Finite time extinction of the Kähler-Ricci flow

Abstract. The goal of this talk is to represent main results in this paper of *Jian Song*. The unnormalized Kähler-Ricci was investigated on a Kähler manifold with a polarized initial Kähler metric. The main results is to prove the unnormalized Kähler-Ricci flow become extinction in finite time if and only if the manifold has positive first Chern class and the initial Kähler class is proportional to the first Chern class of the manifold.

Let X be a compact Kähler manifold of dimension n. We consider the following unnormalized Kähler-Ricci flow starting with a Kähler metric ω_0,

$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t) = -Ric(\omega(t)) \\ \omega|_{t=0} = \omega_0. \end{cases} \tag{1.1}$$

And the normalized Kähler-Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t) = -Ric(\omega(t)) - \omega(t) \\ \omega|_{t=0} = \omega_0. \end{cases} \tag{1.2}$$

(1.3). We call also $g(t)$ is metric respect to $\omega(t)$. Assuming $g(t)$ is a smooth solution of flows (1.1) or (1.2) on X for $t \in [0, T[$. We say that the flow becomes extinct at $t = T$ if the diameter of $(X, g(t))$ tends to 0 as $t \to T$. When the flow becomes extinct at $T < +\infty$, we say that it becomes extinct in finite time. A manifold with the positive first Chern class is called Fano manifold (i.e $c_1(X) > 0$).

(1.4) Main theorem. *Let X be a compact Kähler manifold, $dim_{\mathbb{C}} X = n$. The the unnormalized Kähler-Ricci flow (1.1) with an initial Kähler metric ω_0 in the class of $H^2(X, \mathbb{Z})$ becomes extinct in finite time if and only if X is Fano and the initial Kähler class is proportional to $c_1(X)$.*

In order to prove the main theorem, the author used a collection of results about Kähler-Ricci flow those have been done by Perelman, Tian,... and himself. More precisely, in proof of necessary condition, he used a following result of Perelman

(1.5) Perelman's Theorem. *If X is Fano and the initial Kähler metric lies in $c_1(X)$, the diameter of evolving metrics is uniformly bounded above along the normalized Kähler-Ricci flow $\frac{\partial}{\partial t} g = -Ric(g) + g$.*

Then, after scaling this normalized flow back to the unnormalized flow (1.1) we can see the conclusion. On the other hand, the proving of sufficient condition is contained in the following theorem

(1.6) Theorem. *Let X be compact Kähler manifold of dimension n. Let ω_0 be a Kähler metric on X such that $[\omega_0] \in H^2(X, \mathbb{Z})$. Then the diameter is uniformly bounded below from 0 (i.e $\operatorname{diam}(X, \omega(t)) > const > 0$) along unnormalized Kähler-Ricci flow (1.1) if one of the following conditions holds:*

(i) K_X is semi-ample
(ii) K_X is not nef and K_X^{-1} is not ample.
(iii) K_X^{-1} is ample and $[\omega_0]$ is not proportional to $c_1(X)$.
In particularly, if K_X is semi-ample and the Kodaira dimension of X is positive, the diameter tends to infinity of order $\sqrt{t}$ as $t \to \infty$ along the unnormalized Kähler-Ricci flow (1.1).

(1.7) Remark. The importance of polarized initial metric ω_0 (i.e $\omega_0 \in H^2(X, \mathbb{Z})$) is clear because it permits us to use tools which come from algebraic geometry. For examples base-point-free theorem, rationality theorem....