A new cross theorem for separately holomorphic functions

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1. Introduction. Main result

The *classical cross theorem* is the following result:

Theorem 1.1 ([Sic 1969a], [Sic 1969b], [Zah 1976], [Sic 1981a], [Ngu-Sic 1991], [Ngu-Zer 1991], [Ngu-Zer 1995], [NTV 1997], [Ale-Zer 2001], [Zer 2002]). *For each $f \in \mathcal{O}_s(\boldsymbol{X})$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{f} = f$ on $\boldsymbol{X}$ and $\sup_{\widehat{\boldsymbol{X}}} |\widehat{f}| = \sup_{\boldsymbol{X}} |f|$.*

The aim of this note is to extend the above theorem to a class of more general objects, namely (N, k)–*crosses* $\boldsymbol{X}_{N,k}$ defined for $k \in \{1, \dots, N\}$ as follows:

$$\boldsymbol{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N) := \bigcup_{\substack{\alpha_1,\dots,\alpha_N \in \{0,1\}\\ \alpha_1+\dots+\alpha_N = k}} \mathfrak{X}_\alpha,$$

where

$$\mathfrak{X}_\alpha := \mathfrak{X}_{1,\alpha_1} \times \dots \times \mathfrak{X}_{N,\alpha_N}, \quad \mathfrak{X}_{j,\alpha_j} := \begin{cases} D_j, & \text{if } \alpha_j = 1 \\ A_j, & \text{if } \alpha_j = 0 \end{cases}.$$

Notice that N–fold crosses are just $(N, 1)$–crosses in the above terminology. Obviously, $\boldsymbol{X}_{N,N} = D_1 \times \dots \times D_N$. Thus, if $N = 2$, then in fact we have only $\boldsymbol{X}_{2,1}$.

Recall that the theory of extension of separately holomorphic functions had been first developed for $N = 2$. Then the N–fold case (obtained via induction) was considered as a natural generalization of $\boldsymbol{X}_{2,1}$. In our opinion, each of the crosses $\boldsymbol{X}_{N,k}$ may be considered as a *natural* generalization of $\boldsymbol{X}_{2,1}$. Consequently, one should try to find an analogous of the cross theorem for all (N, k)–crosses.

We say that a function $f : \boldsymbol{X}_{N,k} \longrightarrow \mathbb{C}$ is *separately holomorphic* ($f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$) if for all $a = (a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in \{0, 1\}^N$ with $|\alpha| = k$, the function

$$D^\alpha := \prod_{\substack{j \in \{1,\dots,N\}:\\ \alpha_j = 1}} D_j \ni z \longmapsto f(i_{a,\alpha}(z))$$

is holomorphic, where $i_{a,\alpha} : D^\alpha \longrightarrow \mathfrak{X}_\alpha$, $i_{a,\alpha}(z) := (w_1, \dots, w_N)$, $\quad w_j := \begin{cases} z_j, & \text{if } \alpha_j = 1 \\ a_j, & \text{if } \alpha_j = 0 \end{cases}.$

Put

$$\widehat{\boldsymbol{X}}_{N,k} = \widehat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^N) := \Big\{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^N h^*_{A_j,D_j}(z_j) < k\Big\}.$$

Note that $\widehat{\boldsymbol{X}}_{N,N} = D_1 \times \dots \times D_N$.

Let $\varphi_j : D_j \longrightarrow \widetilde{D}_j$ be the envelope of holomorphy (cf. [Jar-Pfl 2000], Definition 1.8.1). Observe that since φ_j is locally biholomorphic, the set $\widetilde{A}_j := \varphi_j(A_j) \subset \widetilde{D}_j$ is locally pluriregular, $j = 1, \dots, N$. Let

$$\widetilde{\boldsymbol{X}}_{N,k} := \mathbb{X}_{N,k}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N), \quad \widehat{\widetilde{\boldsymbol{X}}}_{N,k} := \widehat{\mathbb{X}}_{N,k}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N).$$

Put

$$\varphi : D_1 \times \dots \times D_N \longrightarrow \widetilde{D}_1 \times \dots \times \widetilde{D}_N, \quad \varphi(z_1, \dots, z_N) := (\varphi_1(z_1), \dots, \varphi_N(z_N)).$$

Note that:

- $\varphi(\boldsymbol{X}_{N,k}) \subset \widetilde{\boldsymbol{X}}_{N,k}$,
- $\varphi(\widehat{\boldsymbol{X}}_{N,k}) \subset \widehat{\widetilde{\boldsymbol{X}}}_{N,k}$ (because $h^*_{\widetilde{A}_j,\widetilde{D}_j} \circ \varphi_j \le h^*_{A_j,D_j}$, $j = 1, \dots, N$).

Our main result is the following *cross theorem for (N, k)–crosses.*

Theorem 1.2. *For every $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{\widetilde{\boldsymbol{X}}}_{N,k})$ such that $\widehat{f} \circ \varphi = f$ on $\boldsymbol{X}_{N,k}$ and $\sup_{\widehat{\widetilde{\boldsymbol{X}}}_{N,k}} |\widehat{f}| = \sup_{\boldsymbol{X}_{N,k}} |f|$.*

Lemma 1.3. *Let G be a Riemann domain over $\mathbb{C}^n$, let $D \subset\subset G$ be a Riemann domain of holomorphy, and let $A \subset D$ be non-pluripolar. Put*

$$\Delta(\mu) := \{z \in D : h^*_{A,D}(z) < \mu\}, \quad 0 < \mu \le 1.$$

Then

$$h^*_{\Delta(r),\Delta(s)} = \max\Big\{0, \frac{h^*_{A,D} - r}{s - r}\Big\} \text{ on } \Delta(s), \quad 0 < r < s \le 1.$$

Lemma 1.4. *Assume additionally that $D_1, \dots, D_N$ are Riemann domains of holomorphy. Then*

$$h^*_{\widehat{\boldsymbol{X}}_{N,k-1},\widehat{\boldsymbol{X}}_{N,k}}(z) = \max\Big\{0, \sum_{j=1}^{N} h^*_{A_j,D_j}(z_j) - k + 1\Big\}, \quad z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}_{N,k}, \quad k \in \{2, \dots, N\}.$$

We do not know whether Lemmas 1.3, 1.4 are true for arbitrary Riemann domains.

2. Basic properties of (N,k)–crosses

Remark 2.1. (a) $A_1 \times \dots \times A_N \subset \boldsymbol{X}_{N,k} \subset \widehat{\boldsymbol{X}}_{N,k}$.
(b) $\boldsymbol{X}_{N,k-1} \subset \boldsymbol{X}_{N,k}$, $\widehat{\boldsymbol{X}}_{N,k-1} \subset \widehat{\boldsymbol{X}}_{N,k}$, $k = 2, \dots, N$.
(c) $\boldsymbol{X}_{N,k} = (\boldsymbol{X}_{N-1,k-1} \times D_N) \cup (\boldsymbol{X}_{N-1,k} \times A_N)$, $k = 2, \dots, N-1$, $N \ge 3$.
(d) $\boldsymbol{X}_{N,k}$ and $\widehat{\boldsymbol{X}}_{N,k}$ are connected.
(e) If $(D_{j,k})_{k=1}^{\infty}$ is a sequence of subdomains of D_j such that $D_{j,k} \nearrow D_j$, $D_{j,k} \supset A_{j,k} \nearrow A_j$, $j = 1, \dots, N$, then $\mathbb{X}_{N,k}((A_{j,k}, D_{j,k})_{j=1}^N) \nearrow \boldsymbol{X}_{N,k}$ and

$$\widehat{\mathbb{X}}_{N,k}((A_{j,k}, D_{j,k})_{j=1}^N) \nearrow \widehat{\boldsymbol{X}}_{N,k}.$$

(f) If $D_1, \dots, D_N$ are domains of holomorphy, then $\widehat{\boldsymbol{X}}_{N,k}$ is a domain of holomorphy.

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