On the completeness of a metric related to the Bergman metric

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We study the completeness of a metric which is related to the Bergman metric of a bounded domain (sometimes called the Burbea metric or Fuks metric). This metric is defined as

$$\tilde{T}_{i\bar{j}}(z) := \left((n+1)T_{i\bar{j}}(z) + \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(T_{p\bar{q}}(z)_{p,q=1,\dots,n})\right).$$

By the well-known formula expressing the Ricci curvature of a Kähler metric this can be interpreted as

$$\tilde{T}_{i\bar{j}}(z) = (n+1)T_{i\bar{j}}(z) - Ric_{i\bar{j}},$$

where $Ric_{i\bar{j}}$ is the Ricci tensor of the Bergman metric. It is well known that the Ricci curvature of the Bergman metric is bounded from above by $n+1$ and hence it follows that $\tilde{T}_{i\bar{j}}$ is positive definite and so it is indeed a metric.

We provide a criterion for its completeness in the spirit of the Kobayashi criterion for the completeness of the Bergman metric. The criterion is

Theorem 1 *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain. If for every $n+1$- tuple of linearly independent $f_0, f_1, \dots, f_n \in L^2_h(\Omega)$ and for every boundary point $z_0 \in \partial\Omega$ and for every sequence $\{z_s\}_{s=1}^\infty \subset \Omega$ of points in Ω with limit (in the Euclidean sense) z_0 there exists a subsequence $\{z_{s_k}\}_{k=1}^\infty$ such that*

$$\lim_{k\to\infty} \left. \frac{\left|\det\begin{pmatrix} f_0(z) & \dots & f_n(z) \\ \frac{\partial f_0}{\partial z_1}(z) & \dots & \frac{\partial f_n}{\partial z_1}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial z_n}(z) & \dots & \frac{\partial f_n}{\partial z_n}(z) \end{pmatrix}\right|^2}{K^{n+1}\det\left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log K\right)} \right|_{z=z_{s_k}} \tag{1}$$
$$< \det\begin{pmatrix} \langle f_0, f_0\rangle_{L^2_h(\Omega)} & \cdots & \langle f_n, f_0\rangle_{L^2_h(\Omega)} \\ \vdots & \ddots & \vdots \\ \langle f_0, f_n\rangle_{L^2_h(\Omega)} & \cdots & \langle f_n, f_n\rangle_{L^2_h(\Omega)} \end{pmatrix},$$

then $\tilde{T}_{i\bar{j}}$ is complete.

The analogy with the Kobayashi criterion enables one to mimic the methods and obtain similar results as those for the Bergman metric.

In particular we prove that in hyperconvex domains our metric is complete.