POLYNOMIAL ESTIMATES, EXPONENTIAL CURVES AND DIOPHANTINE APPROXIMATION

Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and $K := \{(e^z, e^{\alpha z}) : |z| \leq 1\}$. If P is a polynomial of degree n in $\mathbb{C}^n$, normalized by $||P||_K = 1$, we obtain sharp estimates for $||P||_{\mathbb{D}^2}$ in terms of n, where $\mathbb{D}^2$ is closed unit bidisk: we have the following

Theorem 1. *Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and let p_s/q_s, $s \geq 0$, be the convergents to α given by its continued fractions expansion. If $q_s \leq n < q_{s+1}$ then*

$$\max\left\{\frac{n^2 \log n}{2} - n^2, \left[\frac{n}{q_s}\right] \log q_{s+1} - n\right\} \leq e_n(\alpha) \leq \frac{n^2 \log n}{2} + 9n^2 + \frac{n}{q_s} \log q_{s+1},$$

where

$$e_n(\alpha) := \log \sup\{||P||_{\mathbb{D}^2} : P \in \mathcal{P}_n, ||P||_K \leq 1\}.$$

For a given sequence $\varepsilon : \mathbb{N} \to (0, +\infty)$ we introduce the sets:

$$\Gamma(\varepsilon) := \{\alpha \in (0,1) \setminus \mathbb{Q} : \limsup_{s \to +\infty} \varepsilon(q_s) \log q_{s+1} = +\infty\},$$

$$\mathcal{U}(\varepsilon) := \{\alpha \in (0,1) \setminus \mathbb{Q} : \limsup_{n \to +\infty} \varepsilon(n) e_n(\alpha) = +\infty\}.$$

We have the following:

Proposition 2. *i) If ε satisfies $\sum_{n=1}^{+\infty} n\varepsilon(n) < +\infty$ then the set $\Gamma(\varepsilon)$ is polar.*

ii) If ε is given by $\varepsilon(n) = (x(n) n^2 \log n)^{-1}$, $n > 1$, where $x(n) \geq 1$ is an increasing sequence, then $\Gamma(\varepsilon) = (U)(\varepsilon) \subset \mathcal{S}$, where

$$\mathcal{S} := \left\{\alpha \in (0,1) \setminus \mathbb{Q} : \limsup_{s \to +\infty} \frac{\log q_{s+1}}{q_s^2 \log q_s} = +\infty\right\}.$$