Disks with boundaries in totally real manifolds

Definition 1. *We say that a mapping $f \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ is a nearly smooth analytic disk (n.s.a.d.), if there exists $\lambda_0 \in \partial\mathbb{D}$ such that*

$$\forall k : f^{(k)} \text{ extends continuously to } \overline{\mathbb{D}} \setminus \{\lambda_0\}.$$

We say that such f has boundary in a compact set X, if $f(\partial\mathbb{D} \setminus \{\lambda_0\}) \subset X$.

Let L be a compact totally real n-dimensional $\mathcal{C}^\infty$ submanifold of $\mathbb{C}^n$. By $\mathcal{U}_j$ denote the family of all bounded connected components of the set $\mathbb{C} \setminus \pi_j(L)$. For every j fix some family $\mathcal{V}_j \subset \mathcal{U}_j$ containing all but finite many elements of $\mathcal{U}_j$. Define

$$L_j := \pi_j(L) \cup \bigcup \mathcal{V}_j.$$

Our goal is to prove the following

Theorem 2. *There exists a non-constant n.s.a.d. with boundary in L such that*

$$f(\mathbb{D}) \subset L_1 \times \ldots \times L_{n-1} \times \mathbb{C}.$$

The idea of the proof. Fix $r \in (1, \infty) \setminus \mathbb{N}$ and $p \in L$. Define the spaces

$$\begin{aligned}
\widetilde{F} &:= \{f \in \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C}^n) : f(1) = p, f(\partial\mathbb{D}) \subset L\},\\
F &:= \text{the connected component of } \widetilde{F} \text{ containing the constant map } c_p,\\
G &:= \mathcal{C}^r(\mathbb{D}, \mathbb{C}^n)
\end{aligned}$$

and the mapping

$$\Delta : F \ni f \mapsto \frac{\partial f}{\partial \overline{\lambda}} \in G.$$

We use the fact that the space F endowed with the $\mathcal{C}^{r+1}$ norm is a connected smooth Banach manifold, the mapping Δ is a $\mathcal{C}^\infty$ Fredholm mapping of index 0 and $d_{c_p}\Delta$ is an isomorphism.

We use the following

Theorem 3 (Chirka)**.** *If $f \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\gamma \subset \partial\mathbb{D}$ is open and $f^*(\zeta) \in L$ for $\zeta \in \gamma$, then for each k the derivative $f^{(k)}$ extends continuously to the set $\mathbb{D} \cup \gamma$.*

This theorem implies that it is enough to find a non-constant $f \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ with boundary in L which extends continuously to the set $\overline{\mathbb{D}} \setminus \{\lambda_0\}$.

We use the following

Theorem 4 (Vekua)**.** *The mapping $\Psi : \mathcal{C}^r(\mathbb{D}, \mathbb{C}^n) \to \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C}^n)$ given by*

$$\Psi(f)(\lambda) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)}{\lambda - \zeta} \mathcal{L}^n(\zeta)$$

is well-defined and continuous. Moreover, $\frac{\partial}{\partial\overline{\lambda}} \circ \Psi = id$.

The family $\mathcal{W} := \bigcup_j \mathcal{U}_j \setminus \mathcal{V}_j$ consists of finite many open subsets of $\mathbb{C}$, so there exists $\rho > 0$ such that for each $U \in \mathcal{W}$ there exists some open disk $B(b_U, \rho) \subset U$. Fix some $\epsilon \in \left(0, \frac{\rho}{4\|\Psi\|}\right)$. Define

$$\begin{aligned}
G_\epsilon &:= \{g = (g_1, \ldots, g_n) \in G : \|g_j\|_{\mathcal{C}^r} < \epsilon \text{ for } j = 1, \ldots, n-1\},\\
F_\epsilon &:= \text{the connected component of } \Delta^{-1}(G_\epsilon) \text{ containing the map } c_p
\end{aligned}$$

and $\Delta_\epsilon := \Delta|_{F_\epsilon} : F_\epsilon \to G_\epsilon$.

We prove the following

Lemma 5. Δ_ϵ *is not surjective.*

The key point of the proof is to use the following theorem for manifolds F_ϵ and G_ϵ and the mapping Δ_ϵ:

Theorem 6 (Smale)**.** *Let M, N be smooth connected Banach manifolds, and let $\Phi : M \to N$ be a smooth Fredholm mapping of index 0. Suppose that:*

(1) *Φ is a proper mapping,*
(2) *there exist points $x_0 \in M$ and $y_0 \in N$ such that $\Phi^{-1}(\{y_0\}) = \{x_0\}$ and $d_{x_0}\Phi$ is surjective.*

Then Φ is surjective.

By the Lemma 5 we get that one of the conditions (1) or (2) is not fulfilled.

We prove that following lemmas:

Lemma 7. *For every $f = (f_1, \ldots, f_n) \in F_\epsilon$ the following condition holds:*

$$\text{for every } 1 \leq j \leq n-1 \text{ and } U \in \mathcal{U}_j \setminus \mathcal{V}_j \text{ there is } |f_j - b_U| \geq \frac{\rho}{4} \text{ on } \mathbb{D}.$$

Lemma 8. *Let $K \subset \mathbb{C}$ be a compact set, Ω be a connected component of $\mathbb{C} \setminus K$ and $f : \overline{\mathbb{D}} \setminus \{\lambda_0\} \to \mathbb{C}$ be continuous, analytic on $\mathbb{D}$ and such that $f(\partial\mathbb{D} \setminus \{\lambda_0\}) \subset K$. Suppose that $f(\mathbb{D}) \cap \Omega$ is not dense in Ω. Then $f(\mathbb{D}) \cap \Omega = \varnothing$.*

We use the following

Fact 9 (Alexander)**.** *Let $\lambda_0 \in \partial\mathbb{D}$, and let $f, f_k : \overline{\mathbb{D}} \setminus \{\lambda_0\} \to \mathbb{C}^n$ be such that $f_k \to f$ compactly uniformly and $f_k(\partial\mathbb{D} \setminus \{\lambda_0\}) \subset L$. Assume, that:*

(1) *f_k is of class $\mathcal{C}^{r+1}$ on U for each open set $U \subset \mathbb{D}$,* $\operatorname{dist}(U, \lambda_0) > 0$,
(2) $\left\{\frac{\partial f_k}{\partial \overline{\lambda}}\right\}_k$ *is convergent in $\mathcal{C}^r(U, \mathbb{C}^n)$ for each open set $U \subset \mathbb{D}$,* $\operatorname{dist}(U, \lambda_0) > 0$.

Then f is of class $\mathcal{C}^{r+1}$ on U and $f_k \to f$ in $\mathcal{C}^{r+1}(U, \mathbb{C}^n)$ for each open set $U \subset \mathbb{D}$, $\operatorname{dist}(U, \lambda_0) > 0$.

Corollary 10. *If $f, f_k : \overline{\mathbb{D}} \to \mathbb{C}^n$ are such that $f_k \to f$ uniformly on $\overline{\mathbb{D}}$, $f_k(\partial\mathbb{D}) \subset L$, f_k is of class $\mathcal{C}^{r+1}$ on $\mathbb{D}$ and the sequence $\left\{\frac{\partial f_k}{\partial \overline{\lambda}}\right\}_k$ is convergent in $\mathcal{C}^r(\mathbb{D}, \mathbb{C}^n)$, then f is of class $\mathcal{C}^{r+1}$ on $\mathbb{D}$ and $f_k \to f$ in $\mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C}^n)$.*