Stability and Kähler Metrics of Constant Scalar Curvature

Abstract

A brief survey is provided of the problem of finding metrics of constant scalar curvature in a given integral positive cohomology class $c_1(L)$ over a compact complex manifold X. This problem can be viewed as the higher-dimensional analogue of the basic uniformization theorem for Riemann surfaces. When L is the anti-canonical bundle of X, it also reduces to the problem of finding Kähler-Einstein metrics of positive scalar curvature, which is itself open at the present time.

A classic conjecture of Yau asserts that the existence of such metrics should be equivalent to the stability of L in the sense of geometric invariant theory (GIT). The prime candidate for the correct notion of stability is K-stability, first introduced by Tian in terms of the asymptotic behavior of the K-energy along one-parameter degenerations, and then modified by Donaldson. The general concept of stability is explained in some detail in the talk. Stability is an algebraic-geometric concept, the goal of which is to insure that the moduli space of stable objects is a well-behaved Hausdorff space. In the present case, the objects are the orbits of Kodaira imbeddings of X under all bases of the space of holomorphic sections of L^k, for k sufficiently large. Different notions of stability correspond to different choices of line bundles over the Hilbert scheme of projective subvarieties with given Hilbert polynomial. The Hilbert-Mumford numerical criterion reduces the stability criteria to the consideration of one-parameter subgroups (more intrinsically, "test configurations"), and thus to the sign of a single numerical invariant for each one-parameter subgroup. In Donaldson's version of K-stability, the numerical invariant is defined in terms of the weights of the action of the one-parameter subgroup on the central fiber.

A proof of Donaldson's lower bound for the Calabi functional is sketched. This lower bound shows that the existence of constant scalar curvature metrics implies semi K-stability. This result has been strengthened recently by Stoppa to K-stability, thus proving the necessity part of the conjecture.

The sufficiency part is still open. A brief discussion is given of some current efforts in this direction, and in particular of those where pluripotential theory plays a prominent role. This includes approaches based on geodesics and Donaldson's earlier infinite-dimensional GIT program (works of X.X. Chen, Blocki, Phong-Sturm, Song-Zelditch and others), and on the Kähler-Ricci flow (Tian-Zhu, Phong-Sesum-Sturm), where the essential tool is the C^0 estimates of Kolodziej.