Colloquium talk on Thursday October 7, 2010 :

From convexity and curvature to the existence of holomorphic objects

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Abstract: In complex analysis, the relevant concept of convexity is pseudo-convexity; it is defined for a function as the positivity property of its complex Hessian. This leads to the important notions of plurisubharmonic functions and pseudoconvex open sets, which generalize convex functions and convex sets of real geometry, respectively. Given a holomorphic line bundle equipped with a hermitian metric, its curvature tensor is the complex Hessian of the weight associated with the metric. A fundamental theorem of Kodaira (1954) asserts that a line bundle which has a sufficiently large positive curvature has many global holomorphic sections, and moreover that it has vanishing Dolbeault cohomology in higher degrees. Similarly, a strongly pseudoconvex manifold (or Stein manifold) has many global holomorphic sections. Proofs of these results can be obtained through the Bochner identity and global L^2 estimates for d-bar. In 1985, the lecturer developed a theory of holomorphic Morse inequalities which relate the asymptotic behaviour of Dolbeault cohomology groups with certain Monge-Ampère integrals defined by the curvature form. Another basic existence problem is the existence of entire holomorphic mappings $f : \mathbb{C} \to X$ into a projective agebraic variety. When X is a curve, there are non constant such mappings if and only if X is of genus 0 or 1, i.e. of positive or zero curvature. When X is of genus 2 or more, the universal cover is the disc with its negatively curved hyperbolic metric, and there are no such entire maps by Liouville's theorem. Related to this is Faltings' theorem (solution of the Mordell conjecture, 1983), asserting the finiteness of $X(\mathbb{Q})$ for curves of genus at least 2 defined over $\mathbb{Q}$. For higher dimensions $n \geq 2$, a fundamental conjecture of Green, Griffiths and Lang states that if X is a projective variety of general type, there should exist a proper algebraic variety $Y \subset X$ containing all images of non constant entire maps $f : \mathbb{C} \to X$, the number theoretic counterpart being that $X(\mathbb{Q})$ is contained in Y except for finitely many points. Using a probabilistic approach and holomorphic Morse inequalities, we have been able to prove at least that such maps f must satisfy differential algebraic equations $P(f; f', \ldots, f^{(k)}) = 0$. This is done by producing sections of tautological bundles $\mathcal{O}_{X_k}(m)$ on k-jet spaces X_k. The idea is that the mean value of the k-jet curvature converges to minus the Ricci curvature as k tends to $+\infty$ (it is a Monte-Carlo like distribution, where the random variables are the successive normalized derivatives of the jet of the curve f). The Ricci curvature is by assumption negative as X is of general type, and as a consequence the Morse inequalities produce a lot of global differential operators P. One can then use results from value distribution theory and the Ahlfors lemma to conclude that the vanishing property $P(f; f', \ldots, f^{(k)}) = 0$ occurs for all global sections P whose coefficients vanish on an ample divisor of X. The hope is that these equations will ultimately reduce to a purely algebraic equation $P_0(f) = 0$, and that the corresponding algebraic locus $Y = \{P_0(z) = 0\}$ is going to absorb most of the integral points $X(\mathbb{Q})$ when X is defined over $\mathbb{Q}$.

Kraków, 11.10.2010

Seminar talk on Monday October 11, 2010 :
Monge-Ampère operators and holomorphic Morse inequalities

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Abstract: The goal of the talk is to show that there are strong relations between certain Monge-Ampère integrals appearing in holomorphic Morse inequalities, and asymptotic cohomology estimates for tensor powers of holomorphic line bundles. Especially, we prove that these relations hold without restriction for projective surfaces, and in the special case of the volume, i.e. of asymptotic 0-cohomology, for all projective manifolds. These results can be seen as a partial converse to the Andreotti-Grauert vanishing theorem.

More detailed introduction. Let X be a compact complex manifold, $n = \dim_{\mathbb{C}} X$ its complex dimension and $L \to X$ a holomorphic line bundle. In order to estimate the growth of cohomology groups, it is interesting to consider appropriate "asymptotic cohomology functions". Following notation and concepts introduced by A. Küronya, we introduce

(1.1) Definition. *One defines the asymptotic q-cohomology of L to be*

$$\widehat{h}^q(X,L) := \limsup_{k\to+\infty} \frac{n!}{k^n} h^q(X, L^{\otimes k}).$$

Clearly, the definition can also be given for a $\mathbb{Q}$-line bundle L or a $\mathbb{Q}$-divisor D, and in the case $q = 0$ one gets what is usually called the volume of L, namely

$$\mathrm{Vol}(X,L) = \widehat{h}^0(X,L) = \limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, L^{\otimes k}). \tag{1.2}$$

(cf. also work by Demailly-Ein-Lazarsfeld, Boucksom, ...). It can be shown, as was already checked by Küronya in the projective case, that the $\widehat{h}^q$ functional induces a continuous map

$$\mathrm{DNS}_{\mathbb{R}}(X) \ni \alpha \mapsto \widehat{h}^q(X,\alpha)$$

defined on the "divisorial Neron-Severi space" $\mathrm{DNS}_{\mathbb{R}}(X) \subset H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$, i.e. the vector space spanned by real linear combinations of classes of divisors in the real Bott-Chern cohomology group of bidegree $(1,1)$. Here $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ is defined as the quotient of d-closed (p,q)-forms by $\partial\overline{\partial}$-exact (p,q)-forms, and there is a natural conjugation $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{q,p}_{\mathrm{BC}}(X,\mathbb{C})$ which allows us to speak of real classes when $q = p$. Notice that $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ coincides with the usual Dolbeault cohomology group $H^{p,q}(X,\mathbb{C})$ when X is Kähler, and that $\mathrm{DNS}_{\mathbb{R}}(X)$ coincides with the usual Néron-Severi space

$$\mathrm{NS}_{\mathbb{R}}(X) = \mathbb{R} \otimes_{\mathbb{Q}} \big(H^2(X,\mathbb{Q}) \cap H^{1,1}(X,\mathbb{C})\big)$$

when X is projective. It follows from holomorphic Morse inequalities (Demailly, 1985) that asymptotic cohomology can be compared with certain Monge-Ampère integrals.

(1.3) Theorem. *For every holomorphic line bundle L on a compact complex manifold X, one has the "weak Morse inequality"*

(i) $\quad \widehat{h}^q(X,L) \le \inf\limits_{u\in c_1(L)} \int_{X(u,q)} (-1)^q u^n$

where u runs over all smooth d-closed $(1,1)$-forms which belong to the cohomology class $c_1(L) \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$, and $X(u,q)$ is the open set

$$X(u,q) := \big\{z \in X\,;\; u(z) \text{ has signature } (n-q,q)\big\}.$$

Moreover, if we put $X(u,\le q) := \bigcup_{0\le j\le q} X(u,j)$, the "strong Morse inequality"

$$\text{(ii)}\qquad \sum_{j=0}^{q}(-1)^{q-j}\widehat{h}^j(X,L) \le \inf_{u\in c_1(L)} \int_{X(u,\le q)} (-1)^q u^n$$

holds provided that all lim sup*'s involved in the $\widehat{h}^j(X,L)$ are limits.*

It is a natural problem to ask whether the inequalities (1.3) (i) and (1.3) (ii) might not always be equalities. These questions are strongly related to the Andreotti-Grauert vanishing theorem. A well-known variant of this theorem says that if for some integer q and some $u \in c_1(L)$ the form $u(z)$ has at least $n-q+1$ positive eigenvalues everywhere (so that $X(u,\ge q) = \bigcup_{j\ge q} X(u,j) = \emptyset$), then $H^j(X, L^{\otimes k}) = 0$ for $j \ge q$ and $k \gg 1$. We are asking here whether conversely the knowledge that cohomology groups are asymptotically small in a certain degree q implies the existence of a hermitian metric on L with suitable curvature, i.e. no q-index points or only a very small amount of such.

The first goal of the talk is to explain that the answer is positive in the case of the volume functional (i.e. in the case of degree $q = 0$), at least when X is projective algebraic.

(1.4) Theorem. *Let L be a holomorphic line bundle on a projective algebraic manifold. Then*

$$\mathrm{Vol}(X,L) = \inf_{u\in c_1(L)} \int_{X(u,0)} u^n.$$

The proof relies mainly on five ingredients: (a) approximate Zariski decomposition for a Kähler current $T \in c_1(L)$ (when L is big), i.e. a decomposition $\mu^*T = [E] + \beta$ where $\mu : \widetilde{X} \to X$ is a modification, E an exceptional divisor and β a Kähler metric on $\widetilde{X}$; (b) the characterization of the pseudoeffective cone (Boucksom-Demailly-Păun-Peternell, 2004), and the orthogonality estimate

$$E\cdot\beta^{n-1} \le C\big(\mathrm{Vol}(X,L) - \beta^n\big)^{1/2}$$

proved as an intermediate step of that characterization; (c) properties of solutions of Laplace equations to get smooth approximations of $[E]$, (d) log concavity of the Monge-Ampère operator, and finally (e) birational invariance of the Morse infimums. In the case of higher cohomology groups, we are able to treat only the case of projective surfaces :

(1.5) Theorem. *Let $L \to X$ be a holomorphic line bundle on a complex projective surface. Then both weak and strong inequalities* (1.3) (i) *and* (1.3) (ii) *are equalities for $q = 0,\ 1,\ 2$, and the* lim sup*'s involved are limits.*

Thanks to the Serre duality and the Riemann-Roch formula, the (in)equality for a given q is equivalent to the (in)equality for $n-q$. Therefore, on surfaces, the only substantial case which still has to be checked in addition to Theorem 1.4 is the case $q = 1$: this is done by using Grauert' criterion that the intersection matrix $(E_i\cdot E_j)$ is negative definite for every exceptional divisor $E = \sum c_j E_j$. Our statements are of course trivial on curves since the curvature of any holomorphic line bundle can be taken to be constant with respect to any given hermitian metric.

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