widehat{M} \cap \boldsymbol{T}' \subset M$,

(B) *for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ the exists an $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $\boldsymbol{T}' \setminus M$,*

(C) *the set $\widehat{M}$ is minimal in that sense that each point of $\widehat{M}$ is singular with respect to the family $\widehat{\mathcal{F}} := \{\widehat{f} : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$ — cf.* [Jar-Pfl 2000], *§ 3.4,*

(D) *if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, the fiber is thin, then $\widehat{M}$ is analytic in $\widehat{\boldsymbol{X}}$ (and in view of* (C)*, either $\widehat{M} = \varnothing$ or $\widehat{M}$ must be of pure codimension one — cf.* [Jar-Pfl 2000], *§ 3.4),*

(E) *if $M = S \cap \boldsymbol{X}$, where $S \subsetneq U$ is an analytic subset of an open connected neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{X}$, then $\widehat{M} \cap U_0 \subset S$ for an open neighborhood $U_0 \subset U$ of $\boldsymbol{X}$ and $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$ for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$,*

(F) *in the situation of* (E)*, if $U = \widehat{\boldsymbol{X}}$, then $\widehat{M}$ is the union of all one codimensional irreducible components of S.*

Observe that in the situation of (E), if $M = S \cap \boldsymbol{X}$ and (†) is satisfied, then for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, the fiber $M_{(a'_j,\cdot,a''_j)}$ is analytic (in particular, thin) and therefore, by (D), the set $\widehat{M}$ must be analytic.

It has been conjectured (in particular, in [Jar-Pfl 2003b]) that in fact conditions (E–F) are consequences of (A–D). Notice that the method of proof of (E–F) used in [Jar-Pfl 2003a] is essentially different than the one of (A–D) in [Jar-Pfl 2003b]. The aim of this paper is to prove this conjecture which finally leads to a uniform presentation of the cross theorem with singularities. Our main result is the following theorem.

Theorem 1.2. *Properties* (E–F) *follow from* (A–D).

2. Proof of Theorem 1.2

Roughly speaking, the main idea of the proof is to show that if $\widehat{M} \cap \boldsymbol{T}' \subset M$, then $\varnothing \neq \widehat{M} \cap \Omega \subset S$ for an open set $\Omega \subset \widehat{\boldsymbol{X}}$. We will need the following extension theorems (without singularities).

Theorem 2.1. (a) (Classical cross theorem — cf. e.g. [Ale-Zer 2001].) *Under the above assumptions, every function $f \in \mathcal{O}_s(\boldsymbol{X})$ extends holomorphically to $\widehat{\boldsymbol{X}}$.*

(b) (Cross theorem for generalized crosses — cf. [Jar-Pfl 2003b], [Jar-Pfl 2007].) *Under the above assumptions, every function $f \in \mathcal{O}_s(\boldsymbol{T}) \cap \mathcal{C}(\boldsymbol{T})$ extends holomorphically to $\widehat{\boldsymbol{X}}$.*

Remark 2.2. (a) The assumptions in Theorem 2.1(b) may be essentially weakened. Namely, using the same method of proof as in [Jar-Pfl 2003b], one may easily show that every function $f \in \mathcal{O}_s(\boldsymbol{T})$ such that for any $j \in \{1, \dots, N\}$ and $b_j \in D_j$, the function $A'_j \times A''_j \setminus \Sigma_j \ni (z'_j, z''_j) \longmapsto f(z'_j, b_j, z''_j)$ is continuous, extends holomorphically to $\widehat{\boldsymbol{X}}$.

(b) We point out that it is still an *open problem* whether for $N \geq 3$ and arbitrary $\boldsymbol{T}$, Theorem 2.1(b) remains true for every $f \in \mathcal{O}_s(\boldsymbol{T})$.

Remark 2.3. If for all $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, the fiber $M_{(a'_j,\cdot,a''_j)}$ is pluripolar, then the sets $\{(a'_j, a_j, a''_j) \in A'_j \times A_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j,\ a_j \notin M_{(a'_j,\cdot,a''_j)}\}$, $j = 1, \dots, N$, are non-pluripolar (cf. [Jar-Pfl 2007]).

Lemma 2.4. *Let $Q \subset \widehat{\boldsymbol{X}}$ be an arbitrary analytic set of pure codimension one and let $\boldsymbol{T} \subset \boldsymbol{X}$ be an arbitrary generalized cross. Then $Q \cap \boldsymbol{T} \neq \varnothing$.*

Lemma 2.5. *Condition* (F) *follows from* (A–E).

Thus to prove Theorem 1.2 we only need to check that (E) *follows from* (A–D).

Lemma 2.6. *Suppose that* (A–D) *are true and in the situation of* (E) *we know that $\widehat{M} \cap \boldsymbol{X} \subset M$. Then $\widehat{f} = f$ on $\boldsymbol{X} \setminus M$. Thus, the proof of* (E) *reduces to the inclusion $\widehat{M} \cap U_0 \subset S$.*

Lemma 2.7. *If condition* (E) *is true with $U = \widehat{\boldsymbol{X}}$ (and arbitrary other elements), then it is true with general U. Thus to prove Theorem 1.2 we only need to check that* (E) *with $U = \widehat{\boldsymbol{X}}$ follows from* (A–D).

Lemma 2.8. *To prove* (E) *with $U = \widehat{\boldsymbol{X}}$ we may assume that $S = h^{-1}(0)$ with $h \in \mathcal{O}(\widehat{\boldsymbol{X}})$, $h \not\equiv 0$.*

References

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