THE μ-SYNTHESIS PROBLEM

TOMASZ WARSZAWSKI

The problem of μ-synthesis is to construct a holomorphic matrix function F on the unit disc $\mathbb{D}$ satisfying a finite number of interpolation conditions and $\mu(F(\lambda)) \leq 1$, $\lambda \in \mathbb{D}$, where μ is a generalization of functions like the spectral radius r or the operator norm.

In the *spectral Nevanlinna-Pick problem* (*SNP*) one has given distinct points $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and $k \times k$ matrices $W_1, \ldots, W_n$, does there exist an analytic function $F : \mathbb{D} \longrightarrow \mathbb{C}^{k\times k}$ such that

$$F(\lambda_j) = W_j,\ j = 1, \ldots, n$$

and

$$r(F(\lambda)) \leq 1,\ \lambda \in \mathbb{D}?$$

For $k = 1$ such F exists iff the matrix

$$\left[\frac{1-\overline{W}_j W_k}{1-\overline{\lambda}_j \lambda_k}\right]_{j,k=1}^{n} \geq 0$$

and this is a classical result known as Pick's Theorem. The new result is for $n = k = 2$:

THEOREM. *Let $\lambda_1, \lambda_2 \in \mathbb{D}$ be distinct points, let W_1, W_2 be nonscalar 2×2 matrices with $r(W_j) < 1$ and let $s_j = \operatorname{tr} W_j$, $p_j = \det W_j$, $j = 1, 2$. Then the following statements are equivalent:*

(1) there exists an analytic function $F : \mathbb{D} \longrightarrow \mathbb{C}^{2\times 2}$ such that

$$F(\lambda_j) = W_j,\ j = 1, 2$$

and

$$r(F(\lambda)) \leq 1,\ \lambda \in \mathbb{D};$$

(2)

$$\max_{\omega\in\mathbb{T}} \left|\frac{(s_2p_1 - s_1p_2)\omega^2 + 2(p_2 - p_1)\omega + s_1 - s_2}{(s_1 - \overline{s}_2p_1)\omega^2 - 2(1 - p_1\overline{p}_2)\omega + \overline{s}_2 - s_1\overline{p}_2}\right| \leq \left|\frac{\lambda_1 - \lambda_2}{1 - \overline{\lambda}_2\lambda_1}\right|;$$

(3)

$$\left[\frac{\overline{(2-\omega s_j)}(2-\omega s_k) - \overline{(2\omega p_j - s_j)}(2\omega p_k - s_k)}{1-\overline{\lambda}_j\lambda_k}\right]_{j,k=1}^{2} \geq 0,\ \omega \in \mathbb{T}.$$

The *spectral Carathéodory-Fejér problem* (*SCF*) is: given $k\times k$ matrices $V_0, \ldots, V_n$, does there exist an analytic function $F : \mathbb{D} \longrightarrow \mathbb{C}^{k\times k}$ such that

$$F^{(j)}(0) = V_j,\ j = 0, \ldots, n$$

and

$$r(F(\lambda)) \leq 1,\ \lambda \in \mathbb{D}?$$

We will show how these problems are related to the symmetrized bidisc and formulate the most general form of the μ-synthesis problem.

REFERENCES

[1] N. J. YOUNG, *Some analysable instances of μ-synthesis*, to appear.