PROPER HOLOMORPHIC MAPPINGS BETWEEN SYMMETRIZED ELLIPSOIDS

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For $n \geqslant 2$ and $p > 0$ let

$$\mathbb{B}_{p,n} := \{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2p} < 1\}$$

denote the *generalized complex ellipsoid*. We shall write $\mathbb{B}_n := \mathbb{B}_{1,n}$, $\mathbb{T} := \partial \mathbb{B}_1$. Note that $\mathbb{B}_{p,n}$ is bounded, complete Reinhardt domain.

Let $\pi_n = (\pi_{n,1}, \dots, \pi_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as follows

$$\pi_{n,k}(z) = \sum_{1 \leqslant j_1 < \dots < j_k \leqslant n} z_{j_1} \dots z_{j_k}, \quad 1 \leqslant k \leqslant n,\ z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Note that π_n is a proper holomorphic mapping with multiplicity $n!$, $\pi_n|_{\mathbb{B}_{p,n}} : \mathbb{B}_{p,n} \to \pi_n(\mathbb{B}_{p,n})$ is proper too.

The set

$$\mathbb{E}_{p,n} := \pi_n(\mathbb{B}_{p,n})$$

is called the *symmetrized* (p, n)*-ellipsoid*. Note that $\mathbb{E}_{p,n}$ is bounded $(1, 2, \dots, n)$-balanced domain (recall that a domain $D \subset \mathbb{C}^n$ is called the $(k_1, \dots, k_n)$*-balanced*, where $k_1, \dots, k_n \in \mathbb{N}$, if $(\lambda^{k_1} z_1, \dots, \lambda^{k_n} z_n) \in D$ for any $(z_1, \dots, z_n) \in D$ and $\lambda \in \overline{\mathbb{B}}_1$).

Our aim is to give necessary and sufficient condition for existence of the proper holomorphic mappings between the symmetrized ellipsoids.

Here is some notation. Let $\mathfrak{S}_n$ denote the group of permutations of the set $\{1, \dots, n\}$. For $\sigma \in \mathfrak{S}_n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ denote $z_\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$. Next, for any $A \subset \mathbb{C}$ put $A_* := A \setminus \{0\}$, $A_*^n := (A_*)^n$. Moreover, for any $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, $t \in \mathbb{C}$ and $r > 0$ we put $zw := (z_1 w_1, \dots, z_n w_n)$, $tz := (tz_1, \dots, tz_n)$, and $z^r := (z_1^r, \dots, z_n^r)$.

Remark 1. (a) Let $l \in \mathbb{N}$. Observe that $\mathbb{C}^n \ni z \mapsto \pi_n(z^l) \in \mathbb{C}^n$ is a symmetric polynomial mapping. According to the fundamental theorem of symmetric polynomials (see e.g. [7]) there is a unique polynomial mapping $P_l : \mathbb{C}^n \to \mathbb{C}^n$ such that $\pi_n(z^l) = P_l(\pi_n(z))$, $z \in \mathbb{C}^n$. In particular, $P_l(\mathbb{E}_{p,n}) = \mathbb{E}_{p/l,n}$ for any $p > 0$.

(b) Fix $A, B, C \in \mathbb{C}$ and put $L := (L_1, \dots, L_n) : \mathbb{C}^n \to \mathbb{C}^n$, where

$$L_j(z) := A \sum_{k=1}^{n} z_k + B z_j + C, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ j = 1, \dots, n.$$

Observe that $\pi_n \circ L$ is a symmetric polynomial mapping. According to the fundamental theorem of symmetric polynomials there is a unique polynomial mapping $S_L : \mathbb{C}^n \to \mathbb{C}^n$ such that $\pi_n \circ L = S_L \circ \pi_n$.

Now we are in position to formulate our main result.

Theorem 2. *There exists proper holomorphic mapping* $f : \mathbb{E}_{p,n} \to \mathbb{E}_{q,n}$ *iff* $p/q \in \mathbb{N}$. *Furthermore, if* $p/q \in \mathbb{N}$, *the only proper holomorphic mappings* $f : \mathbb{E}_{p,n} \to \mathbb{E}_{q,n}$

(a) *in case* $p \neq 1$, *or* $q \neq 1/(2m)$, $m \in \mathbb{N}$, *or* $n \neq 2$ *are of the form*

$$f = P_{p/q} \circ \phi, \tag{1}$$

where $P_{p/q}$ *is as in Remark 1 (a) and* ϕ *is an automorphism of* $\mathbb{E}_{p,n}$*;*

(b) *in case* $p = 1$, $q = 1/(2m)$, $m \in \mathbb{N}$, *and* $n = 2$ *are of the form (1) or*

$$f = P_m \circ \phi_{III} \circ P_2 \circ \phi_{II},$$

where ϕ_{II} *(resp.* ϕ_{III}*) is the automorphism of* $\mathbb{E}_{1,2}$ *(resp.* $\mathbb{E}_{1/2,2}$*) defined in Corollary 4.*

Corollary 3. *Let* $f : \mathbb{E}_{p,n} \to \mathbb{E}_{p,n}$ *be a proper holomorphic self-mapping. Then* f *is an automorphism.*

Corollary 4. (a) *If* $p \neq 1$ *and* $(p, n) \neq (1/2, 2)$ *then the only automorphisms of* $\mathbb{E}_{p,n}$ *are of the form*

$$\phi_I(z_1, z_2, \dots, z_n) = (\zeta z_1, \zeta^2 z_2, \dots, \zeta^n z_n), \quad (z_1, z_2 \dots, z_n) \in \mathbb{E}_{p,n}, \tag{2}$$

where $\zeta \in \mathbb{T}$.

(b) *The only automorphisms of* $\mathbb{E}_{1,n}$*, are of the form*

$$\phi_{II}(z) = \left(\frac{S_{L_{\varphi_{II}},1}(z)}{n(1 - a_0 z_1)}, \dots, \frac{S_{L_{\varphi_{II}},n}(z)}{n^n(1 - a_0 z_1)^n} \right), \quad z = (z_1, \dots, z_n) \in \mathbb{E}_{1,n}, \tag{3}$$

where $S_{L_{\varphi_{II}}} = (S_{L_{\varphi_{II}},1}, \dots, S_{L_{\varphi_{II}},1})$ *is the polynomial mapping as in Remark 1 (b) induced by* $L_{\varphi_{II}} = (L_{\varphi_{II},1}, \dots, L_{\varphi_{II},n}) : \mathbb{C} \to \mathbb{C}^n$*, where*

$$L_{\varphi_{II},j}(z_1, \dots, z_n) := \zeta_1 \Big(\sum_{k=1}^{n} z_k - n a_0 \Big) + \zeta_2 \sqrt{1 - n a_0^2} \Big(\sum_{k=1}^{n} z_k - n z_j \Big),$$

for some $\zeta_1, \zeta_2 \in \mathbb{T}$, $a_0 \in \mathbb{R}$, $a_0^2 < \frac{1}{n}$.

(c) *The only automorphisms of* $\mathbb{E}_{1/2,2}$ *are of the form (2) or*

$$\phi_{III}(z_1, z_2) = \left(\zeta z_1, \zeta^2 \left(\tfrac{1}{4} z_1^2 - z_2 \right) \right), \quad (z_1, z_2) \in \mathbb{E}_{1/2,2}, \tag{4}$$

where $\zeta \in \mathbb{T}$.

Remark 5. It should be mentioned that the automorphisms of the form (2) are special cases of the automorphisms of the form (3).

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