Cross theorem with singularities pluripolar vs. analytic case

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1. Introduction. Main result

We fix an integer $N \geq 2$ and let $D_j$ be a (connected) Riemann domain of holomorphy over $\mathbb{C}^n$, $j = 1, \ldots, N$. Let $\emptyset \neq A_j \subset D_j$ be locally pluriregular, $j = 1, \ldots, N$.

We will use the following conventions. For arbitrary $B_j \subset D_j$, $j = 1, \ldots, N$, we write $B_j := B_1 \times \cdots \times B_j$, $j = 2, \ldots, N$, $B_j' := B_j \cup \cdots \cup B_N$, $j = 1, \ldots, N - 1$. Thus, for each $j \in \{1, \ldots, N\}$, we may write $\hat{B}_j := B_j \cup \cdots \cup B_N = B_j \times B_j \times B_j''$ (with natural exceptions for $j \in \{1, N\}$). Analogously, a point $a = (a_1, \ldots, a_N) \in D \times \cdots \times D$ will be frequently written as $a = (a_j', a_j'', a_j''')$, where $a_j' := (a_1, \ldots, a_j - 1)$, $a_j'' := (a_j + 1, \ldots, a_N)$ (with obvious exceptions for $j \in \{1, N\}$).

We define an $N$-fold cross

$$X = X((D_j, A_j)_{j=1}^N) := \bigcup_{j=1}^N A_j' \times D_j \times A_j''.$$

One may prove that $X$ is connected.

More generally, for arbitrary pluripolar sets $\Sigma_j \subset A_j' \times A_j''$, $j = 1, \ldots, N$, we define an $N$-fold generalized cross

$$T = T((D_j, A_j, \Sigma_j)_{j=1}^N) := \bigcup_{j=1}^N \left\{ (a_j', z_j, a_j'') \in A_j' \times D_j \times A_j'' : (a_j', z_j, a_j'') \notin \Sigma_j \right\} \subset X.$$

We say that $T$ is generated by $\Sigma_1, \ldots, \Sigma_N$. Obviously, $X = T((D_j, A_j, \emptyset)_{j=1}^N)$.

Observe that any 2-fold generalized cross is in fact a 2-fold cross.

Let $h_{A_j, D_j}$ denote the relative extremal function of $A_j$ in $D_j$, $j = 1, \ldots, N$. Recall that

$$h_{A_j, D_j} := \sup\{u \in PSH(D) : u \leq 1, u|A_j \leq 0\}.$$

Put $\hat{X} := \{(z_1, \ldots, z_N) \in D_1 \times \cdots \times D_N : h_{A_1, D_1}(z_1) + \cdots + h_{A_N, D_N}(z_N) < 1\}$, where * stands for the upper semicontinuous regularization. One may prove that $\hat{X}$ is a (connected) domain of holomorphy and $X \subset \hat{X}$.

Let $M \subset T$ be relatively closed. We say that a function $f : T \setminus M \rightarrow \mathbb{C}$ is separately holomorphic on $T \setminus M$ (we write $f \in \mathcal{O}_s(T \setminus M)$) if for any $j \in \{1, \ldots, N\}$ and $(a_j', a_j'') \in \Sigma_j$, the function $D_j \setminus M(a_j', a_j'') \supset z_j \mapsto f(a_j', z_j, a_j'') \in \mathbb{C}$ is holomorphic in $D_j \setminus M(a_j', a_j'')$.

We are going to discuss the following extension theorem with singularities proved in [Jar-Pfl 2003a], [Jar-Pfl 2003b], see also [Jar-Pfl 2007].

**Theorem 1.1** (Extension theorem with singularities for crosses). Under the above assumptions, let $T \subset X$ be an $N$-fold generalized cross and let $M \subset X$ be a relatively closed set such that

1. for all $j \in \{1, \ldots, N\}$ and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j$, the fiber $M(a_j', a_j'')$ is pluripolar,

Then there exist an $N$-fold generalized cross $T' \subset T$ (generated by pluripolar sets $\Sigma_j' \subset A_j' \times A_j''$ with $\Sigma_j' \supset \Sigma_j$, $j = 1, \ldots, N$) and a relatively closed pluripolar set $\hat{M} \subset \hat{X}$ such that:

(A) $\hat{M} \cap T' \subset M$,

(B) for every $f \in \mathcal{O}_s(X \setminus M)$ the exists an $\hat{f} \in \mathcal{O}(\hat{X} \setminus \hat{M})$ such that $\hat{f} = f$ on $T' \setminus M$,

(C) the set $\hat{M}$ is minimal in that sense that each point of $\hat{M}$ is singular with respect to the family $\hat{F} := \{\hat{f} : f \in \mathcal{O}_s(X \setminus M)\}$ — cf. [Jar-Pfl 2000], § 3.4,

(D) if for any $j \in \{1, \ldots, N\}$ and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j$, the fiber is thin, then $\hat{M}$ is analytic in $\hat{X}$ (and in view of (C), either $\hat{M} = \emptyset$ or $\hat{M}$ must be of pure codimension one — cf. [Jar-Pfl 2000], § 3.4),

(E) if $M = S \cap X$, where $S \subset U$ is an analytic subset of an open connected neighborhood $U \subset \hat{X}$ of $X$, then $\hat{M} \cap U_0 \subset S$ for an open neighborhood $U_0 \subset U$ of $X$ and $\hat{f} = f$ on $X \setminus M$ for every $f \in \mathcal{O}_s(X \setminus M)$,

(F) in the situation of (E), if $U = \hat{X}$, then $\hat{M}$ is the union of all one codimensional irreducible components of $S$. 

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Observe that in the situation of (E), if \( M = S \cap X \) and \((j)\) is satisfied, then for any \( j \in \{1, \ldots, N\} \) and \((a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j \), the fiber \( M(a'_j, a''_j) \) is analytic (in particular, thin) and therefore, by (D), the set \( \hat{M} \) must be analytic.

It has been conjectured (in particular, in [Jar-Pfl 2003b]) that in fact conditions (E–F) are consequences of (A–D). Notice that the method of proof of (E–F) used in [Jar-Pfl 2003a] is essentially different than the one of (A–D) in [Jar-Pfl 2003b]. The aim of this paper is to prove this conjecture which finally leads to a uniform presentation of the cross theorem with singularities. Our main result is the following theorem.

**Theorem 1.2.** Properties (E–F) follow from (A–D).

2. Proof of Theorem 1.2

Roughly speaking, the main idea of the proof is to show that if \( \hat{M} \cap T' \subset M \), then \( \emptyset \neq \hat{M} \cap \Omega \subset S \) for an open set \( \Omega \subset \hat{X} \). We will need the following extension theorems (without singularities).

**Theorem 2.1.** (a) (Classical cross theorem — cf. e.g. [Ale-Zer 2001]) Under the above assumptions, every function \( f \in \mathcal{O}_a(X) \) extends holomorphically to \( \hat{X} \).

(b) (Cross theorem for generalized crosses — cf. [Jar-Pfl 2003b], [Jar-Pfl 2007].) Under the above assumptions, every function \( f \in \mathcal{O}_a(T) \cap C(T) \) extends holomorphically to \( \hat{X} \).

**Remark 2.2.** (a) The assumptions in Theorem 2.1(b) may be essentially weakened. Namely, using the same method of proof as in [Jar-Pfl 2003b], one may easily show that every function \( f \in \mathcal{O}_a(T) \) such that for any \( j \in \{1, \ldots, N\} \) and \( b_j \in D_j \), the function \( A'_j \times A''_j \setminus \Sigma_j \ni (z'_j, z''_j) \mapsto f(z'_j, b_j, z''_j) \) is continuous, extends holomorphically to \( \hat{X} \).

(b) We point out that it is still an open problem whether for \( N \geq 3 \) and arbitrary \( T \), Theorem 2.1(b) remains true for every \( f \in \mathcal{O}_a(T) \).

**Remark 2.3.** If for all \( j \in \{1, \ldots, N\} \) and \((a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j \), the fiber \( M(a'_j, a''_j) \) is pluripolar, then the sets \( \{(a'_j, a_j, a''_j) \in A'_j \times A_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j, a_j \notin M(a'_j, a''_j)\}, j = 1, \ldots, N \), are non-pluripolar (cf. [Jar-Pfl 2007]).

**Lemma 2.4.** Let \( Q \subset \hat{X} \) be an arbitrary analytic set of pure codimension one and let \( T \subset X \) be an arbitrary generalized cross. Then \( Q \cap T \neq \emptyset \).

**Lemma 2.5.** Condition (F) follows from (A–E).

Thus to prove Theorem 1.2 we only need to check that (E) follows from (A–D).

**Lemma 2.6.** Suppose that (A–D) are true and in the situation of (E) we know that \( \hat{M} \cap X \subset M \). Then \( \hat{f} = f \) on \( X \setminus M \). Thus, the proof of (E) reduces to the inclusion \( \hat{M} \cap U_0 \subset S \).

**Lemma 2.7.** If condition (E) is true with \( U = \hat{X} \) (and arbitrary other elements), then it is true with general \( U \). Thus to prove Theorem 1.2 we only need to check that (E) with \( U = \hat{X} \) follows from (A–D).

**Lemma 2.8.** To prove (E) with \( U = \hat{X} \) we may assume that \( S = h^{-1}(0) \) with \( h \in \mathcal{O}(\hat{X}) \), \( h \neq 0 \).

**References**


