PROPER HOLOMORPHIC MAPPINGS BETWEEN COMPLEX ELLIPSOIDS AND GENERALIZED HARTOGS TRIANGLES

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For any bounded domains $D, G \subset \mathbb{C}^n$ let $\operatorname{Prop}(D, G)$ denote the set of proper holomorphic mappings $F : D \to G$, where proper, as usual, means $F^{-1}(K)$ compact in D for every compact $K \subset G$, and let $\operatorname{Aut}(D)$ denote the automorphism group of D, i.e. the set of all biholomorphic self-mappings $F : D \to D$. Moreover, we shall write $\operatorname{Prop}(D) := \operatorname{Prop}(D, D)$.

Our aim is to characterize the sets $\operatorname{Prop}(D, G)$ and $\operatorname{Aut}(D)$ when D, G belong either to the class of the complex ellipsoids or the so-called generalized Hartogs triangles.

Here is some notation. Let Σ_n denote the group of the permutations of the set $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ denote $z_\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ and $\Sigma_n(z) := \{\sigma \in \Sigma_n : z_\sigma = z\}$. We shall also write $\sigma(z) := z_\sigma$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{>0}$ put

$$\Psi_\alpha(z) := z^\alpha := (z_1^{\alpha_1}, \dots, z_n^{\alpha_n}), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

If, moreover, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n_{>0}$ we shall write $\alpha\beta := (\alpha_1\beta_1, \dots, \alpha_n\beta_n)$ and $1/\beta := (1/\beta_1, \dots, 1/\beta_n)$.

Finally, let $\mathbb{U}(n)$ denote the set of unitary mappings $U : \mathbb{C}^n \to \mathbb{C}^n$.

1. Complex ellipsoids

For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$, $n \geq 2$, define the *complex ellipsoid*

$$\mathbb{E}_p := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1 \right\}.$$

Note that $\mathbb{B}_n := \mathbb{E}_{(1,\dots,1)}$ is the unit ball in $\mathbb{C}^n$. We shall write $\mathbb{D} := \mathbb{B}_1$, $\mathbb{T} := \partial\mathbb{D}$. Moreover, if $\alpha/\beta \in \mathbb{N}^n$ then $\Psi_{\alpha/\beta} \in \operatorname{Prop}(\mathbb{E}_\alpha, \mathbb{E}_\beta)$.

Theorem 1 ([7], [4])**.** *Assume that $n \geq 2$, $p, q \in \mathbb{R}^n_{>0}$.*

(a) The following conditions are equivalent
- *(i) $\operatorname{Prop}(\mathbb{E}_p, \mathbb{E}_q) \neq \varnothing$;*
- *(ii) there exists $\sigma \in \Sigma_n$ such that $p_\sigma/q \in \mathbb{N}^n$.*

(b) If $p, q \in \mathbb{N}^n$, then the following conditions are equivalent
- *(i) $F \in \operatorname{Prop}(\mathbb{E}_p, \mathbb{E}_q)$;*
- *(ii) $F = \phi \circ \Psi_{p_\sigma/q} \circ \sigma$, where $\sigma \in \Sigma_n$ is such that $p_\sigma/q \in \mathbb{N}^n$ and $\phi \in \operatorname{Aut}(\mathbb{E}_q)$.*

In particular, $\operatorname{Prop}(\mathbb{E}_p) = \operatorname{Aut}(\mathbb{E}_p)$.

(c) If $0 \leq k \leq n$, $p \in \{1\}^k \times (\mathbb{R}_{>0} \setminus \{1\})^{n-k}$. Then

$$\operatorname{Aut}(\mathbb{E}_p) = \{E_{H,\zeta,\sigma} : H \in \operatorname{Aut}(\mathbb{B}_k),\ \zeta \in \mathbb{T}^{n-k},\ \sigma \in \Sigma_n(p)\},$$

where, for $z = (z', z_{k+1}, \dots, z_n) \in \mathbb{E}_p \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$, $\zeta = (\zeta_{k+1}, \dots, \zeta_n) \in \mathbb{T}^{n-k}$, $a' := H^{-1}(0')$, *and* $\sigma \in \Sigma_n(p)$ *we put*

$$E_{H,\zeta,\sigma}(z) := \left(H(z'), \zeta_{k+1} z_{\sigma(k+1)} \left(\frac{\sqrt{1-\|a'\|^2}}{1-\langle z', a'\rangle} \right)^{1/p_{\sigma(k+1)}}, \dots, \zeta_n z_{\sigma(n)} \left(\frac{\sqrt{1-\|a'\|^2}}{1-\langle z', a'\rangle} \right)^{1/p_{\sigma(n)}} \right).$$

In the general case thesis of Theorem 1 (b) is no longer true (take, for instance, $\Psi_{(2,2)} \circ H \circ \Psi_{(2,2)} \in \operatorname{Prop}(\mathbb{E}_{(2,2)}, \mathbb{E}_{(1/2,1/2)})$, where $H \in \operatorname{Aut}(\mathbb{B}_2)$, $H(0) \neq 0$).

Nevertheless, from the proof of Theorem 1.1 in [4] we easily derive the following

Theorem 2. *Assume that* $n \geq 2$, $p, q \in \mathbb{R}^n_{>0}$. *Then the following conditions are equivalent*

(i) $F \in \operatorname{Prop}(\mathbb{E}_p, \mathbb{E}_q)$;
(ii) $F = \Psi_{p_\sigma/(qr)} \circ \phi \circ \Psi_r \circ \sigma$, *where* $\sigma \in \Sigma_n$ *is such that* $p_\sigma/q \in \mathbb{N}^n$, $r \in \mathbb{N}^n$ *is such that* $p_\sigma/(qr) \in \mathbb{N}^n$, *and* $\phi \in \operatorname{Aut}(\mathbb{E}_{p_\sigma/r})$.

In particular, $\operatorname{Prop}(\mathbb{E}_p) = \operatorname{Aut}(\mathbb{E}_p)$.

2. Generalized Hartogs triangles

Let $n, m \in \mathbb{N}$. For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ and $q = (q_1, \dots, q_m) \in \mathbb{R}^m_{>0}$, define the *generalized Hartogs triangle*

$$\mathbb{F}_{p,q} := \left\{ (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^{n+m} : \sum_{j=1}^n |z_j|^{2p_j} < \sum_{j=1}^m |w_j|^{2q_j} < 1 \right\}.$$

Note that $\mathbb{F}_{p,q}$ is nonsmooth pseudoconvex Reinhardt domain, not containing the origin. Moreover, if $n = m = 1$, then $\mathbb{F}_{1,1}$ is the standard Hartogs triangle.

The problem of characterization of $\operatorname{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$ and $\operatorname{Aut}(\mathbb{F}_{p,q})$ has been investigated in many papers. The necessary and sufficient conditions for the nonemptiness of $\operatorname{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$ are given in [1] for $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$, $n, m \geq 2$, in [2] for $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$, $n, m \geq 2$, and in [8] for $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$, $m = 1$. The explicit form of an $F \in \operatorname{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$ is presented in [8] for $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$, $m = 1$, whereas the description of $\operatorname{Aut}(\mathbb{F}_{p,q})$ may be found in [3] for $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$, $n, m \geq 2$, and in [8] for $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$, $m = 1$.

We consider only the case $n = 1$.

Our first result is the following

Theorem 3. *If* $n = m = 1$, *then for arbitrary* $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}_{>0}$ *the following conditions are equivalent*

(i) $F \in \operatorname{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$;
(ii) $F(z,w) = \begin{cases} \left(\zeta z^k w^{l\tilde{q}/\tilde{p} - kq/p}, \xi w^l\right), & \text{if } q/p \notin \mathbb{N},\ l\tilde{q}/\tilde{p} - kq/p \in \mathbb{Z} \\ \left(\zeta w^{l\tilde{q}/\tilde{p}} B\left(z w^{-q/p}\right), \xi w^l\right), & \text{if } q/p \in \mathbb{N},\ l\tilde{q}/\tilde{p} \in \mathbb{N} \end{cases}$,
where $\zeta, \xi \in \mathbb{T}$, $k, l \in \mathbb{N}$, *and* B *is a finite Blaschke product.*

In particular, $\operatorname{Prop}(\mathbb{F}_{p,q}) \supsetneq \operatorname{Aut}(\mathbb{F}_{p,q})$.

Theorem 3 was proved in [8] for $p, q, \tilde{p}, \tilde{q} \in \mathbb{N}$. Our result gives an affirmative answer to the question posed by the Authors in [6], whether the equivalence (i)$\Leftrightarrow$(ii) remains true for arbitrary $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}_{>0}$. In this case, however, neither the method from [2] (where the assumption $m \geq 2$ is essential) nor the method from [8] (where the assumption $p, q, \tilde{p}, \tilde{q} \in \mathbb{N}$ is essential) can be used. Fortunately, it turns out that one may get our result by careful study of the proof of the main result from [5], where full characterization of nonelementary proper holomorphic mappings between bounded Reinhardt domains in $\mathbb{C}^2$ is given.

Our next result is the following

Theorem 4. *Assume that* $n = 1$, $m \geq 2$, $p, \tilde{p} \in \mathbb{R}_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$.

(a) The following conditions are equivalent

(i) $\operatorname{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}}) \neq \varnothing$;

(ii) $p/\tilde{p} \in \mathbb{N}$ *and there exist* $\sigma \in \Sigma_m$ *such that* $q_\sigma/\tilde{q} \in \mathbb{N}^m$.

(b) The following conditions are equivalent

(i) $F \in \operatorname{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$;

(ii) $F(z, w) = (\zeta z^k, h(w))$, *where* $(z, w) \in \mathbb{F}_{p,q} \cap (\mathbb{C} \times \mathbb{C}^m)$, $\zeta \in \mathbb{T}$, $k \in \mathbb{N}$, $h \in \operatorname{Prop}(\mathbb{E}_q, \mathbb{E}_{\tilde{q}})$, $h(0) = 0$.

In particular, $\operatorname{Prop}(\mathbb{F}_{p,q}) = \operatorname{Aut}(\mathbb{F}_{p,q})$.

(c) If $0 \leq \mu \leq m$, $q \in \{1\}^\mu \times (\mathbb{R}_{>0} \setminus \{1\})^{m-\mu}$, *then*

(1)
$$\operatorname{Aut}(\mathbb{F}_{p,q}) = \{F_{H,\zeta,\xi,\sigma} : H \in \mathbb{U}(\mu),\ \zeta \in \mathbb{T},\ \xi \in \mathbb{T}^{m-\mu},\ \sigma \in \Sigma_{m-\mu}(q_{\mu+1}, \dots, q_m)\},$$

where for $w = (w_1, \dots, w_m)$, $\xi = (\xi_1, \dots, \xi_{m-\mu}) \in \mathbb{T}^{m-\mu}$ *we put*

$$F_{H,\zeta,\xi,\sigma}(z, w) := (\zeta z, H(w_1, \dots, w_\mu), \xi_1 w_{\mu+\sigma(1)} \dots, \xi_{m-\mu} w_{\mu+\sigma(m-\mu)}).$$

Theorem 4 (a) was proved in [1] (for $n, m \geq 2$, $p, \tilde{p} \in \mathbb{N}^n$, and $q, \tilde{q} \in \mathbb{N}^m$) and in [2] (for $n, m \geq 2$, $p, \tilde{p} \in \mathbb{R}^n_{>0}$, and $q, \tilde{q} \in \mathbb{R}^m_{>0}$). Theorem 4 (b) was proved in [3] for $n, m \geq 2$, $p = \tilde{p} \in \mathbb{N}^n$, and $q = \tilde{q} \in \mathbb{N}^m$. Theorem 4 (c) was proved in [3] for $n, m \geq 2$, $p \in \mathbb{N}^n$, and $q \in \mathbb{N}^m$. Part (c) of Theorem 4 gives an affirmative answer to the question posed by the Authors in [6], whether (1) remains true for arbitrary $p \in \mathbb{R}^n_{>0}$, $q \in \mathbb{R}^m_{>0}$ (at least in the case $n = 1$).

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