A new cross theorem for separately holomorphic functions

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1. Introduction. Main result

The classical cross theorem is the following result:

**Theorem 1.1** ([Sic 1969a], [Sic 1969b], [Zah 1976], [Sic 1981a], [Ngu-Sic 1991], [Ngu-Zer 1991], [Ngu-Zer 1995], [NTV 1997], [Ale-Zer 2001], [Zer 2002]). For each $f \in \mathcal{O}_s\(X\)$ there exists exactly one $\tilde{f} \in \mathcal{O}(\tilde{X})$ such that $\tilde{f} = f$ on $X$ and $\sup_{\tilde{X}} |\tilde{f}| = \sup_X |f|$.

The aim of this note is to extend the above theorem to a class of more general objects, namely $(N, k)$-crosses $X_{N,k}$ defined for $k \in \{1, \ldots, N\}$ as follows:

$$X_{N,k} = \bigcup_{\alpha_1, \ldots, \alpha_N \in \{0, 1\}} \mathcal{X}_\alpha,$$

where

$$\mathcal{X}_\alpha := \mathcal{X}_{1, \alpha_1} \times \cdots \times \mathcal{X}_{N, \alpha_N}, \quad \mathcal{X}_{j, \alpha_j} := \begin{cases} D_{j}, & \text{if } \alpha_j = 1 \\ A_{j}, & \text{if } \alpha_j = 0 \end{cases}$$

Notice that $N$-fold crosses are just $(N, 1)$-crosses in the above terminology. Obviously, $X_{N,N} = D_1 \times \cdots \times D_N$. Thus, if $N = 2$, then in fact we have only $X_{2,1}$.

Recall that the theory of extension of separately holomorphic functions had been first developed for $N = 2$. Then the $N$-fold case (obtained via induction) was considered as a natural generalization of $X_{2,1}$. In our opinion, each of the crosses $X_{N,k}$ may be considered as a natural generalization of $X_{2,1}$. Consequently, one should try to find an analogous of the cross theorem for all $(N, k)$-crosses.

We say that a function $f : X_{N,k} \rightarrow \mathbb{C}$ is separately holomorphic ($f \in \mathcal{O}_s(X_{N,k})$) if for all $a = (a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \{0, 1\}^N$ with $|\alpha| = k$, the function

$$D^\alpha := \prod_{j \in \{1, \ldots, N\}} D_j \ni z \mapsto f(i_{\alpha,a}(z))$$

is holomorphic, where $i_{\alpha,a} : D^\alpha \rightarrow \mathcal{X}_\alpha$, $i_{\alpha,a}(z) := (w_1, \ldots, w_N), \quad w_j := \begin{cases} z_j, & \text{if } \alpha_j = 1 \\ a_j, & \text{if } \alpha_j = 0 \end{cases}$.

Put

$$\tilde{X}_{N,k} = \tilde{X}_{N,k}((A_j, D_j)_{j=1}^N) := \{(z_1, \ldots, z_N) \in D_1 \times \cdots \times D_N : \sum_{j=1}^N h_{A_j,D_j}(z_j) < k\}.$$

Note that $\tilde{X}_{N,N} = D_1 \times \cdots \times D_N$.

Let $\varphi_j : D_j \rightarrow \tilde{D}_j$ be the envelope of holomorphy (cf. [Jar-Pfl 2000], Definition 1.8.1). Observe that since $\varphi_j$ is locally biholomorphic, the set $\tilde{A}_j := \varphi_j(A_j) \subset \tilde{D}_j$ is locally pluriregular, $j = 1, \ldots, N$. Let

$$\bar{X}_{N,k} := \mathcal{X}_{N,k}(\bar{A}_j, D_j)_{j=1}^N), \quad \tilde{X}_{N,k} := \tilde{X}_{N,k}(\bar{A}_j, \tilde{D}_j)_{j=1}^N).$$

Put

$$\varphi : D_1 \times \cdots \times D_N \rightarrow \tilde{D}_1 \times \cdots \times \tilde{D}_N, \quad \varphi(z_1, \ldots, z_N) := (\varphi_1(z_1), \ldots, \varphi_N(z_N)).$$

Note that:

- $\varphi(X_{N,k}) \subset \tilde{X}_{N,k}$,
- $\varphi(\bar{X}_{N,k}) \subset \tilde{X}_{N,k}$ (because $h^{\ast}_{A_j,D_j} \circ \varphi_j \leq h^{\ast}_{\bar{A}_j,D_j}, j = 1, \ldots, N$).

Our main result is the following cross theorem for $(N, k)$-crosses.

**Theorem 1.2.** For every $f \in \mathcal{O}_s(X_{N,k})$ there exists exactly one $\tilde{f} \in \mathcal{O}(\tilde{X}_{N,k})$ such that $\tilde{f} \circ \varphi = f$ on $X_{N,k}$ and $\sup_{\tilde{X}_{N,k}} |\tilde{f}| = \sup_{X_{N,k}} |f|$.
Lemma 1.3. Let $G$ be a Riemann domain over $\mathbb{C}^n$, let $D \subset G$ be a Riemann domain of holomorphy, and let $A \subset D$ be non-pluripolar. Put
\[ \Delta(\mu) := \{ z \in D : h_{A,D}^*(z) < \mu \}, \quad 0 < \mu \leq 1. \]
Then
\[ h_{\Delta(r),\Delta(s)}^* = \max \left\{ 0, \frac{h_{A,D}^* - r}{s - r} \right\} \quad \text{on} \ \Delta(s), \quad 0 < r < s \leq 1. \]

Lemma 1.4. Assume additionally that $D_1, \ldots, D_N$ are Riemann domains of holomorphy. Then
\[ h_{X_{N,k-1},X_{N,k}}^*(z) = \max \left\{ 0, \sum_{j=1}^N h_{A_j,D_j}^*(z_j) - k + 1 \right\}, \quad z = (z_1, \ldots, z_N) \in \overline{X}_{N,k}, \ k \in \{2, \ldots, N\}. \]

We do not know whether Lemmas 1.3, 1.4 are true for arbitrary Riemann domains.

2. Basic properties of $(N,k)$-cuses

Remark 2.1. (a) $A_1 \times \cdots \times A_N \subset X_{N,k} \subset \overline{X}_{N,k}$.
(b) $X_{N,k-1} \subset X_{N,k}, \overline{X}_{N,k-1} \subset \overline{X}_{N,k}, k = 2, \ldots, N$.
(c) $X_{N,k} = (X_{N-1,k-1} \times D_N) \cup (X_{N-1,k} \times A_N), k = 2, \ldots, N - 1, N \geq 3$.
(d) $X_{N,k}$ and $\overline{X}_{N,k}$ are connected.
(e) If $(D_{j,k})_{k=1}^\infty$ is a sequence of subdomains of $D_j$ such that $D_{j,k} \not\supset D_j, D_{j,k} \supset A_{j,k} \not\supset A_j, j = 1, \ldots, N$, then $X_{N,k}(A_{j,k}, D_{j,k})_{j=1}^N \not\supset X_{N,k}$ and
\[ \overline{X}_{N,k}(\overline{A}_{j,k}, \overline{D}_{j,k})_{j=1}^N \not\supset \overline{X}_{N,k}. \]
(f) If $D_1, \ldots, D_N$ are domains of holomorphy, then $\overline{X}_{N,k}$ is a domain of holomorphy.

References


