FINELY PLURISUBHARMONIC FUNCTIONS

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The *plurifine topology* $\mathcal{F}$ on $\mathbb{C}^n$ is the weakest topology in which all plurisubharmonic functions are continuous, in analogy with the H. Cartan fine topology on $\mathbb{R}^n$, in particular on $\mathbb{C} \cong \mathbb{R}^2$. The plurifine topology $\mathcal{F}$ is clearly biholomorphically invariant. We consider two concepts of plurifinely plurisubharmonic functions, ($\mathcal{F}$ PSH functions). a strong concept defined by $\mathcal{F}$-local uniform approximation with plurisubharmonic functions, and a weak concept defined by demanding that the restriction to complex lines be finely subharmonic.

We thereby draw on the theory of *finely sub- or superharmonic* functions defined on finely open subsets of $\mathbb{C}$ or of $\mathbb{R}^N$. These are defined by demanding fine uppersemicintinuity and a Mean Value Inequality with respect to swept out point mass to the complement of a sufficiently large set of fine domains. This is in analogy with the swept out point mass to the complement of a ball, which gives the usual harmonic measure on the ball.

The plurifine topology on $\mathbb{C}^n$ induces on each complex line L in $\mathbb{C}^n$ the Cartan fine topology on $L \simeq \mathbb{C}$.

Strong $\mathcal{F}$-plurisubharmonicity is obviously kept under biholomorphically transformations, and strongly $\mathcal{F}$-plurisubharmonic functions are finely subharmonic on the corresponding domain in $\mathbb{R}^N$, because the approximating functions have these properties.

We show that the same results hold for the weak concept. In analogy with ordinary plurisubharmonic functions we have the following.

Theorem 1. *The weakly $\mathcal{F}$-plurisubharmonic functions f may be characterized by being $\mathcal{F}$-upper semicontinuous and such that $f \circ h$ is $\mathbb{R}^{2n}$-finely subharmonic (or identically $-\infty$ in some fine component of its domain of definition) for every $\mathbb{C}$-linear bijection h of $\mathbb{C}^n$.*

Theorem 2. *If f is weakly $\mathcal{F}$-plurisubharmonic on an $\mathcal{F}$-domain Ω, and h is a holomorphic map on a $\mathcal{F}$-domain D such that $h(D) \subset \Omega$, then $f \circ h$ is weakly $\mathcal{F}$-plurisubharmonic on D.*

The weak concepts are closed under $\mathcal{F}$-locally uniform convergence, and seem altogether to be more useful. We do not know whether strongly and weakly $\mathcal{F}$-plurisubharmonic (resp. weakly $\mathcal{F}$-holomorphic) functions are actually the same.

Other results shortly touched upon include the following. The convex cone of all weakly $\mathcal{F}$-plurisubharmonic functions on Ω is closed under $\mathcal{F}$-locally uniform convergence, and stable under pointwise infimum for lower directed families and under pointwise supremum for finite families. Namely, for any $\mathcal{F}$-locally upper bounded family of weakly $\mathcal{F}$-plurisubharmonic functions f_α on Ω, the $\mathcal{F}$-upper semicontinuous regularization f^* of the pointwise supremum $f = \sup_\alpha f_\alpha$ is likewise weakly $\mathcal{F}$-plurisubharmonic, and the exceptional set $\{f < f^*\}$ is pluripolar, as expected from a theorem of Bedford and Taylor. Furthermore, there is a removable singularity theorem for weakly $\mathcal{F}$-plurisubharmonic functions.

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