Classification of Ricci-flat metrics on the cotangent bundles of compact rank-one symmetric spaces

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The Einstein equation is a partial differential equation which is very difficult to solve in general without symmetries. Isometries G of the manifold play a crucial role to solve the Einstein equation, since they often allow to reduce the equation to a set of algebraic equations or an ordinary differential equation which is easy to solve. It is natural to suppose that G is a compact subgroup of the isometry group.

Calabi-Yau manifolds are defined as compact Kähler manifolds satisfying the Einstein equation with zero Einstein constant, namely the Ricci-flat condition. But homogeneous manifolds of a compact group cannot have Ricci-flat metric, since their scalar curvature is positive. So the next class of manifolds with expected G-invariant Ricci-flat solutions (metrics) is the class of cohomogeneity one manifolds for which some compact group G of isometries is not transitive but generic orbits have dimensions less than the whole space dimension by one. Therefore the classification of Ricci-flat manifolds of cohomogeneity one is important and interesting.

Hyper-Kähler manifolds are Ricci-flat. All such cohomogeneity one manifolds were completely classified by Dancer and Swann [J. Geom. Phys., **21** (1997)]: there exists a unique such irreducible complete manifold with the simple group G, the tangent bundle $T\mathbb{C}P^n$ over the projective space $\mathbb{C}P^n = SU(n+1)/S(U(n)\times U(1))$ with the Calabi metric [Ann. Sci. Ecole Norm. Sup.,**12** (1979)].

General discussion and some steps to the classification of Ricci-flat Kähler manifolds of cohomogeneity one was given by Dancer and Wang [Math. Ann.,**312** (1998)]. Higashijima, Kimura and Nitta constructed Ricci-flat metrics and its Kähler potential on the canonical line bundle over an arbitary Kähler-Einstein manifold which is a coset space G/K [Nuclear Physics B, **645** (2002)]. But a classification of complete cohomogeneity one Ricci-flat Kähler metrics still seems far off. Remark also that more general results on existence of complete Ricci-flat metrics on quasiprojective manifolds (not necessary of cohomogeneity one) was obtained by Tian and Yau.

A remarkable class of Ricci-flat manifolds of cohomogeneity one was discovered by Stenzel [Manuscripta Math., **80** (1993)]. Over the last ten years there has been considerable interest to these Ricci-flat Kähler metrics with underlying manifold diffeomorphic to the tangent bundle of the rank-one symmetric space G/K. One source of interest in physics is that there is a "conifold transition" – both the Stenzel metrics and another family of Ricci-flat Kähler metrics have a cone as a common degenerate limit. Also, Cvetic, Gibbons, Lü and Pope [Commun. Math. Phys., **232** (2003)] have studied the harmonic forms on these metrics and found an explicit formula for the Stenzel metrics in terms of hypergeometric functions. Earlier Lee [Pacific J. of Math., **185** (1998)] gave an explicit formula for the Stenzel metrics for classical spaces G/K but in another terms, using the approach of Patrizio and Wong [Math. Ann., **289** (1991)]. Baptista [J. of Geom. and Phys.,

50 (2004)] used the Stenzel metrics on $SL(2,\mathbb{C}) \simeq T(SU(2))$ for holomorphic quantization of the classical symmetries of the metrics. Exploiting the fact that the Stenzel metrics are of cohomogeneity one with respect to an action of the Lie group G on $T(G/K)$, Dancer and Strachan [preprint, arXiv:math.DG/0202297 (2002)] gave a much more elementary and concrete treatment in the case when the homogeneous space G/K is the round sphere $\mathbb{S}^n = SO(n+1)/SO(n)$. Remark also that in the case of the standard sphere $\mathbb{S}^2$, the Stenzel metric coincides with the well known Eguchi-Hanson metric.

The natural question of classification of such Ricci-flat metrics arises. In the talk a classification of metrics from some family of Kähler invariant metrics on $T(G/K)$ is obtained and it is shown, in particular, that Stenzel's metrics are not equivalent and parameterize the corresponding equivalent classes.

Let G be a compact Lie group and $M = G/K$ be a rank-one symmetric space of dimension $\dim M \geqslant 3$ with a homogeneous metric $\mathbf{g}_M$. The tangent bundle TM is a symplectic manifold with the symplectic structure Ω which comes from the canonical symplectic structure on the cotangent bundle T^*M using the metric $\mathbf{g}_M$ to identify these two bundles. In the talk, all complete Ricci-flat Kähler G-invariant metrics $(\mathbf{g}, J, \Omega)$ on the tangent bundle TM with the fixed Kähler form Ω are classified. It is proved that the set of the equivalent classes $\{[(\mathbf{g}^{\gamma_a}, J^{\gamma_a}, \Omega)]\}$ of these metrics can be parameterized by positive numbers $a > 0$. Each class is an orbit of some infinite dimensional group of symplectomorphisms – the homomorphic image of the additive group of even functions $C^\infty_+(\mathbb{R},\mathbb{R})$. This group of symplectomorphisms is constructed for all reductive (not only compact) spaces. The proposed classification also is based on results of our paper [Sbornik: Mathematics, **192** (2001], where all (not only global) G-invariant Kähler structures (J,Ω) were described, and on the idea of Stenzel to use a global holomorphic trivialization of the canonical line bundle $\Lambda^{n,0}(G^{\mathbb{C}}/K^{\mathbb{C}})$ to reduce the non-linear partial differential equation governing the Ricci form to a simple first-order ordinary differential equation for the function γ_a. In the case when $\dim M = 2$ $(M = \mathbb{S}^2)$ constructed metrics are Ricci-flat but our classification possibly is not complete.

The canonical G-equivariant diffeomorphism $G^{\mathbb{C}}/K^{\mathbb{C}} \to T(G/K)$ supplies the tangent bundle $T(G/K)$ with the canonical complex structure J^K_c. The pair (J^K_c, Ω) is a Kähler structure [R. Szőke, Annales Polonici Mathematici, **LXX** (1998)]. In the talk all G-invariant Ricci-flat Kähler structures (J,Ω) on TM will be described. It is shown that for each such structure (J,Ω) there exists a diffeomorphism ψ^γ such that $J = \psi^\gamma_* J^K_c$ and, in particular, for the Ricci-flat case $J^{\gamma_a} = \psi^{\gamma_a}_* J^K_c$.

The G-invariant Kähler structures, constructed by Stenzel, are structures of the type (J^K_c, Ω'_a), where $\Omega'_a = -i\partial\bar\partial f_a$ for some strictly plurisubharmonic function $f_a(v) = f_a(\|v\|)$ (in this case the canonical complex structure is fixed). Here $\|v\| = \sqrt{\mathbf{g}_M(v,v)}$ denotes the norm of a vector $v \in TM$. It is proved that ${\psi^{\gamma_a}}^*\Omega = 2\Omega'$, i.e. the Kähler structures (J^{γ_a}, Ω) $(J^{\gamma_a} = \psi^{\gamma_a}_* J^K_c)$ and $(J^K_c, 2\Omega'_a)$ are diffeomorphic. Moreover, the smooth function $y(r) = r\gamma_a(r)$, where the function $\gamma_a : \mathbb{R} \to \mathbb{R}$ defines the diffeomorphism $\psi^{\gamma_a} : v \mapsto v \cdot \gamma_a(\|v\|)$, and the derivative $f'_a(r)$ are solutions to the same ordinary differential equation discovered by Stenzel.