Kołodziej's subsolution theorem for unbounded pseudoconvex domains

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Bounded domains in $\mathbb{C}^n$

Theorem 1. **[Kołodziej]** *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, and $f \in C(\partial\Omega)$. Let $u \in \mathcal{PSH}(\Omega) \cap L^{\infty}_{loc}(\Omega)$, $\lim_{z\to w} u(z) = f(w)$, for all $w \in \partial\Omega$. If μ is a non-negative and finite measure such that $\mu \leq (dd^c u)^n$, then there exists a uniquely determined bounded plurisubharmonic function v such that $(dd^c v)^n = \mu$ and $\lim_{z\to w} v(z) = f(w)$, for all $w \in \partial\Omega$.*

This theorem was generalized by Åhag, Cegrell, Czyż and Phạm to the maximal class of plurisubharmonic function $\mathcal{E}$, for which the complex Monge-Ampère operator is well defined.

Unbounded domains in $\mathbb{C}^n$

Theorem 2.[Kołodziej] *Given two entire locally bounded, entire plurisubharmonic functions v and w satisfying $w \leq v$, $(dd^c v)^n \leq (dd^c w)^n$ and $\lim_{|z|\to\infty}(v(z)-w(z)) = 0$, one can solve the Monge-Ampère equation $(dd^c u)^n = \mu$ for any measure μ with*

$$(dd^c v)^n \leq \mu \leq (dd^c w)^n .$$

Furthermore, the solution u is unique among functions satisfying $w \leq u \leq v$.

We generalize this theorem to unbounded pseudoconvex domains, and to the following class of plurisubharmonic functions

$$\mathcal{D}(\Omega) := \big\{u \in \mathcal{PSH}(\Omega) : \text{ for all } B(z_0, r) \Subset \Omega \text{ there exists a constant } C = C(B(z_0, r), u) \text{ such that } u \in \mathcal{E} + C\big\} .$$

As an immediate consequence we get the following subsolution theorem.

Theorem 3. **[Åhag, Czyż]** *Let Ω be a bounded or unbounded pseudoconvex domain, and let $u \in \mathcal{D}(\Omega)$ be such that the smallest maximal plurisubharmonic majorant $\tilde{u}$ of u exists. Then for any non-negative Radon measure μ that satisfies $\mu \leq (dd^c u)^n$ there exists $w \in \mathcal{D}(\Omega)$ such that*

$$(dd^c w)^n = \mu \qquad \text{and} \qquad u \leq w \leq \widetilde{u} \quad \text{on } \Omega . \tag{0.1}$$

Furthermore, if μ vanishes on pluripolar sets, then the solution w of (0.1) is uniquely determined.

We end by solving a Monge-Ampère type equation, and proving uniqueness and stability of the solution.

Theorem 4. **[Åhag, Czyż]** *Let Ω be a bounded or unbounded pseudoconvex domain. Let $\varphi \in \mathcal{D}(\Omega)$ be such that the measure $\mu = (dd^c\varphi)^n$ vanishes on pluripolar sets, and assume that the smallest maximal plurisubharmonic majorant $\tilde{\varphi}$ of φ exists. Assume also that $F(x, z) \geq 0$ is a $dx \times d\mu$- measurable function on $\mathbb{R} \times \Omega$ that is continuous in the x variable. If there exists a bounded function g such that*

$$0 \leq F(x, z) \leq g(z) ,$$

then there exists a function $u \in \mathcal{D}(\Omega)$ *that satisfies*

$$(dd^c u)^n = F(u(z), z)\, \mu.$$

Furthermore, if F *is a nondecreasing function in the first variable, then the solution* u *is uniquely determined. Assume that* $0 \leq f, f_j \leq 1$ *are measurable functions such that* $\{f_j\, \mu\}$ *converges to* $\{f\, \mu\}$ *in weak***-topology, as* j *tends to* $+\infty$*, and for each* j *let* u_j *and* u *be solutions of*

$$(dd^c u_j)^n = F(u_j(z), z) f_j(z)\, \mu, \qquad \text{and} \qquad (dd^c u)^n = F(u(z), z) f(z)\, \mu\,.$$

Then we have that $\{u_j\}$ *converges in capacity to* u*, as* j *tends to* $+\infty$*.*