The Hörmander proof of the Bourgain-Milman inequality

(based on a paper by Fedor Nazarov)

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For a convex body $K \subset \mathbb{R}^n$, which additionally is origin-symmetric, one can define the so-called polar body of K

$$K^\circ := \{x \in \mathbb{R} : sup_{y \in K} \langle x, y \rangle \leqslant 1\}.$$

It turns out that the quantity $Vol(K)Vol(K^\circ)$ does not change under linear transforms of the set K. Moreover this quantity can be estimated both from above and below.

In 1939 Mahler proved that

$$\frac{4^n}{(n!)^2} \leqslant Vol(K)Vol(K^\circ) \leqslant 4^n$$

and conjectured that

$$\frac{4^n}{n!} \leqslant Vol(K)Vol(K^\circ) \leqslant \frac{\pi^n}{\Gamma\left(\frac{n}{2}+1\right)^2}.$$

The right hand side inequality was proved by Santalo in 1949, moreover it is known, that the only convex bodies for which equality holds are the ellipsoids.

The left hand side inequality remains an open problem. Bourgain and Milman proved in 1987 that

$$c^n \frac{4^n}{n!} \leqslant Vol(K)Vol(K^\circ),$$

for some constant c, whose actual value is not at all clear. In the talk i will present a proof, that one can choose $c = \frac{\pi^3}{4^3}$. The most significant issue is that the proof of this inequality which comes from real analysis relies on methods from the theory of several complex variables.

The proof is divided into two steps. The first one is to utilize the explicit form of the Bergman kernel of the tube domain

$$T_K := \{x + iy : x \in \mathbb{R}^n, y \in K\},$$

which is

$$K_{T_K}(z,w) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i\langle z-\bar{w},t\rangle}}{J_K(t)}dt,$$

where

$$J_K(t) = \int_K e^{-2\langle x,t\rangle}dx.$$

It turs out that

$$K_{T_K}(0,0) \leqslant \frac{2^n n! Vol(K^\circ)}{(2\pi)^n Vol(K)}$$

The second step is estimating $K_{T_K}(0,0)$ from below. With the help of Hörmander's theory of the $\bar{\partial}$ operator one can construct square integrable holomorphic functions on T_K, which have big values at the point 0 and not too big L^2 norms. Precisely one obtains

$$K_{T_K} \geqslant \left(\frac{\pi^3}{8}\right)^n \frac{1}{Vol(K_{\mathbb{C}})(2\pi)^n},$$

where

$$K_{\mathbb{C}} := \{z \in \mathbb{C}^n : |\langle z,t\rangle| \leqslant 1, \forall t \in K^\circ\}.$$

In the meantime one uses the John's maximal ellipsoid theorem, as well as certain trick called the tensor power trick which is roughly speaking the taking of m- tuple Cartesian product of the convex body K, which allows one to cancel some constants.